***Original Research Article***

# Stability Analysis and Optimal Control of Endemic Malaria Transmission Model with Cost-Effective Strategies


## ABSTRACT

Malaria is an old deadly parasitic disease that transmitted to the human body through the bites of infected female anopheles' mosquitoes and still treat a serious challenge to the global world population. In this study, a non-linear system of ordinary differential equation model that describe the dynamics of endemic malaria transmission is formulated and analyzed. Conditions are derived for the existence of disease-free and endemic equilibria. Using the next generation matrix method, the basic reproduction number $R_0$ is obtained. Stability of the equilibria are analyzed in terms of the basic reproduction number $R_0$. The disease-free equilibrium point is both locally and globally asymptotically stable whenever $R_0$ is less than unity. The disease can persist whenever $R_0$ is greater than unity and at $R_0$ is equal to unity, the conditions for the existence of bifurcations are also derived. Sensitivity analysis is performed and the important parameters that derive the disease dynamics are also identified. Furthermore, optimal combinations of time dependent control measures, namely, vaccination, insecticide treated nets ITN, treatment and indoor residual spray IRS are incorporated to the model. Pontryagins's maximum principle of optimal control theory is used to find the necessary conditions for the controls to be optimal. Results from numerical simulations show that, the use of the combination of all controls or strategy d perform well for the given time period of intervention in reducing the number of individuals with malaria symptoms and the total mosquito populations to zero. From the cost-effective analysis, the combination of the use of vaccination, insecticide treated net ITN, treatment, and indoor residual spray IRS or strategy **d** is the most optimal cost-effective and efficacious strategy.



*Keywords:* Endemic Malaria, Equilibrium, next generation matrix, basic reproduction number, stability, Optimal control


## 1. INTRODUCTION

Malaria is an old deadly parasitic disease that transmitted to the human through the bites of infected female anopheles' squito and still treat a serious challenge to the global world population. According to the world health organization (WHO) fact sheet of 2018, malaria killed an estimated 438; 000 individuals, where over 90 percent deaths occurred in Africa, while 10 and 2 percent deaths were recorded in Asia and the Eastern Mediterranean Region respectively [1].

The burden of malaria disease affects community socio-economic in many ways. Some of these are fertility, population growth, saving and investment, worker productivity, absenteeism, premature mortality and medical costs [2]. In areas where malaria is highly endemic, young children bears a larger burden in terms of the disease morbidity and mortality and affects fetal development during early stage of pregnancy in women due to loss of immunity. The most popularly used as malaria intervention strategies in sub Saharan Afirica and Asia includes the use of chemotherapy, intermittent preventive treatment for children and pregnant women (preventive Doses of sulfadoxine-pyrimethamine (IPT/ST)), use of insecticides treated bed nets and insecticides against the vector, and recently, RTS,S/AS01 vaccine for children aged <5 years. However; despite intesive control efforts,global incidence of malaria is increasing in malaria endemic area. This is due to resistance of parasites against drugs and mosquitoes against insecticides in sub Saharan Afirica [3,4,5].

Mathematical modeling has become an important tool in understanding the complex dynamics of disease transmission and in decision making processes regarding intervention programs for disease control. Concerning malaria disease, Ross (1911) developed the first mathematical model. He focused his study on mosquito control and showed that for the disease to be eliminated the mosquito population should be brought below a certain threshold [5]. Later the idea of Ross is extended by Macdonald to account for super infection [7]. Ngwa, G. A. Shu, W.S., A mathematical model for endemic malaria with variable human and mosquito population [8].Alemu G. W., Boka K. B., P.R. Koya derived and analyzed deterministic model for the inclusion of infected immigrants on the spread and dynamics of malaria transmission [9] , Chiyaka,C., Garira, and W., Dube, S., derived analyzed effects of treatment and drug resistance on the transmission dynamics of malaria in endemic areas [10], J.Tumwiine, S.D.H-Musekwa and F.Nyabadaza were analyzed a mathematical model for the transmission and spread of drug sensitive and resistant malaria strains within human populations [11]. Other studies are carried out by using optimal control theory. Okosun *et al*. derived and analyzed a malaria disease transmission mathematical model that includes treatment and vaccination with waning immunity and applied optimal control to study the impact of a possible vaccination with treatment strategies in controlling the spread of malaria [12],Chernet T. D., and Gemechis F. D., derived and analyzed 'Modeling and optimal control analysis of transmission dynamics of COVID-19'[13],K. O. Okosun and O. D. Makinde Modelling the impact of drug resistance in malaria transmission and its optimal control analysis [14] , E. Bonyah, M.A. Khan,K.O. Okosun, J.F. Gómez-Aguilar present ''Modeling the effects of heavy alcohol consumption on the transmission dynamics of gonorrhea with optimal [15]. Khan,M.A.;Ali,K; Bonyah,E; Okosun,K.O.;Islam,S.; Khan,A., formulate

Mathematical modeling and stability analysis of Wilit Disease with optimal control [16],Makinde and Okosun, were applied optimal control to study the impact of chemo-therapyon malaria disease with infective immigrants.[17],K. O. Okosun, O. Rachid, and N. Marcus, applied optimal control strategies and cost-effectiveness analysis of a malaria model [18]. Temesgen D. K, O. D. Makinde & Legesse L. O. derived and analyzed Optimal Control and Cost Effectiveness Analysis of SIRS Malaria Disease Model with Temperature Variability [19].

In this paper, we study SITS-SI and SIRS-SI endemic malaria transmission model with standard incidence law that was presented by [14]. Furthermore, we modified the model [8] by omitting the incubating class from the system and incorporate four-time dependent control measures and the class infective in treatment individuals. The purpose of this study is

(i) to minimize the number of individuals with malaria symptoms and the total mosquito populations
(ii) to analyze the effects of the different optimal combinations of control strategies and
(iii) to explore the best strategy in terms of minimizing the spread of endemic malaria disease dynamics and costs.

## 2. MODEL DESCRIPTION AND FORMULATION

The populations are subdivided into compartments according to the individual's disease status. The human populations are divided in to four sub class namely, Susceptible $S_h$, infected $I_h$, Infective in treatment $T_h$ , and Recovered $R_h$. Similarly, the mosquito populations are divided in to Susceptible $S_v$ , and Infected $I_v$ .The total population sizes at time t, for humans are denoted and defied by $N_h(t) = S_h(t) + I_h(t) + T_h(t) + R_h(t)$ and $N_v(t) = S_v(t) + I_v(t)$respectively.Note that, $S_h = S_h(t), I_h = I_h(t), T_h = T_h(t), R_h = R_h(t), S_v = S_v(t), I_v = I_v(t)$ and $N_h = N_h(t)$.

The recruitment into the susceptible human populations $S_h$ is assumed to be at constant rate $\Lambda_h$ and corresponds to birth or immigration. The susceptible human populations either die from natural causes at a rate of $\mu_h$ or move to infected class $I_h$ by acquiring malaria through contact with infected mosquitoes $I_v$ with respective rate of force of infection $\lambda_h = \phi\omega\beta_h \frac{I_v}{N_h}$ where, $\beta_h$ is the rate of probability of human getting infected, $\phi$ is the mosquito contact rate with human and $\omega$ is mosquito biting rate. Infected humans $I_h$ individuals are also either die from natural causes and due to disease death with respective rates $\mu_h$ and $\delta_h$ respectively or move to infective in treatment $T_h$ compartment and recovered class $R_h$ with temporary immunity with respective rates $\gamma(1-\pi_1)$ and $\gamma\pi_1$ respectively. Infective in treatment $T_h$ individuals are individuals with malaria disease that

are getting treated under the control. They also either die from natural causes and due to disease death with respective rates $\mu_h$ and $\delta_h$ respectively or move to the susceptible class with fraction of $\varepsilon$ due to the administered drug kills off the parasites. These infected individuals progress to partially immune group (recovered class), either partially immune group losses immunity and becomes again move to susceptible class with respective rate $\theta$ or die from natural death at a rate $\mu_h$ .Susceptible mosquitoes $S_v$ are recruited at the rate $\Lambda_v$. They either die due to natural death at a rate of $\mu_v$ or move to Infected class $I_v$ by acquiring malaria through contact with both infected humans $I_h$ and partially immune group humans $R_h$ with respective rate of force of infection $\lambda_v = \phi\omega\beta_v\left(\frac{I_h+\sigma_h R_h}{N_h}\right)$ where, $\beta_v$ is the Probability of a mosquito getting infected and $\sigma_h$ is the modification parameter. Infected mosquitoes $I_v$ are die because of natural and disease induced death with respective rates $\mu_v$ and $\delta_v$ respectively.

Diagrammatically, we represent the flow of both the human and mosquito populations from one class to the other as follows

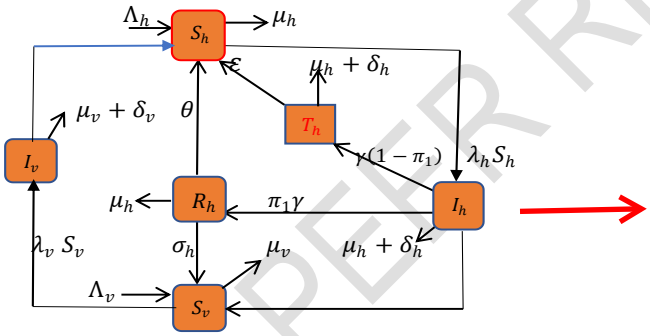



**Figure 1 Flow diagram for the Transmission of Endemic malaria model**

$$\begin{cases} \frac{dS_h}{dt} = \Lambda_h + \varepsilon T_h + \theta R_h - (\lambda_h + \mu_h)S_h \\ \frac{dI_h}{dt} = \lambda_h S_h - (\mu_h + \delta_h + \gamma)I_h \\ \frac{dT_h}{dt} = (1-\pi_1)\gamma I_h - (\delta_h + \mu_h + \varepsilon)T_h \\ \frac{dR_h}{dt} = \pi_1\gamma I_h - (\theta + \mu_h)R_h \\ \frac{dS_v}{dt} = \Lambda_v - (\lambda_v + \mu_v)S_v \\ \frac{dI_v}{dt} = \lambda_v S_v - (\mu_v + \delta_v)I_v \end{cases} \quad (1)$$

$$S_h(0) = S_{0h}\,, I_h(0) = I_{0h}, T_h(0) = T_{0h}\,, R_h(0) = R_{0h}\,, S_v(0) = S_{0v}\,,\ I_v(0) = I_{0v} \quad (2)$$

With some of the following additional assumptions

(i) The susceptible class in both the human and mosquito populations enter into the infective classes by adequate contact with infectious populations not infective in treatment.

(ii) infective individuals in treatment are not infectious to the susceptible populations

(iii) Those infective humans recovered from the disease due to natural immunity and enter into partially immune group

(iv) Those infective individuals in treatment recovered from the disease due to the administered drug kills off the parasites

(v) one part of the recovered class again become susceptible to the disease

(vi) No recovered compartment for mosquitoes.

## 3. Basic Property of the Model

### 3.1 Positivity of the Model

**Theorem 1** Every solution of (1) with initial conditions (2) exists in the interval $[0,\infty)$ and $S_h(t) > 0, I_h(t) > 0, T_h(t) > 0, R_h(t) > 0$, $S_v(t) > 0$ and $I_v(t) > 0$ for all $t \geq 0$.

Proof. To show positivity of solutions, it is enough to show that each of the trajectories of system (1) is non-negative for all $t \geq 0$.

Since the right-hand side of system(1) is completely continuous and locally Lipschitzian on C, the solution $\left(S_h(t),\ I_h(t),\ T_h(t)\ , R_h(t), S_v(t),\ I_v(t)\right)$ of (1) with initial condition (2) exists and unique on $[0,\ k)$ where $0 < k < +\infty$.

It follows from the first equation of system (1) that, the differential inequality describing the evolution of the susceptible human population over time $t$ is given by

$$\frac{dS_h}{dt} \geq \Lambda_h - \left(\phi\omega\beta_h \left(\frac{I_v}{N_h}\right)(t) + \mu_h\right) S_h(t)$$

$$\frac{d}{dt}\left[S_h(t) exp\left\{\mu_h t + \int_0^t \phi\omega\beta_h \left(\frac{I_v}{N_h}\right)(s)\, ds\right\}\right] \geq \Lambda_h exp\left\{\mu_h t + \int_0^t \phi\omega\beta_h \left(\frac{I_v}{N_h}\right)(s)\, ds\right\}$$

Hence,

$$S_h(t) exp\left\{\mu_h t + \int_0^t \phi\omega\beta_h \left(\frac{I_v}{N_h}\right)(s)\, ds\right\} - S_{0h} \geq \int_{\tilde{t}}^t \Lambda_h exp\left\{\mu_h t + \int_0^t \phi\omega\beta_h \left(\frac{I_v}{N_h}\right)(\psi) d\psi\right\} dt$$

Thus,

$$S_h(t) \geq S_{0h} exp\left[-\left\{\mu_h t + \int_0^t \phi\omega\beta_h \left(\frac{I_v}{N_h}\right)(S_h)\, ds\right\}\right] + exp\left[-\left\{\mu_h t + \int_0^t \phi\omega\beta_h \left(\frac{I_v}{N_h}\right)(s)\, ds\right\}\right] \times$$

$$\int_0^t \Lambda_h exp\left\{\mu_h t + \int_0^t \phi\omega\beta_h \left(\frac{I_v}{N_h}\right)(\psi) d\psi\right\} dt > 0.$$

From the second equation of system (1) we have,

$\frac{dI_h}{dt} \geq -(\mu_h + \delta_h + \gamma) I_h(t)$ is equivalent to $I_h(t) \geq exp[-(\mu_h + \delta_h + \gamma)t] > 0$ .

From the third equation of system (1) we have,

$\frac{dT_h}{dt} \geq -(\mu_h + \delta_h + \epsilon) I_h(t)$ is equivalent to $T_h(t) \geq exp[-(\mu_h + \delta_h + \varepsilon)t] > 0.$

From the fourth equation of system (1) we have,

$\frac{dR_h}{dt} \geq -(\mu_h + \theta) I_h(t)$ is equivalent to $R_h(t) \geq exp[-(\mu_h + \theta)t] > 0.$

From the fifth equation of system (1) we have,

$$\frac{dS_v}{dt} \geq \Lambda_v - \left(\int_0^t \phi\omega\beta_v \left(\frac{I_h + \sigma_h R_h}{N_h}\right)(t) + \mu_v\right) S_v$$

$$\frac{d}{dt}\left[S_v(t) exp\left\{\mu_v t + \int_0^t \phi\omega\beta_v \left(\frac{I_h + \sigma_h R_h}{N_h}\right)(s)\, ds\right\}\right] \geq \Lambda_v exp\left\{\mu_h t + \int_0^t \phi\omega\beta_v \left(\frac{I_h + \sigma_h R_h}{N_h}\right)(s)\, ds\right\}$$

Hence,

$$S_v(t) exp\left\{\mu_v t + \int_0^t \phi\omega\beta_v \left(\frac{I_h + \sigma_h R_h}{N_h}\right)(s)\, ds\right\} - S_{0v} \geq$$

$$\int_{\tilde{t}}^t \Lambda_v exp\left\{\mu_v t + \int_0^t \phi\omega\beta_v \left(\frac{I_h + \sigma_h R_h}{N_h}\right)(\psi) d\psi\right\} dt$$

Thus,

$S_v(t) \geq$
$S_{0h} exp\left[-\left\{\mu_v t + \int_0^t \phi\omega\beta_v \left(\frac{I_h+\sigma_h R_h}{N_h}\right)(s)\, ds\right\}\right] + exp\left[-\left\{\mu_v t + \int_0^t \phi\omega\beta_h \left(\frac{I_h+\sigma_h R_h}{N_h}\right)(s)\, ds\right\}\right] \times$
$\int_0^t \Lambda_v exp\left\{\mu_v t + \int_0^t \phi\omega\beta_v \left(\frac{I_h+\sigma_h R_h}{N_h}\right)(\psi) d\psi\right\} dt > 0.$
From the sixth equation of system (1) we have,
$\frac{dI_v}{dt} \geq -(\mu_v + \delta_v) I_v(t)$ is equivalent to $I_v(t) \geq exp[-(\mu_v + \delta_v)t] > 0$ .
Therefore; we can see that $S_h(t) > 0$, $I_h(t) > 0$, $T_h(t) > 0, R_h(t) > 0, S_v(t) > 0$, $I_v(t) > 0$ for all $t \geq 0$.

## 3.2 Invariant Region

**Theorem 2** The feasible region $\Gamma$ defined by

$\Gamma = \{\Gamma_h \times \Gamma_v\} \subset \{\mathbb{R}_+^4 \times \mathbb{R}_+^2\}$ where, $\Gamma_h = \left\{(S_h, \quad I_h, \quad T_h, \quad R_h) \in \mathbb{R}_+^4 : N_h \leq \frac{\Lambda_h}{\mu_h}\right\}$ and

$\Gamma_v = \left\{(S_v, \quad I_v,) \in \mathbb{R}_+^2 : N_v \leq \frac{\Lambda_v}{\mu_v}\right\}$, with initial conditions $S_h(0) = S_{0h}$ , $I_h(0) = I_{0h}$, $T_h(0) = T_{0h}$ , $R_h(0) = R_{0h}$ , $S_v(0) = S_{0v}$ , $I_v(0) = I_{0v}$ , is bounded.

Proof: Let $N_h(t) = S_h(t) + I_h(t) + T_h(t) + R_h(t)$ and $N_v(t) = S_v(t) + I_v(t)$.

The feasible region of both the human and mosquito populations are determined by the feasible region of $N_h(t)$ and $N_v(t)$ respectively as follows

(i) The feasible region of $N_h(t)$: Total sum of human compartments of (1) leads to

$\frac{dN_h}{dt} = \Lambda_h - \mu_h N_h(t) - \delta_h(I_h(t) + T_h(t))$

$\frac{dN_h}{dt} \leq \Lambda_h - \mu_h N_h(t)$

$\frac{dN_h}{dt} + \mu_h N_h(t) \leq \Lambda_h$

The resulting differential inequality can be solved by separation of variables to give,

$\int \frac{d}{dt}(N_h e^{\mu_h t}) \leq \int \Lambda_h e^{\mu_h t}$

Taking the initial conditions $t = 0$ and denoting $N_h(0)$ by $N_{0h}$, then the complete solution $N_h(t) \leq \frac{\Lambda_h}{\mu_h} + \left(N_{0h} - \frac{\Lambda_h}{\mu_h}\right) e^{-\mu_h t}$.

As $t \to \infty$ ,$0 < N_h \leq \frac{\Lambda_h}{\mu_h}$ .So if $N_{0h} \leq \frac{\Lambda_h}{\mu_h}$, then $\lim_{t\to\infty} N_h(t) \leq \frac{\Lambda_h}{\mu_h}$ .This means that $\frac{\Lambda_h}{\mu_h}$ is upper bound of $N_h$.On the other hand if $N_{0h} > \frac{\Lambda_h}{\mu_h}$ ,then $N_h(t)$ will decrease to $\frac{\Lambda_h}{\mu_h}$ Thus $N_{0h} \leq N_h(t) \leq \frac{\Lambda_h}{\mu_h}$. Therefore; the total human population is bounded.

(ii) Invariant region of $N_v(t)$: total sum of mosquito compartments of the system of equations (1) leads to $\frac{dN_v}{dt} = \Lambda_v - \mu_v N_v - \delta_v I_v$

$\frac{dN_v}{dt} \leq \Lambda_v - \mu_v N_v(t)$

$\frac{dN_v}{dt} + \mu_v N_v(t) \leq \Lambda_v$

the resulting differential inequality can be solved by separation of variables to give,

$\int \frac{d}{dt}(N_v e^{\mu_v t}) \leq \int \Lambda_v e^{\mu_v t}$ .

Taking the initial conditions $t = 0$ and denoting $N_v(0)$ by $N_{0v}$, then the complete solution

$N_v(t) \leq \frac{\Lambda_v}{\mu_v} + \left(N_{0v} - \frac{\Lambda_v}{\mu_v}\right) exp(-\mu_v t)$.

As $t \to \infty$ ,$0 < N_v \leq \frac{\Lambda_v}{\mu_v}$ .So if $N_{0v} \leq \frac{\Lambda_v}{\mu_v}$, then $\lim_{t\to\infty} N_v(t) \leq \frac{\Lambda_v}{\mu_v}$ .This means that $\frac{\Lambda_v}{\mu_v}$ is upper bound of $N_v$.On the other hand if $N_{0v} > \frac{\Lambda_v}{\mu_v}$ ,then $N_v(t)$ will decrease to $\frac{\Lambda_v}{\mu_v}$ .Thus $N_{0v} \leq N_v(t) \leq \frac{\Lambda_v}{\mu_v}$.

Therefore; the total mosquito population is bounded.

Thus, the solutions of the model variables representing human populations $\{(S_h, I_h, T_h, R_h)\}$ are confined in the feasible region $\Gamma_h = \left\{(S_h(t), I_h(t), T_h(t), R_h(t)) \in \mathbb{R}_+^4 : N_h \leq \frac{\Lambda_h}{\mu_h}\right\}$. Similarly, the solutions of the model variables representing mosquito populations $\{(S_v, I_v)\}$ are confined in the feasible region $\Gamma_v = \left\{(S_v, I_v) \in \mathbb{R}_+^2 : N_v \leq \frac{\Lambda_v}{\mu_v}\right\}$ .This shows that the feasible region of the model system (1) is bounded and is given by $\Gamma = \{S_h(t), I_h(t), T_h(t), R_h(t), S_v(t), I_v(t)\} \in \mathbb{R}_+^6$ or equivalent to $\Gamma = \{\Gamma_h \times \Gamma_v\} \subset \{\mathbb{R}_+^4 \times \mathbb{R}_+^2\}$.

Thus, in $\Gamma$ the model (1) is well-posed epidemiologically and mathematically. Hence, it is sufficient to study the dynamics of the model in $\Gamma$.

## 4. Disease-free Equilibrium and Basic Reproduction Number, Disease-free Stability

The disease-free equilibrium point of the model is its steady state solutions without infection or disease.

Consider the disease free-equilibrium points denoted and given by: $E_0 = (S_h^0 \quad I_h^0 \quad T_h^0 \quad R_h^0 \quad S_v^0 \quad I_v^0) =$

Where, $S_h^0$ , $I_h^0$, $T^0$, $R_h^0$ , $S_v^0$ and $I_v^0$ are the components of $E_0$ and $I_h^0 = T_h^0 = R_h^0 = I_v^0 = 0$ and the non-infectious are obtained by setting $\frac{dS_h}{dt} = \frac{dV_s}{dt} = \frac{dS_v}{dt} = 0$ for the malaria model system (1) and after computing the resultant gives $S_h^0 = \frac{\Lambda_h}{\mu_h}$ , and $S_v^0 = \frac{\Lambda_v}{\mu_v}$.Therefore;

$$E_0 = \left(\frac{\Lambda_h}{\mu_h} \quad 0 \quad 0 \quad 0 \quad \frac{\Lambda_v}{\mu_v} \quad 0\right). \tag{3}$$

The basic reproduction number denoted by $R_0$ is the average number of secondary infectious infected by an infective individual during his or her whole course of disease [20]. We use the next generation matrix method by van den Driessche and Watmough [21] to derive the basic reproduction

number $R_0$ of (1).The infectious compartment of model (1) are, $I_h, R_h,$ and $I_v$. To apply the method [21], let the system (1) be rearranged by beginning with the infected classes as follows:

Let $X = (I_h \quad T_h \quad I_v \quad S_h \quad R_h \quad S_v)^T$

$$F(X_i) = \begin{pmatrix} \frac{\phi\omega\beta_h I_v}{N_h} S_h \\ 0 \\ 0 \\ \frac{\phi\omega\beta_v(I_h+\sigma_h R_h)}{N_h} S_v \end{pmatrix} \quad \text{and} \quad V(X_i) = \begin{pmatrix} (\mu_h+\delta_h+\gamma)I_h \\ (\mu_h+\delta_h+\varepsilon)T_h - \gamma(1-\pi_1)I_h \\ (\mu_h+\theta)R_h - \gamma\pi_1 I_h \\ (\mu_v+\delta_v)I_v \end{pmatrix}$$

The new infection matrix $F$ and the transition matrix $V$ are given, respectively, by

$$F = \frac{\partial F(X_i)}{\partial X_i}(E_0) = \begin{pmatrix} 0 & 0 & 0 & \phi\omega\beta_h \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{\phi\omega\beta_v\Lambda_v\mu_h}{\Lambda_h\mu_v} & 0 & \frac{\sigma_h\phi\omega\beta_v\Lambda_v\mu_h}{\Lambda_h\mu_v} & 0 \end{pmatrix} \quad \text{and} \quad V = \frac{\partial V(X_i)}{\partial X_i}(E_0) =$$

$$\begin{pmatrix} J_1 & 0 & 0 & 0 \\ -\gamma(1-\pi_1) & J_2 & 0 & 0 \\ -\gamma\pi_1 & 0 & J_3 & 0 \\ 0 & 0 & 0 & J_4 \end{pmatrix}$$

Where, $J_1 = \mu_h + \delta_h + \gamma$ , $J_2 = \mu_h + \delta_h + \varepsilon$ , $J_3 = \theta + \mu_h$, $J_4 = \mu_v + \delta_v$

$$FV^{-1} = \begin{pmatrix} 0 & 0 & 0 & \frac{\phi\omega\beta_h}{(\mu_v+\delta_v)} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{\phi\omega\beta_v\mu_h\Lambda_v(J_3+\sigma\gamma\pi_1)}{J_1J_3\Lambda_h\mu_v} & 0 & \frac{\sigma_h\phi\omega\beta_v\Lambda_v\mu_h}{J_3\Lambda_h\mu_v} & 0 \end{pmatrix} \text{ and}$$

The basic reproduction number of (1) is the dominant eigen value of the next generation matrix $FV^{-1}$ which is given by

$$R_0 = \sqrt{\frac{\phi^2\omega^2\beta_h\beta_v\mu_h\Lambda_v((J_3+\sigma_h\gamma\pi_1))}{\Lambda_h\mu_v J_1 J_3 J_4}} \tag{4}$$

## 4.1 Local Stability of Disease-Free Equilibrium Point

**Theorem3** the disease-free equilibrium point $E_0$ is locally asymptotically stable if $R_0 < 1$ and unstable if $R_0 > 1$.

Proof:

The local stability of the system is determined by the signs of the eigenvalues and it is further proved by linearizing to obtain its Jacobian at disease-free steady-state points so that The Jacobian matrix of (1) at disease free equilibrium point $E_0$ defined and given by

$$J(E_0) = \begin{pmatrix} -\mu_h & 0 & \varepsilon & \theta & 0 & -J_{16} \\ 0 & -J_1 & 0 & 0 & 0 & J_{26} \\ 0 & \gamma(1-\pi_1) & -J_2 & 0 & 0 & 0 \\ 0 & \gamma\pi_1 & 0 & -J_3 & 0 & 0 \\ 0 & -J_{52} & 0 & -J_{54} & -\mu_v & 0 \\ 0 & J_{62} & 0 & J_{64} & 0 & -J_4 \end{pmatrix}. \tag{5}$$

Where, $J_3 = \theta + \mu_h$, $J_{16} = J_{26} = \phi\omega\beta_h$ , $J_{52} = J_{62} = \frac{\phi\omega\beta_v\mu_h\Lambda_v}{\Lambda_h\mu_v}$, $J_{54} = J_{64} = \frac{\sigma_h\phi\omega\beta_v\mu_h\Lambda_v}{\Lambda_h\mu_v}$

$Det(J(E_0) - \lambda I) = 0$ if and only if $\lambda_1 = -\mu_h < 0, \lambda_2 = -\mu_v < 0$ , $\lambda_3 = -J_2 < 0$, and

$$a_0\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0$$

(6)

Where,

$$a_0 = 1,$$

$$a_1 = J_1 + J_3 + J_4$$

(7)

$$a_2 = J_4(J_1 + J_3) + J_1J_3 - J_{26}\ J_{52}$$

$$a_3 = (1 - R_0^2)J_1J_3J_4$$

By the principle of Ruth-Hurwitz criteria [22], (6) has negative real eigenvalues if and only if $a_1 > 0,\ a_3 > 0$ and $a_1\ a_2 > a_3$. Clearly we see that, $a_1 > 0$ because of it is the sum of positive variables, but $a_3 > 0$ if and only if $1 - R_0^2 > 0$ which is equivalent to $R_0 < 0$ and hence, all eigenvalues of the determinant of (5) will have negative real eigenvalues. Therefore; the disease-free equilibrium point $E_0$ is locally asymptotically stable.

## 5. Existence of Endemic Equilibrium and Bifurcation

Let $E^* = (S_h^*\ \ I_h^*\ \ T_h^*\ \ R_h^*\ \ S_v^*\ \ I_v^*)$ be a non-trivial endemic equilibrium point of (1),that is all components of $E_*$ are positive.

If we set (1) to zero we get the following

$$S_h^* = \frac{J_1J_2J_3\Lambda_h}{J_1J_2J_3\mu_h + \left(\mu_hK + J_3\delta_h(J_2 + \gamma(1-\pi_1))\right)\lambda_h^*},\ I_h^* = \frac{\lambda_h^*S_h^*}{J_1},\ T_h^* = \frac{\gamma(1-\pi_1)\lambda_h^*S_h^*}{J_1J_2},\ R_h^* = \frac{\pi_1\gamma\lambda_h^*S_h^*}{J_1J_3}$$

$$S_v^* = \frac{\Lambda_v}{\mu_v + \lambda_v^*},\ I_v^* = \frac{\Lambda_v\lambda_v^*}{(\mu_v + \lambda_v^*)J_4}, \quad (9)$$

$$\lambda_h^* = \phi\omega\beta_h\frac{I_v^*}{N_h^*} \quad (10)$$

$$\lambda_v^* = \phi\omega\beta_v\frac{(I_h^* + \sigma_hR_h^*)}{N_h^*} \quad (11)$$

Where, $N_h^* = S_h^* + I_h^* + T_h^* + R_h^*$ and substituting (9) in to (11) we get

$$\lambda_v^* = \frac{\phi\omega\beta_vJ_2(J_3 + \sigma_h\gamma\pi_1)\lambda_h^*}{J_1J_3(J_1J_2J_3 + K\lambda_h^*)} \quad (12)$$

Again substituting (9) and (12) respectively in to (10) we get

$$\lambda_h^*(a_0(\lambda_h^*)^2 + b_0\lambda_h^* + c_0) = 0 \quad (13)$$

$$a_0 = \Lambda_hJ_2J_1^2J_3^2K(K\mu_v + \phi\omega\beta_v(J_3 + \sigma_h\gamma\pi_1)J_2)$$

$$b_0 = J_1^2J_2^2J_3^3\phi\omega\beta_v(J_3 + \sigma_h\gamma\pi_1)\left(J_1J_3J_4 - \delta_h\phi\omega\beta_v\left(J_2 + \gamma(1-\pi)\right)\right) + J_1^4J_2^3J_3^4J_4\mu_v\Lambda_h\left((1 - R_0^2) + K\right)$$

$$= J_1^4J_2^3J_3^4J_4\mu_v\Lambda_h(1 - R_0^2) \quad (14)$$

Where, $R_0$ is the basic reproduction number, $K = J_2J_3 + \gamma(\pi_1J_2 + (1-\pi)J_3)$.

Equation (13) admits a trivial solution $\lambda_h^* = 0$ which corresponds to the disease-free equilibrium point (DFEP). Now we assume $\lambda_h^* \neq 0$ the existence of endemic equilibria is

regulated by the quadratic equation $a_0(\lambda_h^*)^2 + b_0\lambda_h^* + c_0 = 0$. The coefficient $a_0$ in (14) is always positive and $c_0$ is positive if $R_0 < 1$ and negative if $R_0 > 1$. So, the sign of $b_0$ and $c_0$ will decide about the positive solution of (13). For the case when $R_0 > 1$, two solutions can be obtained for (13), that are positive and negative. For the case when considering $c_0$= 0 if and only if $R_0$ = 1, then a solution of the form $\lambda_h^* = \frac{-b_0}{a_0}$ exists when $b_0 < 0$ .It follows that the number of endemic equilibria of (1) is depend on the coefficient $a_0$, $b_0$ and $c_0$ as follows:

**Theorem 4** The system (1) has

(i) a unique endemic equilibrium if $c_0 < 0$ if and only if $R_0 > 1$.

(ii) a unique endemic equilibrium if $b_0 < 0$ and $c_0 = 0$ or ( $b_0 < 0$ , $c_0 > 0$ and $b_0^2 - 4a_0c_0 = 0$ ).

(iii) Two endemic equilibrium if $c_0 > 0$ and $b_0 < 0$ and $b_0^2 - 4a_0c_0 > 0$.

(iv) otherwise no endemic equilibrium

‘

Condition (iii) of theorem 4, indicate the occurrence of multiple endemic equilibria for $R_0 < 1$.From epidemiological perspective this implies that the elimination of the disease in the population is no longer guaranteed by the condition $R_0 < 1$.

Here also, if we put for the value of $\Lambda_h = 100$ , $\theta = 0.03$ , $\delta_h = 0.068$ , $\mu_v = 0.1429, \gamma = 0.98$ , $\pi_1 = 0.9$ , $\varepsilon = 0.9$ and use table 2 for the other parameters values , the two roots are presented graphically as shown in figure 2. Where, the green line represents stable equilibrium and the red line represents unstable equilibrium.

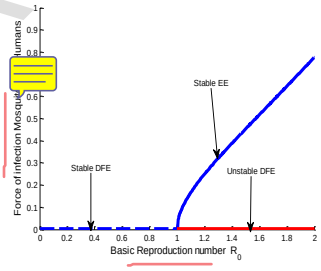



**Figure 2:**When we plot the basic reproduction number $R_0$ versus the force of infection mosquitoes to humans, we note stable disease free region when $\lambda_h^* = 0$ and when $R_0 = 1$, the force of infection mosquitoes to humans starts to increase in stable endemic region where we note that the disease start to spread again and hence, forward bifurcation.

## 5.1 Existence of Back ward Bifurcation

To show the existence of bifurcation of (1) , we employ the method developed in Gumel and Song, 2008; Castillo-Chavez and Song, 2004 [23, 24, 25].We also assume as a

summary that the normal form representing the dynamics of the system on the Centre manifold theory as summary it is given by $\dot{\mu} = a\mu^2 + b\xi\mu$, where,

$$a = \frac{v}{2}.D_{xx}f(x_0,0)w^2 = \frac{1}{2}\sum_{k,i,j=1}^{n} v_k w_i w_j \frac{\partial^2 f_k}{\partial x_i \eth x_j}(x_0,0) \neq 0 \text{ for j =1,2…,n} \quad (15)$$

$$b = V.D_{x\xi}f(x_0,0)w = \sum_{k,i=1}^{n} v_k w_i \frac{\partial^2 f_k}{\partial x_i \eth \xi}(x_0,0) \neq 0 \text{ for i =1,2…,n} \quad (16)$$

Where,

$\xi$ Denotes a bifurcation parameter to be chosen, $f_k$s Denote the right hand side of (1), $x$ Denotes the state vector, $x_0$ Denotes the disease-free equilibrium $E_0$, $D_x$ Denotes the differential operator with respect to $x$, $D_\xi$ Denotes the differential operator with respect to $\xi$ , and $w$ , $v$ denotes the right and left eigenvectors respectively corresponding to the null eigenvalue of the Jacobian matrix of (1), evaluated at $x_0$ for $\xi = 0$.

To analyze the bifurcation of (1), let we choose the rate of transmission of infection from an infectious mosquito to a susceptible human $\beta_h$ as the bifurcation parameter. We observe that $R_0 = 1$ is equivalent to:

$$\beta_h = \beta_h^* = \frac{\Lambda_h \mu_v J_1 J_3 J_4}{\phi^2 \omega^2 \beta_v \mu_h \Lambda_v (\sigma_h \gamma \pi + J_3)} \quad . \quad (17)$$

and the linearized Jacobian matrix evaluated at $E_0$ and $\beta_h^*$ is denoted and given by

$$J(E_0) = \begin{pmatrix} -\mu_h & 0 & \varepsilon & \theta & 0 & -J_{16}^* \\ 0 & -J_1 & 0 & 0 & 0 & J_{26}^* \\ 0 & \gamma(1-\pi_1) & -J_2 & 0 & 0 & 0 \\ 0 & \gamma\pi_1 & 0 & -J_3 & 0 & 0 \\ 0 & -J_{52} & 0 & -J_{54} & -\mu_v & 0 \\ 0 & J_{62} & 0 & J_{64} & 0 & -J_4 \end{pmatrix}. \quad (18)$$

Where, $J_3 = \theta + \mu_h$, $J_{16}^* = J_{16}^* = \phi\omega\beta_h^*$ , $J_{52} = J_{62} = \frac{\phi\omega\beta_v\mu_h\Lambda_v}{\Lambda_h\mu_v}$, $J_{54} = J_{64} = \frac{\sigma_h\phi\omega\beta_v\mu_h\Lambda_v}{\Lambda_h\mu_v}$

$Det(J(E_0) - \lambda I) = 0$ if and only if $\lambda_1 = -\mu_h < 0, \lambda_2 = -\mu_v < 0$ , $\lambda_3 = -J_2 < 0$, and

$\lambda(g_0\lambda^3 + g_1\lambda^2 + g_2\lambda + g_3) = 0$

(19)

Where,

$g_0 = 1,$

$g_1 = J_1 + J_3 + J_4$

(20)

$g_2 = J_4(J_1 + J_3) + J_1J_3 - J_{26}\ J_{52}$

$g_3 = (1 - R_0^2)J_1J_3J_4$

If we also substitute 1(one) for $R_0$ in to (19), then (1) will have a simple zero eigenvalue and the other eigenvalues have negative real parts. Therefore; the disease-free equilibrium point $E_0$ is a non- hyperbolic. To compute the coefficients (15) and (16), we determine the right and left eigenvectors corresponding to the zero eigenvalue. Thus, the components of the right eigenvectors denoted by $w_i$, for $i = 1, \dots, 6$ are given by

$$\begin{cases} -\mu_h w_1 + \varepsilon w_3 + \theta w_4 - J_{16}^* w_6 = 0 \\ -J_1 w_2 + J_{26}^* w_6 = 0 \\ \gamma(1-\pi_1) w_2 - J_2 w_3 = 0 \\ \gamma\pi w_2 - J_3 w_4 = 0 \\ -J_{52}(w_2 + w_4) - \mu_v w_5 = 0 \\ J_{52}(w_2 + w_4) - J_4 w_6 = 0 \end{cases} \tag{21}$$

Where, $J_{16}^* = J_{26}^* = \phi\omega\beta_h^*$, $J_{52}^* = \frac{\phi\omega\beta_v\mu_h\Lambda_v}{\Lambda_h\mu_v}$,

$$w_1 = \frac{J_{26}^*\gamma(\pi_1 J_2\theta + \sigma_h(1-\pi_1)J_3) - J_{16}^* J_1 J_2 J_3}{\mu_h J_1 J_2 J_3} w_6 \ , w_2 = \frac{J_{26}^*}{J_1} w_6 \ , \ w_3 = \frac{\gamma(1-\pi_1)J_{26}^*}{J_1 J_2} w_6 \ ,$$

$$w_4 = \frac{\pi_1\gamma J_{26}^*}{J_1 J_3} w_6 \ , \ w_5 = -\frac{J_{26}^* J_{52}(J_3 + \sigma_h\gamma\pi_1)}{J_1 J_3 \mu_v} w_6 \ , \ w_6 = w_6 > 0. \tag{22}$$

And the components of the left eigenvectors denoted by $v_i$, for $i = 1, \dots, 6$ are given by

$$\begin{cases} -\mu_h v_1 = 0 \\ -J_1 v_2 + \gamma(1-\pi_1)v_3 + \pi_1\gamma v_4 - J_{52}(v_5 - v_6) = 0 \\ \alpha v_1 - J_2 v_3 = 0 \\ \theta v_1 - J_3 v_4 - \sigma_h J_{54}(v_5 - v_6) = 0 \\ -\mu_v v_5 = 0 \\ -J_{16}^* v_1 + J_{26}^* v_2 - J_4 v_6 = 0 \end{cases} \tag{23}$$

$$v_1 = v_3 = v_5 = 0, v_2 = \frac{J_{52}(J_3 + \sigma_h\gamma\pi_1)}{J_1 J_3} v_6, \ v_4 = \frac{\sigma_h J_{52}}{J_3} v_6, \ v_6 = v_6 > 0, \tag{24}$$

Let we make the following change of state variables $S_h = x_1, I_h = x_2, T_h = x_3, R_h = x_4, \ S_v = x_5, I_v = x_6$ and using the vector notation $x = (x_1, \ x_2, x_3, \ x_4, \ x_5, \ x_6,)^T$.The system (1) can then be written in the form $\frac{dx}{dt} = F(x)$ where, $F = (f_1, \ f_2, f_3, \ f_4, \ f_5, \ f_6,)^T$ as shown below

$$\begin{cases} \frac{dx_1}{dt} = f_1 = \Lambda_h + \varepsilon x_3 + \theta x_4 - \left(\phi\omega\beta_h^* \frac{x_6}{x_1 + x_2 + x_3 + x_4} + \mu_h\right) x_1 \\ \frac{dx_2}{dt} = f_2 = \phi\omega\beta_h^* \left(\frac{x_6}{x_1 + x_2 + x_3 + x_4}\right) x_1 - (\mu_h + \delta_h + \gamma) x_2 \\ \frac{dx_3}{dt} = f_3 = \gamma(1-\pi_1)x_2 - (\delta_h + \mu_h + \varepsilon)x_3 \\ \frac{dx_4}{dt} = f_4 = \gamma\pi_1 x_2 - (\theta + \mu_h)x_4 \\ \frac{dx_5}{dt} = f_5 = \Lambda_v - \left(\phi\omega\beta_v \frac{x_2 + \sigma_h x_4}{x_1 + x_2 + x_3 + x_4} + \mu_v\right) x_5 \\ \frac{dx_6}{dt} = f_6 = \left(\phi\omega\beta_v \frac{x_2 + \sigma_h x_4}{x_1 + x_2 + x_3 + x_4}\right) x_5 - (\mu_v + \delta_v) x_{76} \end{cases} \tag{25}$$

$$\frac{\partial^2 f_2}{\partial I_v \partial I_h}(x_0, 0) = \frac{\partial^2 f_2}{\partial I_v \partial T_h}(x_0, 0) = \frac{\partial^2 f_2}{\partial I_v \partial R_h}(x_0, 0) = -\frac{\phi\omega\beta_h^*}{S_h^0}, \frac{\partial^2 f_6}{\partial I_h^2}(x_0, 0) = -\frac{2\phi\omega\beta_v S_v^0}{(S_h^0)^2}, \frac{\partial^2 f_6}{\partial R_h^2}(x_0, 0) =$$

$$-\frac{2\sigma\phi\omega\beta_v S_v^0}{(S_h^0)^2}$$

$$,\frac{\partial^2 f_6}{\partial I_h \eth S_h}(x_0,0)=\frac{\partial^2 f_6}{\partial I_h \eth T_h}(x_0,0)=-\frac{\phi\omega\beta_v S_v^0}{(S_h^0)^2},\frac{\partial^2 f_6}{\partial I_h \eth S_v}(x_0,0)=\frac{\phi\omega\beta_v}{S_h^0},\frac{\partial^2 f_6}{\partial R_h \eth I_h}(x_0,0)=-\frac{\phi\omega\beta_v S_v^0}{(S_h^0)^2}(1+\sigma_h)$$

$$\frac{\partial^2 f_6}{\partial R_h \eth S_h}(x_0,0)=\frac{\partial^2 f_6}{\partial R_h \eth T_h}(x_0,0)=-\frac{\sigma\phi\omega\beta_v S_v^0}{(S_h^0)^2}$$

(26)

$$\frac{\partial^2 f_2}{\partial I_v \eth \beta_h^*}(x_0,0)=\phi\omega$$

(27)

Thus,

$$a=v_2w_2w_6\frac{\partial^2 f_2}{\partial I_v \eth I_h}(x_0,0)+v_2w_3w_6\frac{\partial^2 f_2}{\partial I_v \eth T_h}(x_0,0)+v_2w_4w_6\frac{\partial^2 f_2}{\partial I_v \eth R_h}(x_0,0)+$$

$$v_6w_2^2\frac{\partial^2 f_6}{\partial I_h^2}(x_0,0)+v_6w_4^2\frac{\partial^2 f_6}{\partial R_h^2}(x_0,0)+w_1v_6\left(w_2\frac{\partial^2 f_6}{\partial I_h \eth S_h}(x_0,0)+w_4\frac{\partial^2 f_6}{\partial R_h \eth S_h}(x_0,0)\right)+$$

$$w_3v_6\left(w_2\frac{\partial^2 f_6}{\partial I_h \eth T_h}(x_0,0)+w_4\frac{\partial^2 f_6}{\partial R_h \eth T_h}(x_0,0)\right)+w_5v_6\left(w_2\frac{\partial^2 f_6}{\partial I_h \eth S_v}(x_0,0)+w_4\frac{\partial^2 f_6}{\partial R_h \eth S_v}(x_0,0)\right)+$$

$$v_6w_2w_4\frac{\partial^2 f_6}{\partial I_h \eth R_h}(x_0,0)+v_6w_2w_4\frac{\partial^2 f_6}{\partial R_h \eth I_h}(x_0,0) \quad (28)$$

and

$$b=v_2w_6\frac{\partial^2 f_2}{\partial I_v \eth \beta_h^*}(x_0,0)$$

(29)

After substituting (22), (24) and (26) respectively in to (28), then the coefficient $a$ in terms of $w_6$ and $v_6$ is given by

$$a=\frac{\Lambda_v\mu_h^2\phi^3\omega^3{\beta_h^*}^2\beta_v}{\mu_v(\mu_h+\delta_h+\alpha)\Lambda_h^2(\theta+\mu_h)^2(\mu_h+\delta_h+\gamma)^2}v_6w_6^2\Delta_0$$

(30)

Where,

$$\Delta_0=\frac{(J_3+\sigma_h\gamma\pi_1)J_1J_2J_3}{\mu_h}-2J_3(1-\pi)\gamma+J_2\left((1+J_3+\mu_v\mu_h\pi\gamma)J_3+\gamma\pi_1\left((1+\sigma_h^2\gamma\pi_1)+(\sigma_h+1)\right)\right)-(J_3+\sigma_h\gamma\pi_1)\left(\frac{\mu_hJ_2\phi\omega\beta_v+\gamma(\theta\pi J_2+\alpha(1-\pi)J_3)\mu_v}{\mu_v\mu_h}\right)$$

Similarly, after substituting (22), (24) and (27) respectively in to (29), then the coefficient $b$ in terms of $w_6$ and $v_6$ is given by

$$b=\frac{\phi^2\omega^2\beta_v(J_3+\sigma_h\gamma\pi_1)S_v^0}{J_1J_3S_h^0}w_6v_6>0$$

(31)

Clearly, the coefficient $b$ is positive since all the parameters are non-negative. Thus, the local dynamics of the system (1) around $\mathrm{E}_0$, for $\beta_h=\beta_h^*$ is depends on the sign of the coefficient $a$. Similar to theorem [26] we also established the following theorem.

**Theorem 5** The system (1) will undergo backward bifurcation at $R_0 = 1$ if the coefficient $a$ is positive ($\Delta_0 > 0$) otherwise it will exhibit a forward bifurcation if $a$ is negative ($\Delta_0 < 0$).

## 6 Global Stability

### 6.1 Global Stability of Disease-Free Equilibrium Point

**Theorem 6** the disease-free equilibrium point $E_0$ is globally asymptotically stable if $R_0 \leq 1$ and unstable if $R_0 > 1$.

Proof:

To proof the disease-free equilibrium point $E_0$,we use the Lyapunov function technically. We construct a Lyapunov function $L$ such that

$$L = J_4(J_3 + \sigma_h\gamma\pi_1)I_h + J_1J_4R_h + \phi\omega\beta_h(J_3 + \sigma_h\gamma\pi_1)I_v$$

$$\frac{dL}{dt} = J_4(J_3 + \sigma_h\gamma\pi_1)\frac{dI_h}{dt} + J_1J_4\frac{dR_h}{dt} + \phi\omega\beta_h(J_3 + \sigma_h\gamma\pi_1)\frac{dI_v}{dt} \quad (32)$$

After substituting $\frac{dI_h}{dt}$, $\frac{dR_h}{dt}$ and $\frac{dI_v}{dt}$ from (1) and simplifying it , then we get

$$\frac{dL}{dt} = \phi^2\omega^2\beta_h\beta_v(J_3 + \sigma_h\gamma\pi_1)\frac{(I_h+\sigma_hR_h)}{N_h}S_v - J_1J_3J_4(I_h + \sigma_hR_h)\,.$$

Since $\frac{dS_h}{dt} \leq \frac{\Lambda_h}{\mu_h} = S_h^0 = N_h^0$ , $\frac{dS_v}{dt} \leq \frac{\Lambda_v}{\mu_v} = S_v^0$ (9) is equivalent to (33)

$$\frac{dL}{dt} = \frac{\phi^2\omega^2\beta_h\beta_v(\sigma_h\gamma\pi_1+J_3)\Lambda_v\mu_h}{(1-\varphi)\Lambda_h\mu_v}(I_h + \sigma_hR_h) - J_1J_3J_4(I_h + \sigma R_h) \quad (34)$$

Since $R_0^2 = \frac{\phi^2\omega^2\beta_h\beta_v\mu_h\Lambda_v\mu_h(J_3+\sigma_h\gamma\pi_1)}{\Lambda_h\mu_vJ_1J_3J_4}$ , (10) is also equivalent to

$$\frac{dL}{dt} = (R_0^2 - 1)J_1J_3J_4(I_h + \sigma_hR_h) \quad (35)$$

So, $\frac{dL}{dt} \leq 0$ if $(R_0^2 - 1) \leq 0$ which leads to $R_0 \leq 1$. $\frac{dL}{dt} = 0$ if and only if $I_h = R_h = 0$ or $R_0^2 = 1$.

Therefore; by Lasalle's invariant principle [27],every solution to equations of the model (1) with initial conditions in $\Gamma$ approaches to the disease-free equilibrium point $E_0$ at time $t$ leads infinity whenever, $R_0 \leq 1$.Hence,the disease-free equilibrium $E_0$ is globally asymptotically stable if $R_0 \leq 1$ and unstable if $R_0 > 1$.

The epidemiological implication of theorem 6 is that the elimination of the endemic malaria disease is possible regardless of initial condition (2) of the sub-population of the model(1) whenever $R_0 \leq 1$.

## 7. Sensitivity Analysis

The use of Sensitivity analysis enable us to identify the model parameters that have great influence on the basic reproduction number $R_0$.The method developed by Chitnis $et\ al$ [28, 29] is used to compute the sensitivity index of each parameters that has relation with basic reproduction number $R_0$ The sensitivity indices of each parameters that have great influence on the basic reproduction number $R_0$ is computed as follows:

Let $p$ be a parameter in $R_0$ , then the sensitivity of $p$ is given by $\Upsilon_p^{R_0} = \frac{\partial R_0}{\partial p} \times \frac{p}{R_0}$ .

**Table 1: Table for summary of sensitivity analysis**

| Symbol | Description | Sensitivity indices |
|---|---|---|
| $\Lambda_h$ | Humans recruitments either due to birth or immigration | -0.0021 |
| $\mu_h$ | Human natural death rate | +0.0002 |
| $\Lambda_v$ | Mosquitoes recruitments due to birth | +0.5000 |
| $\mu_v$ | Mosquito natural death rate | -0.0042 |
| $\phi$ | Mosquito contact rate with human | +1.0000 |
| $\omega$ | Mosquito biting rate | +1.0000 |
| $\gamma$ | Per capita rate of recovery from malaria | -0.0002 |
| $\theta$ | Per capita rate of human immunity loss | -0.0017 |
| $\sigma_h$ | Modification parameter | +0.00000039659 |
| $\delta_h$ | Humans death rate due to malaria disease | -0.0019 |
| $\delta_v$ | Mosquitoes death rate due to malaria disease | -0.0001 |
| $\beta_h$ | the rate of transmission of infection from an infectious mosquito to a susceptible human | +0.5000 |
| $\beta_v$ | the rate of transmission of infection from an infectious human to a susceptible mosquito | +0.5000 |

Parameters ( $\phi$ , $\omega$ , $\beta_h$ , $\Lambda_v$ , $\sigma_h$ , $\mu_h$ and $\beta_v$) have direct relations with $R_0$ hence if their magnitude is increased ,then they can expand the disease but parameters ( $\delta_h$ , $\delta_v$ , $\mu_v$, $\gamma$, $\theta$ and $\Lambda_h$) have inverse relations with $R_0$ hence if their magnitude is increased, then they can reduce the burden of the disease in the community.

## 8. Analysis of the Model with Optimal Control

In this section, we consider model system (1) and incorporate optimal combinations of time dependent control measures namely, (i) prevention measure vaccination $u_1(t) = u_1$ for the purpose of immunization of an individual to infection who are requited to the susceptible human populations due to birth or immigration ,(ii) the use of insecticide treated bed net (ITN) $u_2(t) = u_2$ as preventive measure i.e., to reduce the number of bites from mosquitoes as they physically provide a barrier between the infectious mosquitoes and the susceptible humans, and also to reduce the population of the mosquitoes by killing them after they land on the treated net. (iii) treatment with drugs $u_3(t) = u_3$, treating individuals who developed symptoms of the disease, and (iv) the use of indoor residual spray (IRS), $u_4(t) = u_4$ as preventive measure i.e., insecticide spray on the breeding site of mosquitoes reduces the number of mosquito populations by killing these rest indoors after feeding. The controls are practiced on time interval $[t_0 , t_f]$, where $t_0$ and $t_f$ are initial and final time respectively. After incorporating the above controls in to the basic model (1) we get the following modified state equations:

$$\begin{cases} \frac{dS_h}{dt} = (1-u_1)\Lambda_h + \big(\varepsilon + (1-\tau u_3)\big)T_h + \theta R_h - \big((1-u_2)\lambda_h + \mu_h\big)S_h \\ \frac{dI_h}{dt} = (1-u_2)\lambda_h S_h - (\mu_h + \delta_h + \gamma + \tau u_3)I_h \\ \frac{dT_h}{dt} = \gamma(1-\pi_1)I_h - \big(\delta_h + \mu_h + \varepsilon + (1-\tau u_3)\big)T_h \\ \frac{dR_h}{dt} = (\pi_1\gamma + \tau u_3)I_h - (\theta + \mu_h)R_h \\ \frac{dS_v}{dt} = \Lambda_v - \big((1-u_2)\lambda_v + \mu_v + \delta u_2 + \beta u_4\big)S_v \\ \frac{dI_v}{dt} = (1-u_2)\lambda_v S_v - (\mu_v + \delta_v + \delta u_2 + \beta u_4)I_v \end{cases} \quad (36)$$

Here the following objective function $J$ is used to minimize the number of infected and infective in treatment human populations and total mosquito populations while keeping the costs of applying the controls $u_1, u_2, u_3$ and $u_4$ as low as possible.

The form of the objective function is defined based on the approach [30,31].It is also denoted and defined by

$$J = min \int_0^{t_f} \left(B_1 I_h + B_2 T_h + B_3 I_v + \frac{1}{2}\sum_1^4 n_i u_i^2\right) dt \quad (37)$$

Where, $i$ =1,2,3,4 , $B_1$ , $B_2$, and $B_3$ and $n_1, n_2$ , $n_3$ and , $n_4$ are coefficients associated to the state variable and controls respectively. Following the approach [30 ,31], the cost of the controls have been chosen quadratic. Thus, the goal is to find, an optimal control quadruple, $u_1^*$ , $u_2^*$, $u_3^*$ and $u_4^*$ such that

$J(u_1^*, u_2^*, u_3^*, u_4^*) = min\{J(u_1, u_2, u_3, u_4): u_1, u_2, u_3, u_4 \epsilon U\}$ (38)

Where, $\Phi = \{u_1(t) \quad u_2(t) \quad u_3(t) \quad u_4(t) \quad : 0 \le u_i < 1, i = 1,2,\dots,4, 0 \le t \le t_f\}$ is the control set.

The Pontryagins's Maximum Principle [32] converts the system (36) with (37) and (38) into a problem of minimizing pointwise the Hamiltonian H with respect to $u_1, u_2, u_3$ and $u_4$

$$H = (S_h, I_h, T_h, R_h, S_v\ I_v, t) = L(I_h, T_h, I_v, u_1, u_2, u_3, u_4, t) + \lambda_1 \frac{dS_h}{dt} + \lambda_2 \frac{dI_h}{dt} + \lambda_3 \frac{dT_h}{dt} + \lambda_4 \frac{dR_h}{dt} + \lambda_5 \frac{dS_v}{dt} + \lambda_6 \frac{dI_v}{dt} \quad (39)$$

Where, $L(I_h, T_h, I_v, u_1, u_2, u_3, u_4, t) = B_1 I_h + B_2 T_h + B_3 I_v + \frac{1}{2}\sum_1^4 n_i u_i^2$ for i=1,2,3,4 and $\lambda_i$, for i= 1,2,3,4,5,6 are adjoint variables. Using the exitance result for the optimal control [33], we established the following theorem as

**Thereom7** There exists a set of an optimal control $u_i^* = (u_1^*, u_2^*, u_{3,}^* u_4^*)$ and corresponding state solution, $S_h^*$, $I_h^*$, $T_h^*$ $R_h^*$ $S_v^*$ and $I_v^*$ that minimizes $J(u_1, u_2, u_{3,} u_4)$ over $\Phi$ subject to (36).Further , there exists adjoint functions $\lambda_1(t), \dots, \lambda_6(t)$ , and $u_1(t), \dots u_4(t)$ satisfying

$$\begin{cases} \frac{d\lambda_1}{dt} = \mu_h\lambda_1 + (1-u_2)\lambda_h(\lambda_1 - \lambda_2) \\ \frac{d\lambda_2}{dt} = -B_1 + (\mu_h + \delta_h)\lambda_2 + \gamma(\lambda_2 - \lambda_3 - \lambda_4) + \tau u_3(\lambda_2 - \lambda_4) + \frac{(1-u_2)\phi\omega\beta_v}{N_h} S_v \left(1 - \frac{I_h}{N_h}\right)(\lambda_5 - \lambda_6) \\ \frac{d\lambda_3}{dt} = -B_2 + (\mu_h + \delta_h)\lambda_3 + \big(\varepsilon + (1-\tau u_3)\big)(\lambda_3 - \lambda_1) \\ \frac{d\lambda_4}{dt} = \mu_h\lambda_4 + \theta(\lambda_4 - \lambda_1) + \frac{(1-u_2)\phi\omega\beta_v}{N_h} S_v \left(1 - \frac{\sigma_h R_h}{N_h}\right)(\lambda_5 - \lambda_6) \\ \frac{d\lambda_5}{dt} = (1-u_2)\lambda_v(\lambda_5 - \lambda_6) + (\mu_v + \delta u_2 + \beta u_4)\lambda_5 \\ \frac{d\lambda_6}{dt} = -B_3 + (1-u_2)\frac{\phi\omega\beta_h S_h}{N_h}(\lambda_1 - \lambda_2) + (\mu_v + \delta_v + \delta u_2 + \beta u_4)\lambda_6 \end{cases} \quad (40)$$

with transversality conditions

$$\lambda_i(t_f) = 0 \quad \text{for} \quad i = \ 1,2,3,4,5, \tag{41}$$

Further, the optimal controls $u_1^*$ , $u_2^*$, $u_3^*$ and $u_4^*$ are given by

$$u_1^* = min\left\{max\left(0, \frac{\Lambda_h(\lambda_1-\lambda_4)}{n_1}\right), 1\right\}$$

$$u_2^* = min\left\{max\left(0, \frac{\lambda_h(\lambda_2-\lambda_1)S_h+\lambda_v(\lambda_5-\lambda_6)S_v+\delta(S_v\lambda_5+I_v\lambda_6)}{n_2}\right), 1\right\} \tag{42}$$

$$u_3^* = min\left\{max\left(0, \frac{\tau(\lambda_2-\lambda_4)I_h+\tau(\lambda_1-\lambda_3)T_h}{n_3}\right), 1\right\}$$

$$u_4^* = min\left\{max\left(0, \frac{\beta(S_v\lambda_5+I_v\lambda_6)}{n_4}\right), 1\right\}.$$

Proof:

The existence of the optimal control follows from Fleming and Rischel [33] due to convexity of the integrand objective functional $J$ in (37) with respect to $u_i, i = 1,2,3,4$ over the convex and closed control set $\Phi$ and the system (36) satisfies the and Lipchitz property with respect to state variables since the state solutions are bounded. The differential equation (40) governing the adjoint variables $\lambda_1, \lambda_2, \ldots, \lambda_6$ are obtained by partial differentiation of the Hamiltonian H given by (39) with respect to the corresponding state variables that is, $\frac{d\lambda_1}{dt} = -\frac{\partial H}{\partial S_h}$, $\frac{d\lambda_2}{dt} = -\frac{\partial H}{\partial I_{hs}}$, $\frac{d\lambda_3}{dt} = -\frac{\partial H}{\partial I_{hr}}$, $\frac{d\lambda_4}{dt} = -\frac{\partial H}{\partial T_h}$, $\frac{d\lambda_5}{dt} = -\frac{\partial H}{\partial R_h}$, $\frac{d\lambda_6}{dt} = -\frac{\partial H}{\partial S_v}$ with terminal conditions(41).The characterization of optimal control given by (42) is obtained by partial derivative of the Hamiltonian H (39) with respect to each control $u_i$ and solving $\frac{\partial H}{\partial u_i} = 0$ , for $i = 1,2,3,4$.

## 9. Numerical simulation

In this section, numerical simulations are performed to confirm with our analytical results stated in the proof of theorem 3,4,5,6 and the optimality system which is characherized by the state system(39) and the adjoint system (46) was solved numerically by applying Runge Kutta fourth order schemes of the approach [34]. The implimentation of the scheme was done using MATLAB packege.

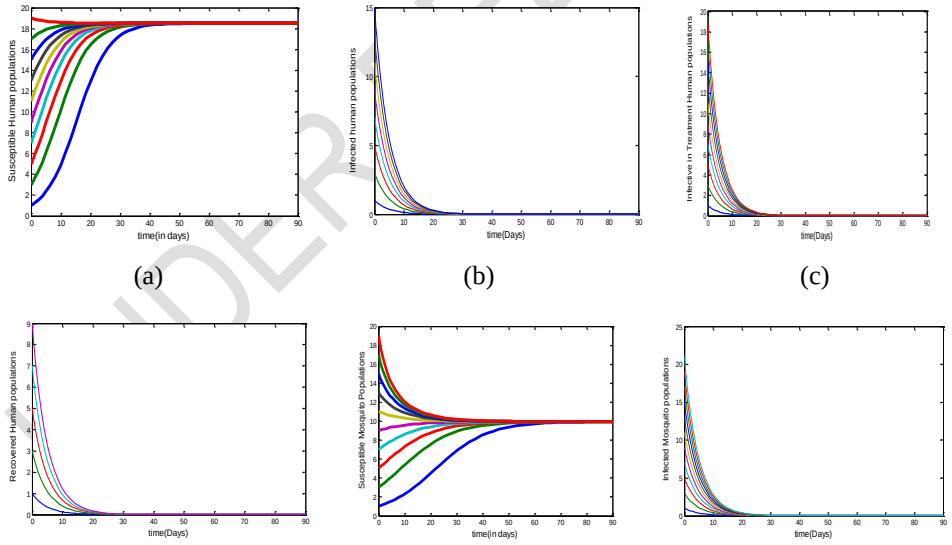
**Tables 2 : Table for parameter values**

| Parameter | Value | Source |
|---|---|---|
| $\beta_h$ | 0.8333 | [35] |
| $\Lambda_h$ | 0.00099 | [35] |
| $\varepsilon$ | (0, 1) | Assumed |
| $\mu_h$ | 0.00005447 | [36] |
| $\delta_h$ | 0.0500 | [37] |
| $\theta$ | 0.01672 | [29] |
| $\tau$ | 0.5000 | Assumed |
| $\omega$ | 0.2000 | [38] |
| $\phi$ | 0.6000 | [39] |
| $\beta_v$ | 0.4800 | [29] |
| $\Lambda_v$ | 50.00 | Assumed |
| $\mu_v$ | 0.0714 | [29] |
| $\gamma$ | 0.005 | [40] |
| $\delta_v$ | 0.0100 | [40] |

| Parameter | Value | Source |
|---|---|---|
| $\sigma_h$ | 0.1000 | [41] |
| $\beta$ | 0.2500 | Assumed |
| $\delta$ | 0.2500 | Assumed |
| $\pi_1$ | 0.0230 | Assumed |

Initial values that we used for simulation of the optimal control are: $S_h(0)$ = 800, $I_h(0)$ = 60, $T_h(0)$ = 10, $R_h(0)$=10, $S_v(0) = 5000$, and $I_v(0)$ = 100. And. Due to the lack of the available literatures and data, as an example, we assumed cost of weight factor of controls are $n_1 =$, $n_2 = n_3 = n_4 = 4$ .

First we considered the stability analysis in case when $R_0 = 0.8274039204 < 1$. Using parameter values given in table2 and replacing value of $\Lambda_v$ in the table 2 by $\Lambda_v = 0.71$and for different initial conditions, the dynamics of the model (1) is presented in figures 3(a)-(f) .These figures show that only the susceptible human $(S_h^0 = 18.1751422802)$ and mosquito $(S_v^0 = 9.943977591)$ populations exists and the infective in treatment class, recovered class, infected human and mosquito populations $(I_h^0 = T_h^0 = R_h^0 = I_v^0 = 0)$ that is, tends to the disease free equilibrium ( $E_0$) as time t tends to infinity. This numerical result supports the result stated in theorem3 i.e., the disease free equilibrium is locally asymptotical stable if $R_0 < 1$.

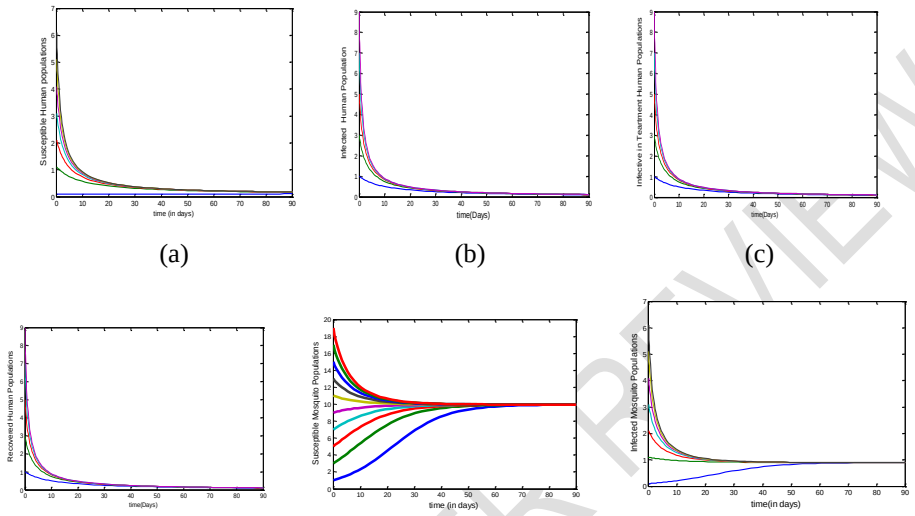



(a) (b) (c)

(d) (e) (f)

**Figure 3** Time series plots Susceptible humans,( (a) Infected humans,(b), Infective in Treatment humans (c), Recovered humans (d) Susceptible mosquitoes,(e) and (f) Infected mosquitoes for $R_0 = 0.8274039204 < 1$ with various initial conditions

Second, we considered the stability analysis in case when $R_0 = 6.9434194577 > 1$ using the parameter values given in table2 and for different initial conditions, the dynamics of the model (1) is presented in figures 4(a)-(f). These figures show that, susceptible humans, infected humans, infective in treatment humans, recovered humans, susceptible and infected mosquito populations exist ( $S_h^* = 0.96988308, I_h^* = 0.0160824661, T_h^* = 0.00000567, R_h^* = 0.000000407, S_v^* = 9.9513, I_v^* = 0.9003$ ) that is the population tends to endemic equilibrium $E^*$ when $R_0 > 1$. This shows t hat the endemic equilibrium $E^*$ is locally asymptotically stable if $R_0 > 1$, which is also supports our analytical results stated in theorem4.




(a) (b) (c)

(d) (e) (f)

**Figure 4** Time series plots Susceptible humans,( (a) Infected humans,(b), Infective in Treatment humans (c), Recovered humans (d) Susceptible mosquitoes,(e) and (f) Infected mosquitoes for $R_0 = 6.9434194577 > 1$ with various initial conditions.

Third for the model (1), we proposed optimal combinations of the aforementioned control strategies as alternative choose to minimize the spread of endemic malaria disease dynamics. So as to do this, we introduced different optimal combination strategies in our model and numerically compare their effects on malaria infeected populatons.Thus,the proposed optimal combinations and numerical result analysis for $R_0 = 6.9434194577 > 1$ are as follows

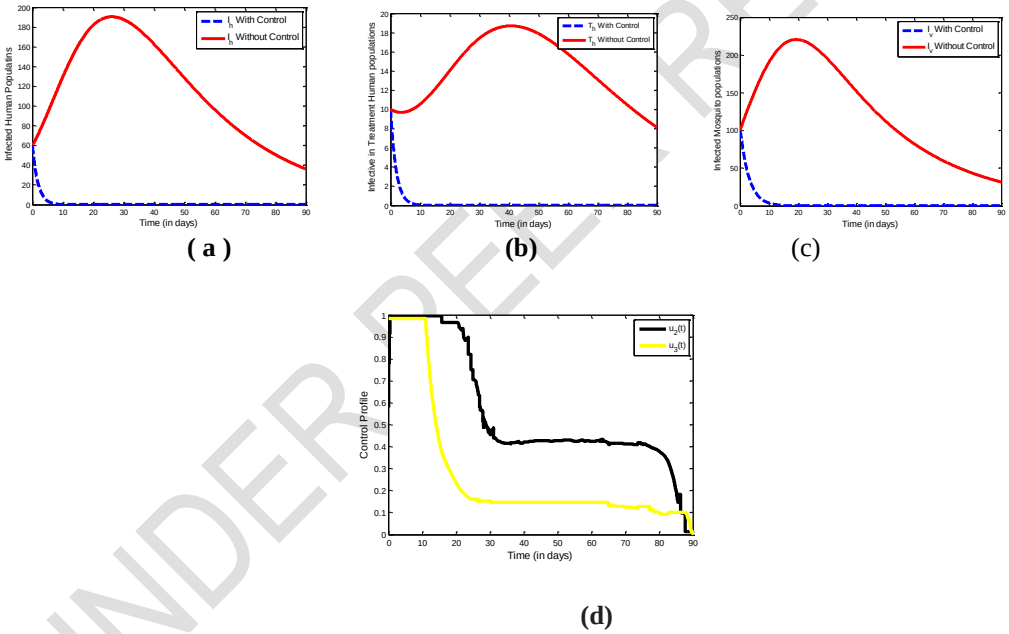
**9.1 Strategy *a*:** Combination of use of insecticide treated net ITN and treatment of infective individuals

**9.2 Strategy *b*:** Combination of use of vaccination, insecticide treated net ITN and treatment of infective individuals

**9.3 Strategy *c*:** Combinations of use of insecticide treated nets ITN, indoor residual spray IRS for vector control and treatment of infective individuals

**9.4 Strategy *d*:** Combinations of use of vaccination, insecticide treated nets ITN and indoor residual spray IRS and treatment of infective individuals

**Strategy $a$:** *Control with the preventive of insecticide treated net ITN and indoor residual spray IRS,* $(u_2 \neq 0, u_3 \neq 0)$. In this strategy, we compare the strategy a situation where no control ( $u_1 = 0$ , $u_2 = 0, u_3 = 0, u_4 = 0$) was used with the application of strategy $a$. It can be seen from the figures 5a,5b and 5c that there is a significant decrease in the number of infected and infective in treatment human populations and infected mosquito populations compared to the strategy with no control at a given time respectively. With the application of strategy $a$ , $I_h$ , $T_h$ and $I_v$ in between time $t = 8,\ 8,$ and 11 days respectively will be eliminated from the system. The control profile shown in figure 5d shows that, control $u_2$ is at 60% initially and increased up to the maximum of 100% before steadily decline to the lower bound within 90 days while control $u_3$ is at the maximum of 100% initially before dropping gradually to the lower bound within 90 days. This suggests that, a high effort is required for the use of medical treatment $u_3$ and there is a low effort for the uses of insecticide treated net $u_2$ under this strategy.



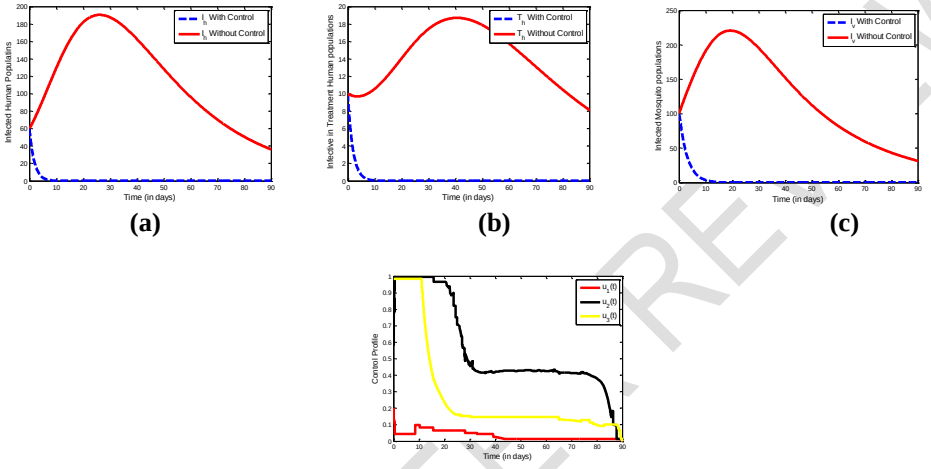

**( a )** **(b)** (c)

**(d)**

**Figure 5 Simulations of the model Showing the effects of a combination of ITN and treatment**

**Strategy $b$:** *Control with the preventive of vaccination, insecticide treated net ITN and treatment* ( $u_1 \neq 0$ , $u_2 \neq 0, u_3 \neq 0$ ). In this strategy, we compare the strategy a situation where no control ( $u_1 = 0$ , $u_2 = 0, u_3 = 0, u_4 = 0$) was used with the application of strategy $d$. It can be seen from the figures 6a, 6b and 6c that there is a significant decrease in the number of infected and infective in treatment human populations and infected mosquito populations compared to the strategy with no control at a given time respectively.

With the application of strategy $b$ , $I_h$ , $T_h$ and $I_v$ in between time $t = 8$ ,8 and 10days respectively will be eliminated from the system. This result is a bit more promising than strategy $a$.The control profile shown in figure 6d shows that, control $u_2$ is at 60% initially and increased up to the maximum of 100% before steadily decline to the lower bound within 90 days while controls $u_1$ and $u_3$ are at the maximum of 20% and 100% initially before dropping gradually to the lower bound within 90 days respectively. This suggests that, a high effort is required for the use of medical treatment $u_3$ and there is a low effort for the uses of preventive controls vaccination $u_1$ and insecticide treated net $u_2$ under this strategy.

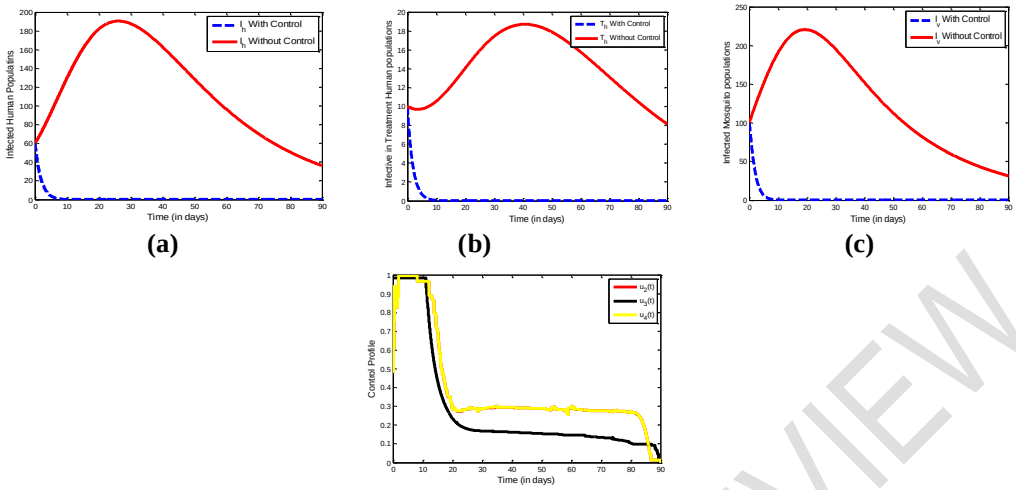



**(a)** **(b)** **(c)**

**(d)**

**Figure 6 Simulations of the model Showing the effects of a combination of Vaccination , ITN and treatment**

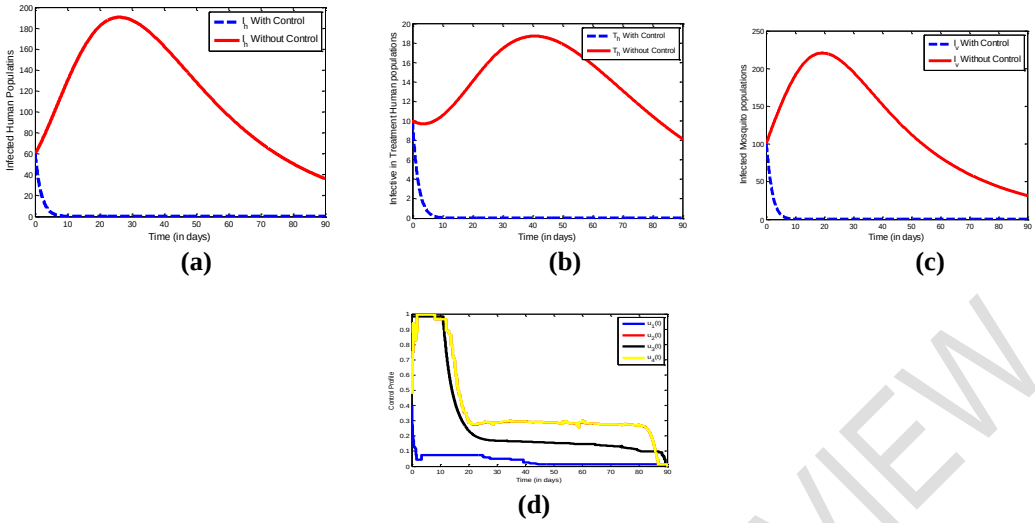
**Strategy $c$:** *Control with the preventive of insecticide treated net ITN, indoor residual spray IRS, and treatment* $(u_2 \neq 0, u_3 \neq 0, u_4 \neq 0)$ . In this strategy, we compare the strategy a situation where no control ( $u_1 = 0$ , $u_2 = 0, u_3 = 0, u_4 = 0$) was used with the application of strategy $d$. It can be seen from the figures 7a, 7b and 7f that there is a dramatic decrease in the number of infected and infective in treatment human populations and infected mosquito populations compared to the strategy with no control at a given time respectively. With the application of strategy $\boldsymbol{c}$ , $I_h$ , $T_h$ and $I_v$ will be eliminated from the system almost within time $t = 8$, days . This result is a bit more promising than strategy $a$. The control profile shown in figure 7d shows that, both control $u_2$ and $u_4$ are at 60% initially and increased up to the maximum of 100% before steadily decline and converges to the lower bound within 90 days while control $u_3$ is at the maximum of 100% initially before dropping gradually to the lower bound within 90 days. This suggests that, a high effort is required for the use of medical treatment $u_3$ and there is a low effort for the uses of insecticide treated net $u_2$, and indoor residual spray IRS $u_4$ under this strategy.




**(a)** **(b)** **(c)** **(d)**

**Figure 7 Simulations of the model Showing the effects of a combination of ITN , treatment and IRS**

**Strategy $d$:** *Control with the preventive of vaccination, insecticide treated net ITN, indoor residual spray IRS, and treatment* ( $u_1 \neq 0$ , $u_2 \neq 0, u_3 \neq 0, u_4 \neq 0$) . In this strategy, we compare the strategy a situation where no control ( $u_1 = 0$ , $u_2 = 0, u_3 = 0, u_4 = 0$) was used with the application of strategy $d$. It can be seen from the figures 8a,8b and 8c that there is a dramatic decrease in the number of infected and infective in treatment human populations and infected mosquito populations compared to the strategy with no control at a given time respectively. With the application of strategy $d$ , $I_h$ , $T_h$ and $I_v$ will be eliminated from the system within time $t = 7$ days. This result is a bit more promising than when strategy $a$ except possibly strategies $b$ and $c$ which yield almost the same results. The control profile shown in figure 8d shows that, both control $u_2$ and $u_4$ are at 60% initially and increased up to the maximum of 100% before steadily decline and converges to the lower bound within 90 days while controls $u_1$ and $u_3$ are at the maximum of 40% and 100% initially before dropping gradually to the lower bound within 90 days respectively. This suggests that, a high effort is required for the use of medical treatment $u_3$ and there is a low effort for the uses of preventive controls vaccination $u_1$ , insecticide treated net $u_2$, and indoor residual spray IRS $u_4$ under this strategy.




**(a)** **(b)** **(c)** **(d)**

**Figure 8 Simulations of the model Showing the effects of a combination of** vaccination, ITN, treatment and IRS

## 10. Cost Effectiveness Analysis

To analyze the cost-effectiveness of the strategies, we employ the approach of incremental cost-effective ratio (ICER) in [18]. The ICER is used to compare the cost and the health outcomes of two alternative intervention strategies that compete for the same resources.

Based on values of ICER of the strategy , the alternative that are more expensive and less ineffective are then excluded. This is done after simulating the optimal control model and then ranking strategies in order of increasing effectiveness measured as the total infection averted. We calculate ICER based on strategy $k(k = a, b, c\ d)$ by using the following formula.

ICER($a$)= $\frac{Cost\ of\ intervention(a) - Cost\ of\ intervention(b)}{Effect\ of\ intervention(a) - Effect\ of\ intervention(b)}$ .

The total infection averted by optimal strategy $k$ during the time period $t_f$ is denoted and defined as

$$TA_k = \frac{B_1 A_{kI_h} + B_3 A_{kT_h} + B_3 A_{kI_v}}{B_1 + B_2 + B_3}. \quad (43)$$

Where,

$$A_{kI_h} = t_f I_h(0) - \int_{t_0}^{t_f} I_h^*(t)dt \quad (44)$$

$$A_{kT_h} = t_f T_h(0) - \int_{t_0}^{t_f} T_h^*(t)dt$$

$$A_{kI_v} = t_f I_v(0) - \int_{t_0}^{t_f} I_v^*(t)dt$$

And $A_{kI_h}$ , $A_{kT_h}$ and $A_{kI_v}$ are the total infected and infective in treatment human populations and infected mosquito populations respectively averted by optimal strategy $k$ during the time period final time $t_f$, $I_{hs}^*$ , $T_h^*$, and $I_v^*$ are the optimal solution associated with optimal controls $(u_1^*, u_2^*, u_3^*, u_4^*)$ (39) and $I_{hs}(0), T_h(0)$ and $I_v(0)$ are the corresponding initial values. Note that these initial values are obtained as the equilibrium proportion of the system (39) with no post-exposure intervention $(u_1 = u_2 = u_3 = u_4 = 0)$. The total cost associated with a strategy is denoted and given by

$$C_{kT} = \int_{t_0}^{t_f}\left[n_1 u_1^*(t)S_h^*(t) + n_2 u_2^*(t)\left(S_h^*(t) + I_h^*(t) + S_v^*(t) + I_v^*(t)\right) + n_3 u_3^*(t)\left(S_h^*(t) + I_h^*(t) + T_h^*(t)\right) + n_4 u_4^*(t)\left(S_v^*(t) + I_v^*(t)\right)\right]dt \quad (45)$$

Where, $n_1$ corresponds to the per person unit cost following preventive control of vaccination intervention, $n_2$ corresponds to the per person unit cost of preventive control of ITN intervention, $n_3$ corresponds to the per person unit cost of treatment intervention, and $n_4$ corresponds to the per person unit cost of preventive control of IRS intervention. Parameters values from table1 are used to estimate the total infection averted and total cost presented in table 3.

**Table 3 Cases Averted and Total Costs¨**

| Strategy k | $A_{kI_h}$ | $A_{kT_h}$ | $A_{kI_v}$ | $TA_k$ | $Costs(\$)C_{kT}$ |
|---|---|---|---|---|---|
| $d$ | 5395.293 | 899.973 | 8993.5470 | 5995.5570 | 494140 |
| $c$ | 5395.293 | 899.973 | 8993.5560 | 5995.5610 | 502060 |
| $b$ | 5394.609 | 899.964 | 8992.3500 | 5994.7950 | 669410 |
| $a$ | 5394.618 | 899.964 | 8992.3590 | 5994.8020 | 669900 |

Table 3 incorporates ICER, it is presented as follows; first we rearrange control strategies from table 2 in increasing order of effectiveness $(TA_k)$. Next, we compute incremental effectiveness $\triangle TA_k$ and incremental costs $\Delta C_{Tk}$ .The ICER is calculated by dividing $\Delta C_{Tk}$ to $\triangle TA_k$ where $k = a, b, c, d$ and their ICER are as follows

ICER$(\boldsymbol{b}) = \frac{\Delta C_{Tb}}{\Delta TA_b} = \frac{669410}{5994.7950} = 111.6652028968463$,

ICER$(\boldsymbol{a}) = \frac{C_{Ta} - C_{Tb}}{TA_a - Tb} = \frac{\Delta C_{Ta}}{\Delta TA_a} = 70{,}000$,

ICER$(\boldsymbol{d}) = \frac{C_{Td} - C_{Ta}}{TA_d - TA_a} = \frac{\Delta C_{Td}}{\Delta TA_d} =$ -23,2794.701986755,

ICER$(\boldsymbol{c}) = \frac{C_{Tc} - C_{Td}}{TA_c - TA_d} = \frac{\Delta C_{Tc}}{\Delta TA_c} =$ 1,980,000.

**Table** 4 Incremental cost-effectiveness ratios of different optimal control strategies

| Strategy k | $TA_k$ | $\Delta TA_k$ | $Costs(\$)C_{Tk}$ | $\Delta C_{Tk}$ | ICER $(\Delta C_{Tk}/\Delta TA_k)$ |
|---|---|---|---|---|---|
| $b$ | 5994.7950 | 5994.7950 | 669410 | 669410 | 111.6652028968463 |
| $a$ | 5994.8020 | 0.007 | 669900 | 490 | 70,000 |
| $d$ | 5995.5570 | 0.755 | 494140 | -175760 | -23,2794.70198675 |

| | | | | | |
|---|---|---|---|---|---|
| $c$ | 5995.5610 | 0.004 | 502060 | 7920 | 1,980,000 |

**Table 5** Incremental cost-effectiveness ratios for optimal control strategies $a$ , $b$ and $d$

| Strategy k | $TA_k$ | $\Delta TA_k$ | $Costs(\$)C_{Tk}$ | $\Delta C_{Tk}$ | | ICER $(\Delta C_{Tk}/\Delta TA_k)$ |
|---|---|---|---|---|---|---|
| $b$ | 5994.7950 | 5994.7950 | 669410 | 669410 | | 111.6652028968463 |
| $a$ | 5994.8020 | 0.007 | 669900 | 490 | | 70000 |
| $d$ | 5995.5570 | 0.755 | 494140 | -175760 | | -232794.701986755 |

**Table 6** Incremental cost-effectiveness ratios for optimal control strategies $b$ and $d$

| Strategy k | $TA_k$ | $\Delta TA_k$ | $Costs(\$)C_{Tk}$ | $\Delta C_{Tk}$ | | ICER $(\Delta C_{Tk}/\Delta TA_k)$ |
|---|---|---|---|---|---|---|
| $b$ | 5994.7950 | 5994.7950 | 669410 | 669410 | | 111.6652028968463 |
| $d$ | 5995.5570 | 0.755 | 494140 | -175760 | | -232794.701986755 |

Finally, the comparison results reveals that strategy $\boldsymbol{d}$ is cheaper than strategy $\boldsymbol{b}$. Therefore, strategy $\boldsymbol{d}$ (the use of vaccination, insecticide treated net ITN, treatment for infected and infective in treatment individuals, and indoor residual spray IRS is the best of all possible strategies due to its more effectiveness and less cost or due to its cost-effectiveness and healthy benefits.

## 11. Conclusion

In this study, a non-linear system of ordinary differential equation model that describe the dynamics of endemic malaria transmission is formulated and analyzed. Conditions are derived for the existence of disease-free and endemic equilibria. A basic reproduction number $R_0$ is exists and the disease-free equilibrium point is both locally and globally asymptotically stable whenever $R_0$ is less than unity. The disease can persist whenever $R_0$ is greater than unity and at $R_0$ is equal to unity, the conditions for the existence of bifurcations are also derived. Using the model parameter when we plot $R_0$ versus the force infection mosquitoes to humans the model exhibits forward bifurcation at $R_0$ is equal to unity. Sensitivity analysis is also performed and hence, (i) An increase of magnitude of the indices of parameter with positive indices increases the magnitude of the associated basic reproduction number,(ii) An increase of magnitude of the indices of parameter with negative indices decreases the magnitude of the associated basic reproduction number. Furthermore, optimal combinations of time dependent control measures, namely; vaccination, insecticide treated nets ITN, treatment and indoor residual spray IRS are incorporated to the model. Pontryagins's maximum principle of optimal control theory is used to find the necessary conditions for the controls to be optimal. The optimality system which is characherized by the state system(36) and the adjoint system (40) was solved numerically by applying Runge Kutta fourth order schemes. Results from numerical simulations show that the use of the

combination of all controls or strategy **d** perform well for the time period of intervention. The combination of the use of vaccination, insecticide treated net ITN, treatment, and indoor residual spray IRS or strategy **d** is the most optimal cost-effective and efficacious strategy in controlling the disease dynamics. Finally, we note that with the strict application of either one of the incorporated combinations of optimal control strategies, it is possible to reduce the number populations with malaria symptoms to zero in the given time. Further note that, application of optimal control strategy to endemic malaria disease dynamics is not only minimize the number populations with malaria symptoms and costs related to infectious and control measures but also minimize the spread of the disease dynamics in the community.

## 12.Recommendations

Here, we recommend to malaria control policy makers, health care workers and any concerning body can use the incorporated strategy in this paper to reduce the burden of the endemic malaria disease dynamics in community.

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