Analysis of Generalization of a Problem of Vieta's Descend Method with Examples and Computing Support

Abstract. Let c and d be fixed integer numbers. Assume that $(a^2 + b^2 + c)$ is divisible by $(ab + d)$ for some natural numbers a and b . Then the value of the fraction $k\ (= \frac{a^2+b^2+c}{ab+d})$ remains the same. Statement of this kind will be proved in pp. 1-3 and illustrated on some examples in pp. 3-10. The general method of proofs will be unified and simplified. Computing support will be provided: in pages

11-19 a simple program code is defined with the help of which one can hunt for natural numbers a, b with the same integer values of c , d and k. Here, a number of examples are given as well.

Keywords: Vieta's formulae, Vieta's descend method, divisibility, descending pairs of integers.

Introduction

The last problem of 1988 International Mathematical Olympiad and other interesting remarks in [1-2] have inspired me to check problems of this type:

Problem. Let c and d be fixed integers. Assume that

$$(a^2 + b^2 + c) \text{ is divisible by } (ab + d) \tag{1}$$

for some natural numbers a and b . Then the value of the fraction

$$k \doteq \frac{a^2+b^2+c}{ab+d} \tag{2}$$

does not change, no matter how a and b $(\in \mathbb{N})$ vary.

When solving problems of this kind, some parts of all proofs can be done in a unified manner. Recognizing this fact forced me to formulate two lemmas and the so-called "proof-finishing procedure".

Rough Sketch of the Proof of our Problem.

Without loss of generality we can assume that $a \geq b$.

First, we check two simple cases number 1 and 2.1 .

Case 1: $a = b$. We have $k = \frac{2a^2+c}{a^2+d} = 2 + \frac{c-2d}{a^2+d} \in \mathbb{Z}$.

Case 2.1: $a > b = 1$. Then we have $k = \frac{a^2+1+c}{a+d} = a + \frac{c+1-ad}{a+d} \in \mathbb{Z}$.

Then in Case 2.2: $a > b \geq 2$, we will use Lemma 1, Lemma 2 and the so-called finishing procedure.

For $ab + d \neq 0$, (2) is equivalent to $a^2 + b^2 + c = abk + dk$.

By replacing a by x , we obtain the quadratic equation

$x^2 - kbx + (b^2 - dk + c) = 0$ with roots $x_1 (= a)$ and x_2 . (3)

Vieta's formulae apply to give

$a + x_2 = bk$ and $ax_2 = b^2 - dk + c$

from which we get $x_2 = bk - a = \frac{b^2-dk+c}{a} (\in \mathbb{Z})$.

On the other hand, $a > b \Leftrightarrow a = b + p$ for some $p \in \mathbb{N}$.

Lemma 1. If $a > b, ab + d > 0$ and $S \doteq 2bd + p(b^2 + d) - bc > 0$ for $a, b, p \in \mathbb{N}$ and $c, d \in \mathbb{Z}$, then $x_2 < b$.

Proof. $x_2 = bk - a < b \Leftrightarrow b\frac{a^2+b^2+c}{ab+d} < a + b \Leftrightarrow$

$\Leftrightarrow b(a^2 + b^2 + c) < (a + b)(ab + d) \Leftrightarrow S \doteq 2bd + p(b^2 + d) - bc > 0$. ■

Lemma 2. Assume that $a > b$ and $T \doteq b^2(b^2 + c) + d^2 \neq 0$ for $a, b, p \in \mathbb{N}$.

If we replace a by b , and b by x_2 in (2), then the value of k remains the same:

$k = \frac{a^2+b^2+c}{ab+d} = \frac{b^2+ {x_2}^2+c}{bx_2+d}$.

Proof. We form $\frac{b^2+ {x_2}^2+c}{bx_2+d} = \frac{b^2+(bk-a)^2+c}{b(bk-a)+d} = \frac{(b^2k^2-abk+dk)-abk-dk+a^2+b^2+c}{b^2k-ab+d} =$

$= k + \frac{a^2+b^2+c-k(ab+d)}{b^2k-ab+d} = k + \frac{a^2+b^2+c-(\frac{a^2+b^2+c}{ab+d})(ab+d)}{b^2k-ab+d} = k$, provided

$b^2k - ab + d = b^2\left(\frac{a^2+b^2+c}{ab+d}\right) - ab + d = \frac{b^2(b^2+c)+d^2}{ab+d} \neq 0$, i.e.

$T \doteq b^2(b^2 + c) + d^2 \neq 0$ [for $ab + d \neq 0$]. ■

Procedure Finishing (C, K)

So, we can replace the ordered pair (a, b) by the pair (b, x_2) with $a > b > x_2$, and the value of k remains the same. After a finite number of "descending" and "k-preserving" steps $(a, b) \searrow (b, x_2)$, we will arrive at case **C**, where $k = \boldsymbol{K}$, whenever $(ab + d)|(a^2 + b^2 + c)$. ■

Let me present some examples of applications of our Lemmas and "proof-finishing" procedure. These examples will cover many combinations of $sign(c)$ and $sign(d)$ but, in each example, $k \in \mathbb{N}$.

Example 1. If $(a^2 + b^2 + 2)$ is divisible by $(ab + 4)$ for some $a, b \in \mathbb{N}$, then $\frac{a^2+b^2+2}{ab+4} = 12$.

Proof. Without loss of generality we can assume that $a \geq b$.

Case 1: $a = b$. We have

$k = \frac{2a^2+2}{a^2+4} = 2 - \frac{6}{a^2+4} \in \mathbb{Z} \implies (a^2 + 4)|6$. Then

$a^2 + 4 = \{1 \ OR \ 2 \ OR \ 3 \ OR \ 6\} \implies a^2 = \{-3 \ OR \ -2 \ OR \ -1 \ OR \ 2\}$ contradicting $a \in \mathbb{N}$.

Case 2: $a > b$.

Case 2.1: $a > b = 1$. Then we have

$k = \frac{a^2+3}{a+4} = a - 4 + \frac{19}{a+4} \in \mathbb{Z} \implies (a + 4)|19$. Thus,

$a + 4 = \{1 \ OR \ 19\}$.

$a + 4 = 1 \implies a = -3$, contradicting $a \in \mathbb{N}$.

$a + 4 = 19 \implies a = 15$. Then $k = \frac{15^2+3}{15+4} = 12$.

Case 2.2: $a > b \geq 2$. Lemma 1 applies to give $x_2 < b$, because $ab + 4 > 0$ and

$S = 2b(4) + p(b^2 + 4) - b(2) = 6b + p(b^2 + 4) > 0$ for $b, p \in \mathbb{N}$.

Then, Lemma 2 applies to give $k = \frac{a^2+b^2+2}{ab+4} = \frac{b^2+ {x_2}^2+2}{bx_2+4}$, because

$T = b^2(b^2 + 2) + 4^2 > 0$, for $b \in \mathbb{N}$.

Finishing (2.1, 12) applies, whenever $(ab + 4)|(a^2 + b^2 + 2)$. ■

The first few pairs (a, b) in Ex. 1 are as follows:

15, 1; 37, 3; 179, 15; 441, 37; 2133, 179; 5255, 441.

Example 2. If $(a^2 + b^2 + 1)$ is divisible by $(ab + 3)$ for some $a, b \in \mathbb{N}$,

then $\frac{a^2+b^2+1}{ab+3} = 6$.

Proof. Without loss of generality we can assume that $a \geq b$.

Case 1: $a = b$. Then

$k = \frac{2a^2+1}{a^2+3} = 2 - \frac{5}{a^2+3} \in \mathbb{Z} \implies (a^2 + 3)|5$. Then

$a^2 + 3 = \{1 \ OR \ 5\} \implies a^2 = \{-2 \ OR \ 2\}$, contradicting $a \in \mathbb{N}$.

Case 2: $a > b$.

Case 2.1: $a > b = 1$. Then we have

$k = \frac{a^2+2}{a+3} = a - 3 + \frac{11}{a+3} \in \mathbb{Z} \implies (a + 3)|11$. So,

$a + 3 = \{1 \ OR \ 11\}$.

$a + 3 = 1 \implies a = -2$, contradicting $a \in \mathbb{N}$.

On the other hand, $a + 3 = 11 \implies a = 8$. Then $k = \frac{8^2+2}{8+3} = 6$.

<u>Case 2.2</u>: $a > b \geq 2$. Then Lemma 1 applies to give $x_2 < b$, because $ab + 3 > 0$ and $S = 2b(3) + p(b^2 + 3) - b(1) = 5b + p(b^2 + 3) > 0$ for $b, p \in \mathbb{N}$. Then, Lemma 2 applies to give $k = \frac{a^2+b^2+1}{ab+3} = \frac{b^2+ {x_2}^2+1}{bx_2+3}$, because $T = b^2(b^2 + 1) + 3^2 > 0$, for $b \in \mathbb{N}$.

Finishing (2.1, 6) applies, whenever $(ab + 3)|(a^2 + b^2 + 1)$. ■

The first few pairs (a, b) in Ex. 2 are as follows:

8, 1; 13, 2; 47, 8; 76, 13; 274, 47; 443, 76; 1597, 274; 2582, 443; 9308, 1597.

<u>Example 3.</u> If $(a^2 + b^2 - 6)$ is divisible by $(ab + 4)$ for some $a, b \in \mathbb{N}$, then $\frac{a^2+b^2-6}{ab+4} = 4$.

<u>Proof</u>. Without loss of generality we can assume that $a \geq b$.

<u>Case 1</u>: $a = b$. Then we have

$k = \frac{2a^2-6}{a^2+4} = 2 - \frac{14}{a^2+4} \in \mathbb{Z} \implies (a^2 + 4)|14$. Then

$a^2 + 4 = \{1 \ OR \ 2 \ OR \ 7 \ OR \ 14\} \implies a^2 = \{-3 \ OR \ -2 \ OR \ 3 \ OR \ 10\}$, contradicting $a \in \mathbb{N}$.

<u>Case 2</u>: $a > b$.

<u>Case 2.1</u>: $a > b = 1$. Then we have

$k = \frac{a^2-5}{a+4} = a - \frac{4a+5}{a+4} = a - 4 + \frac{11}{a+4} \in \mathbb{Z} \implies (a + 4)|11$.

Then $a + 4 = \{1 \ OR \ 11\}$.

$a + 4 = 1 \implies a = -3$, contradicting $a \in \mathbb{N}$.

$a + 4 = 11 \implies a = 7$. Then $k = \frac{7^2+1-6}{7+4} = 4$.

<u>Case 2.2</u>: $a > b \geq 2$. Then Lemma 1 applies to give $x_2 < b$, because

$ab + 4 > 0$ and $S = 2b(4) + p(b^2 + 4) - b(-6) = 14b + p(b^2 + 4) > 0$

for $b, p \in \mathbb{N}$. Then, Lemma 2 applies to give $k = \frac{a^2+b^2-6}{ab+4} = \frac{b^2+x_2{}^2-6}{bx_2+4}$ because,

for $b \geq 3$: $T = b^2(b^2 - 6) + 4^2 > 0$;

for $b = 2$: $T = 2^2(2^2 - 6) + 4^2 = 8 > 0$; and

for $b = 1$: $T = 1^2(1^2 - 6) + 4^2 = 11 > 0$.

Thus, for $b \in \mathbb{N}$, $T > 0$.

Finishing (2.1, 4) applies, whenever $(ab + 4)|(a^2 + b^2 - 6)$. ■

The first few pairs (a, b) in Ex. 3 are as follows:

7, 1; 13, 3; 27, 7; 49, 13; 101, 27; 183, 49; 377, 101; 683, 183; 1407, 377;

2549, 683; 5251, 1407; 9513, 2549.

<u>**Example 4.**</u> If $(a^2 + b^2 - 11)$ is divisible by $(ab + 3)$ for some $a, b \in \mathbb{N}$,

then $\frac{a^2+b^2-11}{ab+3} = 3$.

<u>**Proof**</u>. Without loss of generality we can assume that $a \geq b$.

<u>Case 1</u>: $a = b$. Then

$k = \frac{2a^2-11}{a^2+3} = 2 - \frac{17}{a^2+3} \in \mathbb{Z} \implies (a^2 + 3)|17$. Then

$a^2 + 3 = \{1 \ OR \ 17\} \implies a^2 = \{-2 \ OR \ 14\}$, contradicting $a \in \mathbb{N}$.

<u>Case 2</u>: $a > b$.

<u>Case 2.1</u>: $a > b = 1$. Then we have

$k = \frac{a^2+1-11}{a+3} = a - 3 - \frac{1}{a+3} \in \mathbb{N} \implies (a+3)|1$. Then

$a + 3 = 1 \implies a = -2$, contradicting $a \in \mathbb{N}$.

<u>Case 2.2</u>: $a > b = 2$. Then $k = \frac{a^2+2^2-11}{2a+3} = \frac{a^2-7}{2a+3} \in \mathbb{Z}$. Checking the values $a = 3, 4, 5, 6, 7$, none of them will do, i.e. $k \notin \mathbb{Z}$. The first value that will do is $a = 8$, for which $k = \frac{8^2-7}{2(8)+3} = 3$.

<u>Case 2.3</u>: $a > b \geq 2$. Then Lemma 1 applies to give $x_2 < b$, because $ab + 3 > 0$ and $S = 2b(3) + p(b^2+3) - b(-11) = 17b + p(b^2+3) > 0$ for $b, p \in \mathbb{N}$. Then, Lemma 2 applies to give $k = \frac{a^2+b^2-11}{ab+3} = \frac{b^2+{x_2}^2-11}{bx_2+3}$ because,

for $b \geq 4$: $T = b^2(b^2-11) + 3^2 > 0$;

for $b = 3$: $T = 3^2(3^2-11) + 3^2 = -9 \neq 0$;

for $b = 2$: $T = 2^2(2^2-11) + 3^2 = -19 \neq 0$; and

for $b = 1$: $T = 1^2(1^2-11) + 3^2 = -1 \neq 0$.

Thus, for $b \in \mathbb{N}$, $T \neq 0$.

Finishing (2.2, 3) applies, whenever $(ab+3)|(a^2+b^2-11)$. ■

The first few pairs (a, b) in Ex. 4 are as follows:

8, 2; 22, 8; 58, 22; 152, 58; 398, 152; 1042, 398; 2728, 1042; 7142, 2728.

<u>Example 5.</u> If (a^2+b^2-6) is divisible by $(ab+8)$ for some $a, b \in \mathbb{N}$, then $\frac{a^2+b^2-6}{ab+8} = 44$.

<u>Proof</u>. Without loss of generality we can assume that $a \geq b$.

<u>Case 1</u>: $a = b$.

Then $k = \frac{2a^2-6}{a^2+8} = 2 - \frac{22}{a^2+8} \in \mathbb{Z} \implies (a^2 + 8)|22$, and

$a^2 + 8 = \{1 \ OR \ 2 \ OR \ 11 \ OR \ 22\} \implies a^2 = \{-7 \ OR \ -6 \ OR \ 3 \ OR \ 14\}$, contradicting $a \in \mathbb{N}$.

Case 2: $a > b$.

Case 2.1: $a > b = 1$. Then we have

$k = \frac{a^2-5}{a+8} = a - 8 + \frac{59}{a+8} \in \mathbb{Z} \implies (a + 8)|59$. Then

$a + 8 = \{1 \ OR \ 59\}$.

$a + 8 = 1 \implies a = -7$, contradicting $a \in \mathbb{N}$.

$a + 8 = 59 \implies a = 51$. Then $k = \frac{51^2-5}{51+8} = 44$.

Case 2.2: $a > b \geq 2$. Then Lemma 1 applies to give $x_2 < b$, because

$ab + 8 > 0$ and $S = 2b(8) + p(b^2 + 8) - b(-6) = 22b + p(b^2 + 8) > 0$

for $b, p \in \mathbb{N}$.

Then, Lemma 2 applies to give $k = \frac{a^2+b^2-6}{ab+8} = \frac{b^2+ {x_2}^2-6}{bx_2+8}$ because,

for $b \geq 3$: $T = b^2(b^2 - 6) + 8^2 > 0$;

for $b = 2$: $T = 2^2(2^2 - 6) + 8^2 = 56 > 0$; and

for $b = 1$: $T = 1^2(1^2 - 6) + 8^2 = 59 > 0$.

Thus, for $b \in \mathbb{N}$, $T \neq 0$.

Finishing (2.1, 44) applies, whenever $(ab + 8)|(a^2 + b^2 - 6)$. ■

The first few pairs (a, b) in Ex. 5 are as follows:

51, 1; 309, 7; 2243, 51.

Example 6. If $(a^2 + b^2)$ is divisible by $(ab - 1)$ for some $a, b \in \mathbb{N}$,

then $\frac{a^2+b^2}{ab-1} = 5$.

Proof. Without loss of generality we can assume that $a \geq b$.

Case 1: $a = b$. Then

$k = \frac{2a^2}{a^2-1} = 2 + \frac{2}{a^2-1} \in \mathbb{Z} \implies (a^2-1)|2$. Then

$a^2 - 1 = \{1\ OR\ 2\} \implies a^2 = \{2\ OR\ 3\}$, contradicting $a \in \mathbb{N}$.

Case 2: $a > b$.

Case 2.1: $a > b = 1$. Then we have

$k = \frac{a^2+1}{a-1} = a + 1 + \frac{2}{a-1} \in \mathbb{Z} \implies (a-1)|2$. Then

$a - 1 = \{1\ OR\ 2\} \implies a = \{2\ OR\ 3\}$.

Case 2.1.1: $a = 2$, $b = 1$. Then $k = \frac{2^2+1}{2-1} = 5$.

Case 2.1.2: $a = 3$, $b = 1$. Then $k = \frac{3^2+1}{3-1} = 5$.

Case 2.2: $a > b \geq 2$. Then $a = b + p$ for some $p \in \mathbb{N}$.

If $p = 1$, then $a = b + 1$, and we have

$ab - 1 = (b^2 + b - 1) \nmid a^2 + b^2 = 2(b^2 + b - 1) + 3$, since

$b^2 + b - 1 \geq 2^2 + 2 - 1 = 5$.

For $p \geq 2$, Lemma 1 applies to give $x_2 < b$, because

$ab - 1 \geq 2a - 1 > 0$ and $S = 2b(-1) + p(b^2 - 1) - b(0) =$

$= b(bp-2) - p \geq 2(2p-2) - p = 3p - 4 \geq 3(2) - 4 = 2 > 0$.

Then, Lemma 2 applies to give $k = \frac{a^2+b^2}{ab-1} = \frac{b^2+{x_2}^2}{bx_2-1} \in \mathbb{N}$, because

$T = b^4 + 1 > 0$, for $\in \mathbb{N}$.

Finishing (2.1, 5) applies, whenever $(ab-1)|(a^2+b^2)$. ■

The first few pairs (a, b) in Ex. 6 are as follows:

2, 1; 3, 1; 9, 2; 14, 3; 43, 9; 67, 14; 206, 43; 321, 67; 987, 206; 1538, 321; 4729, 987; 7369, 1538.

Example 7. If $(a^2 + b^2 - 1)$ is divisible by $(ab - 1)$ for some $a, b \in \mathbb{N}$,

then $\frac{a^2+b^2-1}{ab-1} = 4$.

Proof. Without loss of generality we can assume that $a \geq b$.

Case 1: $a = b$. Then

$k = \frac{2a^2-1}{a^2-1} = 2 + \frac{1}{a^2-1} \in \mathbb{Z} \implies (a^2 - 1)|1 \implies a^2 = 2$, contradicting $a \in \mathbb{N}$.

Case 2.1: $a > b = 1$. Then we have

$k = \frac{a^2}{a-1} = a + 1 + \frac{1}{a-1} \in \mathbb{Z} \implies (a - 1)|1 \implies a = 2$. Then $k = \frac{2^2}{2-1} = 4$.

Case 2.2: $a > b \geq 2$. Then Lemma 1 applies to give $x_2 < b$, because

$ab - 1 \geq 2a - 1 > 0$ and $S = 2b(-1) + p(b^2 - 1) - b(-1) =$

$= b(pb - 1) - p \geq 2(2p - 1) - p = 3p - 2 \geq 1 > 0$, for $a, b, p \in \mathbb{N}$.

Then, Lemma 2 applies to give $k = \frac{a^2+b^2-1}{ab-1} = \frac{b^2+ {x_2}^2-1}{bx_2-1}$ because,

for $b \geq 2$, $T = b^2(b^2 - 1) + 1 > 0$;

for $b = 1$, $T = 1 > 0$. So, $T > 0$, for $b \in \mathbb{N}$.

Finishing (2.1, 4) applies, whenever $(ab - 1)|(a^2 + b^2 - 1)$. ■

The first few pairs (a, b) in Ex. 7 are as follows:

2, 1; 7, 2; 26, 7; 97, 26; 362, 97; 1351, 362; 5042, 1351.

Remarks

Our Lemma 1 does not immediately apply when proving the following statement:

if $(ab - 1)|(a^2 + b^2 + 5)$ for some $a, b \in \mathbb{N}$, then $\frac{a^2+b^2+5}{ab-1} = 10$.

On the other hand, one can prove that

$k = 12$ for $c = -14$ and $d = 6$;

$k = 108$ for $c = -14$ and $d = 12$;

k = 4 for c = 2 and d = 0 ;

k is a complete square for c = 0 and d = 1 (See [1,2]).

Computing Support

1., Hunting for natural numbers a, b with the same values of c, d and k

If somebody wants to hunt for some interesting integer values of k (= $\frac{a^2+b^2+c}{ab+d}$) , then he/she can use the following jbasic program code:

```
Rem "Vieta jump general input n c d output a b k.bas"
[start]
print chr$(13)
print "n:"
input n
print "c:"
input c
print "d:"
input d
print chr$(13)
print "c          d"
print c,d
print chr$(13)
print "a            b            k"
print chr$(13)
for b = 2 to n
for a = 1 to b
```

```
q=(a*a+b*b+c)/(a*b+d)
if q<>int(q) then goto [L1]
print a,b,q
[L1]
next a
next b
goto [start]
end
```

Input: n (the upper bound for natural numbers a, b) , c and d.

Output: a, b, k (obeying the relations $a, b \in N$ and $\frac{a^2+b^2+c}{ab+d} = k \in Z$).

Some examples are listed below [with $0<a<b$]:

c	**d**
2	4

a	**b**	**k**
1	15	12
3	37	12
15	179	12
37	441	12
179	2133	12
441	5255	12

c	**d**
1	3

a	b	k
1	8	6
2	13	6
8	47	6
13	76	6
47	274	6
76	443	6
274	1597	6
443	2582	6

c	d
-6	4

a	b	k
1	7	4
3	13	4
7	27	4
13	49	4
27	101	4
49	183	4
101	377	4
183	683	4
377	1407	4

683	2549	4
1407	5251	4

c	**d**
-11	3

a	**b**	**k**
2	8	3
8	22	3
22	58	3
58	152	3
152	398	3
398	1042	3
1042	2728	3
2728	7142	3

c	**d**
-6	8

a	**b**	**k**
1	51	44
7	309	44
51	2243	44
309	13589	44

c	**d**	
0	-1	
a	**b**	**k**
1	2	5
1	3	5
2	9	5
3	14	5
9	43	5
14	67	5
43	206	5
67	321	5
206	987	5
321	1538	5
987	4729	5
1538	7369	5

c	d	
-1	-1	
a	**b**	**k**
1	2	4
2	7	4
7	26	4
26	97	4
97	362	4
362	1351	4
1351	5042	4

c	d	
-14	6	
a	**b**	**k**
1	17	12
5	61	12
17	203	12
61	727	12
203	2419	12

c	d
-14	12

a	b	k
1	119	108
11	1189	108
119	12851	108

c	d
2	0

a	b	k
1	3	4
3	11	4
11	41	4
41	153	4
153	571	4
571	2131	4
2131	7953	4

c	d
1	0

a	b	k

1	2	3
2	5	3
5	13	3
13	34	3
34	89	3
89	233	3
233	610	3
610	1597	3
1597	4181	3

c	**d**
5	-1

a	**b**	**k**
1	2	10
1	8	10
2	19	10
8	79	10
19	188	10
79	782	10
188	1861	10
782	7741	10

In our last example, **k** is a square number:

c	**d**
0	1

a	**b**	**k**$(= m^2)$
2	8	4
3	27	9
8	30	4
4	64	16
30	112	4
5	125	25
6	216	36
27	240	9
7	343	49
112	418	4
8	512	64
9	729	81
10	1000	100
64	1020	16
11	1331	121
418	1560	4

12	1728	144
240	2133	9
13	2197	169
14	2744	196
125	3120	25
15	3375	225
16	4096	256
17	4913	289
1560	5822	4
18	5832	324
19	6859	361
216	7770	36
20	8000	400

{In some cases here, we have $b = a^3$ and $a^6 + a^2 = a^2(a^4 + 1)$, so $k = a^2$.}

2., Exploiting recursivity

Equation (3) gives us a recursive formula with the help of which we can list the first few terms of an increasing sequence $a_1, b_1 = a_2, b_2 = a_3, \ldots$ obeying (2), with the same value of k. Numerical calculations can be performed by using, e.g., the following jbasic program code:

```
Rem "Vieta's descend recursive formula"
[start]
print "c="
input c
print "d="
input d
print "k="
input k
print chr$(13)
[L1]
print "a="
input a
[L2]
b=(a*k+sqr(a^2*(k^2-4)-4*c+4*d*k))/2
print b
a=b
if a<10^9 then goto [L2]
goto [L1]
end
```

Input: c , d, k and a_1 .

Output: $b_1 = a_2, b_2 = a_3, b_3 = a_4, \ldots$ (obeying the relations $a, b \in N$ and $\frac{a^2+b^2+c}{ab+d} = k \in Z$).

Find some numerical results as follows.

c=2, d=4, k=12

1, 15, 179, 2133, 25417, 302871, 3609035, 43005549, 5.12457553e8;

3, 37, 441, 5255, 62619, 746173, 8891457, 1.05951311e8;

c=1, d=3, k=6

1, 8, 47, 274,1597, 9308, 54251, 316198, 1842937, 10741424, 62605607, 3.64892218e8;

2,13, 76, 443,2582, 15049, 87712, 511223, 2979626, 17366533, 1.01219572e8;

c=-6, d=4, k=4

1, 7, 27,101, 377, 1407,5251, 19597, 73137,272951, 1018667,3801717, 14188201, 52951087, 1.97616147e8;

3, 13, 49, 183,683,2549,9513, 35503, 132499, 494493, 1845473, 6887399, 25704123, 95929093, 3.58012249e8;

c=-11, d=3, k=3

2, 8, 22, 58,152, 398, 1042, 2728, 7142, 18698, 48952, 128158, 335522, 878408, 2299702, 6020698, 15762392, 41266478, 1.08037042e8;

c=-6, d=8, k=44

1, 51, 2243, 98641, 4337961, 1.90771643e8;

7, 309, 13589,597607, 26281119,1.15577163e9;

c=0, d=-1, k=5

2, 9, 43, 206, 987, 4729, 22658, 108561, 520147, 2492174, 11940723, 57211441, 2.74116482e8;

3, 14, 67, 321, 1538, 7369, 35307, 169166, 810523, 3883449, 18606722, 89150161, 4.27144083e8;

c=-1, d=-1, k=4

1, 2, 7, 26, 97, 362, 1351, 5042, 18817, 70226, 262087, 978122, 3650401, 13623482, 50843527, 1.89750626e8;

c=-14, d=6, k=12

1, 17, 203, 2419, 28825, 343481, 4092947, 48771883, 5.81169649e8;

5, 61, 727,8663,103229, 1230085, 14657791, 1.74663407e8;

c=-14, d=12, k=108

1, 119, 12851, 1387789, 1.49868361e8;

11, 1189, 128401, 13866119, 1.49741245e9;

c=2, d=0, k=4

1, 3, 11, 41, 153, 571, 2131, 7953, 29681, 110771, 413403,1542841, 5757961, 21489003, 80198051, 2.99303201e8;

c=1, d=0, k=3

1, 2, 5, 13, 34, 89, 233, 610, 1597, 4181, 10946, 28657, 75025, 196418, 514229, 1346269, 3524578, 9227465, 24157817, 63245986, 1.65580141e8;

c=5, d=-1, k=10

2, 19, 188, 1861, 18422, 182359, 1805168, 17869321, 1.76888042e8;

8, 79, 782, 7741, 76628, 758539, 7508762, 74329081, 7.35782048e8 .

References

[1] https://en.wikipedia.org/wiki/Vieta_jumping

[2] https://brilliant.org/wiki/vieta-root-jumping (2018)