# NUMERICAL SOLUTION OF NONLINEAR EQUATIONS BY A TWELTH-ORDER ITERATIVE METHOD WITH MEMORY

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Abstract. Some real life mathematical problems can be converted in the form of nonlinear equations. Solving such problems by analytical approaches is di¢ cult in many situations. Hence numerical solution is the best way in this case. In this paper, a twelth-order iterative scheme for solving nonlinear equations is presented and analyzed in terms of e¢ ciency. The new scheme is derived from the well-known King.s method with order of convergence eight. We extend eighth-order King.s method to an iterative method with memory of order 12:16 by using famous Newton.s interpolating polynomial of degree 6 to avoid the derivative used in King.s method. The new derived method is a three-step and is totally derivative free with twelth order of convergence. The method requires four functional evaluations at each iteration introducing high e¢ ciency index of (12:16)

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4 = 1:8673: Convergence order of new method is also studied. It is achieved by using matrix method of Herzburger. Numerical results are also provided to support theoretical analysis. Comparison of the derived scheme with previously well-known iterative schemes of the same order is also presented. As di¤erent schemes of same order has e¢ ciency index of (12)

1

6 = 1:5131 because they requires six functional evaluations at each iteration, hence the proposed scheme is better than other schemes.

## 1. Introduction

It is well known that a broad class of problems that appear in many .elds of pure and applied research may be explored in the general context of nonlinear equations. Due to their signi.cance, a number of numerical approaches for solving nonlinear equations have been proposed and examined under certain circumstances. The construction of these numerical methods used a variety of methodologies, including the Taylor series, the homotopy perturbation method and its variants, the quadrature formula, the variational iteration method, and the decomposition method. Solving nonlinear equation f(x) = 0 using iterative schemes is a classical problem in .eld of numerical computation [1]-[6] and references therein. Iterative schemes also has a great importance in Nanotechnology, in which solution of fractional di¤erential-di¤erence equations[15] is obtained using these schemes. In [16], Nasir ali et al. proposed a new iterative scheme for solving important nonlinear equations in .eld of fractional calculus.

Jarratt and Ostrowski presented somewell-known two-point techniques. King [11] presented one of the most famous optimal 4thorder iterative method. But .aws of this scheme is that it requires .rst derivative in each step. Many authors

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modi.ed King.s method to get more accurate results such as Chun [9] introduced King.s like methods of order four, but computing the .rst derivative within the iteration is also needed.

Chun and Lee presented a new 4th order optimal root-.nding method to solve non-linear equations which describe the conjugacy classes and dynamics of the presented optimal method for complex polynomials of degree two and three. They obtained Jarratt.s scheme of fourth order as a special case. Behl [2]-[4] introduced a fourth-order derivative-free scheme which is a modi.cation in King.s method by using weight functions.

To achieve more accuracy with less computation many authors introduced Optimal methods of eighth-order of convergence i.e Chun et al. [9] Cordero et al. [7]-[8], Behl et al.[2]-[4] and Geum et al. [10]. Geum and Neta [10] developed a 16th order simple root .nding optimal method with general weight functions.

Our aim is to present a derivative-free iterative method with memory of twelth order using King.s iterative schemes and ste¤ensen approaches. The new scheme is derived from the well-known King.s method with order of convergence eight. We extend eighth-order King.s method to an iterative method with memory of order

12:16 by using famous Newton's interpolating polynomial of degree 6 to avoid the derivative used in King's method. The new derived method is a three-step and is totally derivative free with twelth order of convergence. This paper is organized in the following manner.

In the next section we establish our new method and gave convergence analysis. In section 3, solution of some numerical examples with their comparison to other well-known iterative schemes are presented. Section 4 is a smart conclusion.

## 2. Establishment of new scheme of twelth-order

We start with king's [11] technique that is one of important family for finding solution of nonlinear problems.

$$y_m = x_m - \frac{f(x_m)}{f'(x_m)}, \quad x_{m+1} = y_m - \frac{f(y_m)}{f'(x_m)} \cdot \frac{f(x_m) + \beta f(y_m)}{f(x_m) + (\beta - 2) f(y_m)}, \quad (m = 0, 1, \ldots), \ \beta \in R;$$

Here $x_0$ is a preliminary approximation of a simple zero $\xi$ of f. First we derive an optimal derivative free scheme of 2 point method having convergence order 4. We consider Steffensen's scheme for the 1st & 2nd step approximation $f'(x_m)$

$$f'(x_n) \approx \frac{f[y_m, w_m]}{G(t_m)},$$

while $w_m = x_m - \alpha f(x_m)$; $\alpha \neq 0$; $f[y_m, w_m] = \frac{f(y_m) - f(w_m)}{y_m - w_m}$; $t_m = \frac{f(y_m)}{f(x_m)}$. Here G is actual function.

Hence, we obtain a fourth order scheme,

$$\begin{cases} y_m = x_m - \frac{\alpha f(x_m)^2}{f(x_m) - f(w_m)}, \\ x_{m+1} = y_m - \frac{f(x_m) + \beta f(y_m)}{f(x_m) + (\beta - 2) f(y_m)} \cdot \frac{f(y_m)}{f[y_m, w_m]} G(t_m). \end{cases} \tag{1}$$

Now we derive eighth-order scheme by adding Newton step in scheme (1), we have



$$\begin{cases} y_m = x_m - \frac{\alpha f(x_m)^2}{f(x_m) - f(w_m)}, \\ z_m = y_m - \frac{f(x_m) + \beta f(y_m)}{f(x_m) + (\beta - 2) f(y_m)} \cdot \frac{f(y_m)}{f[y_m, w_m]} G(t_m), \\ x_{m+1} = z_m - \frac{f(z_m)}{f'(z_m)}. \end{cases} \tag{2}$$

It is observed that function is evaluated many times to make it derivative-free and optimal method. We approximate $f'(z_m)$ with Newton's interpolation of degree

3 at the point $x_m$; $y_m$; and $z_m$:

$N_3(t; z_m; y_m; x_m; w_m) = f(z_m) + f[z_m; y_m](t - z_m)$
$+ f[z_m; y_m; x_m](t - z_m)(t - y_m) + f[z_m; y_m; x_m; w_m]$
$(t - z_m)(t - y_m)(t - x_m)$ :

It can be seen
$N_3(z_m) = f(z_m)$ ; and $N_3'$
$(t)\,|_{t=z_m} = f'(z_m)$ :

So,
$N_3'$
$(z_m) =$

$\frac{d}{dt} N_3(t) \Big|_{t=z_m}$

$= f[z_m; y_m] + f[z_m; y_m; x_m](z_m - y_m) + f[z_m; y_m; x_m; w_m]$
$(z_m - y_m)(z_m - x_m)$ :

hence we get

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>>:

$y_m = x_m - \frac{f(x_m)^2}{f(x_m) - f(w_m)}$ ;

$z_m = y_m - \frac{f(x_m)+f(y_m)}{f(x_m)+(-2)f(y_m)} : \frac{f(y_m)}{f[y_m; w_m]} G(t_m)$ :

$x_{m+1} = z_m - \frac{f(z_m)}{f[z_m; y_m]+f[z_m; y_m; x_m](z_m - y_m)+f[z_m; y_m; x_m; w_m](z_m - y_m)(z_m - x_m)}$ :

(3)

which is iterative scheme with memory of order 8. Now we extend this method to achieve convergence order 12.

This is done by using the speed acceleration parameters in scheme (3): If _ 6= 1=f0 (_) rate of convergence of scheme (3) is 8: When _ = 1=f0 (_) rate of convergence of method (3) could be twelve. Because the value of f0 (_) is unavailable, we apply an approximation f0 (_) t f0 ($x_m$). Our objective is to create a with memory method that includes parameter calculations _ = $\_m$ as iteration progresses by $\_m$ = 1=f0 (_) for m = 1; 2; 3:::: Initial value $\_0$ should be selected before begning of iteration process. Here we use some characters !;O and _ according to Traub.s iterative sccheme.

If $\lim_{m \to 1} f(x_m) = C$; we write down $f(x_m)$ ! C or f ! C; where C is a non zero contants. If $L_m$ ! C; we write f = O (g) or f _ C (g) :

By approximating f0 (_) with $N_4'$
($x_m$) we get

$\_m = \frac{1}{N_4'(x_m)}$ ;

Here N
_
$_4(t_m) := N_4(t; x_m; z_{m-1}; y_{m-1}; w_{m-1}; x_{m-1})$ is Newton.s interpolation polynomial of 4th degree, place with 5 approximation $(x_m; z_{m-1}; y_{m-1}; w_{m-1}; x_{m-1})$ :



$N_4'$
$(x_m) =$

_
d

dt
$N_4(t)$
_
$t=x_m$
$= f[x_m; z_{m-1}] + f[x_m; z_{m-1}; y_{m-1}](x_m - z_{m-1})$
$+ f[x_m; z_{m-1}; y_{m-1}; w_{m-1}](x_m - z_{m-1})(x_m - y_{m-1})$
$+ f[x_m; z_{m-1}; y_{m-1}; w_{m-1}; x_{m-1}](x_m; z_{m-1})(x_m - y_{m-1})$
$(x_m - w_{m-1})$ :
We approximate in (2) $f_0(z_m)$ with Newton.s interpolation of degree 6 at the point $z_m; y_m; w_m; x_m; z_{m-1}$ and $y_{m-1}$:
$N_5(t; z_m; y_m; w_m; x_m; z_{m-1}; y_{m-1}) = f(z_m) + f[z_m; y_m](t - z_m) + f[z_m; y_m; w_m]$
$(t - z_m)(t - y_m) + f[z_m; y_m; w_m; x_m](t - z_m)$
$(t - y_m)(t - w_m) + f[z_m; y_m; w_m; x_m; z_{m-1}]$
$(t - z_m)(t - y_m)(t - w_m)(t - x_m)+$
$f[z_m; y_m; w_m; x_m; z_{m-1}; ; y_{m-1}](t - z_m)$
$(t - y_m)(t - w_m)(t - x_m)(t - z_{m-1})$ :
It is clear that,
$N_5(z_m) = f(z_m)$ ; and $N_{05}$
$(t)\, j_{t=z_m} = f_0(z_m)$ :
Then,
$N_{05}$
$(z_m) =$
_
d
dt
$N_5(t)$
_
$t=z_m$
$= f[z_m; y_m] + f[z_m; y_m; w_m](z_m - y_m) + f[z_m; y_m; w_m; x_m]$
$(z_m - y_m)(z_m - w_m) + f[z_m; y_m; w_m; x_m; z_{m-1}](z_m - y_m)(z_m - w_m)(z_m - x_m)$
$+ f[z_m; y_m; w_m; x_m; z_{m-1}; y_{m-1}](z_m - y_m)(z_m - w_m)(z_m - x_m)(z_m - z_{m-1})$ :
and hence we get,
$y_m = x_m - {}_m f(x_m)^2$
$f(x_m) - f(w_m)$ ;
$z_m = y_m - f(x_m) + f(y_m)$
$f(x_m) + (-2) f(y_m)$ : $f(y_m)$
$f[y_m; w_m] G(t_m)$ ;
$x_{m+1} = z_m - f(z_m)$
$f[z_m; y_m] + f[z_m; y_m; w_m](z_m - y_m) + ::: + f[z_m; y_m; w_m; x_m; z_{m-1}; y_{m-1}](z_m - y_m):::(z_m - w_{m-1})$ :
(4)
We denote the above scheme with AM12 which is a with memory method of convergence order 12.16. Now, we will prove the convergence order by applying the matrix method of Herzberger.
Theorem 1. Let $x_0$ be a starting value which is close enough to 0 of f(x) and the iterative scheme AM12 has 2 parameters which are repeatedly computed by the outline, then the scheme AM12 has 12:164414 order of convergence.
Proof. By using Herzberger.s matrix method we will .nd R-order of convergence which states that the spectral radius of matrix $M_{(u)} = (t_{p;q})(1 \_ p; q \_ u)$ related to a with-memory 1 step r-point scheme $x_k = \_(x_{k-1}; x_{k-2}; :::; x_{k-u})$ is the lower bound of its rate of convergence. The elements of this method are as:
$t_{p;q}$ = no.of functional evaluations needed at point $x_{k-q} = 1; 2; :::; u$
$t_{p;q-1} = 1$ for $p = 2; 3; :::; u$
$t_{p;q} = 0$, otherwise.

Moreover, the spectral radius of product of the matrices $B_1; B_2; :::B_m$ is the lower bound of order of an $m-$step method $\_ = \_{}_1; \_{}_2; :::; \_{}_m$ where the matrices $B_k$ correspond to the iteration step $\_{}_k; 1 \_ K \_ m$: From the above equations we develop the associated matrices as follow:

$X_{k+1} = \_1 (z_m; y_m; w_m; x_m; z_{m-1}; y_{m-1})$ :

$M_1 =$

2
6666664
1 1 1 1 0 0
1 0 0 0 0 0
0 1 0 0 0 0
0 0 1 0 0 0
0 0 0 1 0 0
0 0 0 0 1 0
3
7777775

$Z_m = \_2 (y_m; w_m; x_m; z_{m-1}; y_{m-1}; w_{m-1})$ :

$M_2 =$

2
6666664
1 1 1 0 0 0
1 0 0 0 0 0
0 1 0 0 0 0
0 0 1 0 0 0
0 0 0 1 0 0
0 0 0 0 1 0
3
7777775

$Y_m = \_3 (w_m; x_m; z_{m-1}; y_{m-1}; w_{m-1}; x_{m-1})$ :

$M_3 =$

2
6666664
1 1 0 0 0 0
1 0 0 0 0 0
0 1 0 0 0 0
0 0 1 0 0 0
0 0 0 1 0 0
0 0 0 0 1 0
3
7777775

$W_m = \_4 (x_m; z_{m-1}; y_{m-1}; w_{m-1}; x_{m-1}; z_{m-2})$ :

$M_4 =$

2
6666664
1 1 1 1 1 1
1 0 0 0 0 0
0 1 0 0 0 0
0 0 1 0 0 0
0 0 0 1 0 0
0 0 0 0 1 0
3
7777775

$M_1 \_ M_2 =$

2
6666664
2 2 2 0 0 0
1 1 1 0 0 0
1 0 0 0 0 0
0 1 0 0 0 0

0 0 1 0 0 0
0 0 0 1 0 0
3
7777775
$M_1$ _$M_2$ _$M_3$ =
2
6666664
4 4 0 0 0 0
2 2 0 0 0 0
1 1 0 0 0 0
1 0 0 0 0 0
0 1 0 0 0 0
0 0 1 0 0 0
3
7777775

Hence we obtained,
$M_{(4)}$ = $M_1$ _$M_2$ _$M_3$ _$M_4$
=
2
6666664
8 4 4 4 4 4
4 2 2 2 2 2
2 1 1 1 1 1
1 1 1 1 1 1
1 0 0 0 0 0
0 1 0 0 0 0
3
7777775
The eigen values of $M_{(4)}$ are
$\_{1}$ = 12:164414
$\_{2}$ = □0:164414003
$\_{3}$ = 2:0470017e$_{□15}$ □ 3:78075673e$_{□81}$
$\_{4}$ = 2:75578744e$_{□15}$ □ 3:78075673e$_{□81}$
$\_{5}$ = 1:22388505e$_{□16}$ + 1:13323081e$_{□16}$
Hence spectral radius of $M_{(4)}$matrix is 12:164414 which is convergence order of the method. _
3. Numerical Examples and comparison
3.1. Method 1. Consider G as weight function
G($t_m$) = 1 □ $t_m$; (5)
where $t_m$ = $_{f(y_m)}$
$_{f(x_m)}$ : Hence we get the following scheme denoted by AM.1.
$y_m$ = $x_m$ □
$_{m}$f ($x_m$)$_2$
f ($x_m$) □ f ($w_m$)
;$w_m$ = $x_m$ □ $_{m}$f ($x_m$) ; $\_{m}$ =
1
$N_{05}$
($x_m$)
;
$z_m$ = $y_m$ □
f ($x_m$) + f ($y_m$)
f ($x_m$) + ( □ 2) f ($y_m$)
:
f ($x_m$) □ f ($y_m$)
f ($x_m$)

$$\cdot \frac{f(y_m)}{f[y_m;w_m]},$$

$$x_{m+1} = z_m - \left[\frac{f(z_m)}{f[z_m;y_m] + f[z_m;y_m;w_m](z_m - y_m) + \cdots + f[z_m;y_m;w_m;x_m;z_{m-1};y_{m-1}](z_m - y_m)\cdots(z_m - w_{m-1})}\right]$$

### 3.2. Method 2. Consider G as a weight function

$$G(t_m) = 1 - \frac{t_m}{1 + t_m}, \quad \text{where } t_m = \frac{f(y_m)}{f(x_m)}.$$

$$y_m = x_m - \frac{\beta_m f(x_m)^2}{f(x_m) - f(w_m)}, \quad w_m = x_m - \beta_m f(x_m), \quad \beta_m = \frac{1}{N'_5(x_m)}$$

$$z_m = y_m - \frac{f(x_m)+f(y_m)}{f(x_m)+(-2)f(y_m)} \cdot \frac{f(x_m)}{f(x_m)+f(y_m)} \cdot \frac{f(y_m)}{f[y_m;w_m]},$$

$$x_{m+1} = z_m - \frac{f(z_m)}{f[z_m;y_m]+f[z_m;y_m;w_m](z_m-y_m)+\cdots+f[z_m;y_m;w_m;x_m;z_{m-1};y_{m-1}](z_m-y_m)\cdots(z_m-w_{m-1})}.$$

we call this method AM.2.



### 3.3. Method 3. Choose weight function G

$$G(t_m) = \frac{1 - 2t_m}{1 - t_m}, \quad \text{where } t_m = \frac{f(y_m)}{f(x_m)}$$

$$y_m = x_m - \frac{\beta_m f(x_m)^2}{f(x_m) - f(w_m)}, \quad w_m = x_m - \beta_m f(x_m), \quad \beta_m = \frac{1}{N'_5(x_m)}$$

$$z_m = y_m - \frac{f(x_m)+f(y_m)}{f(x_m)+(-2)f(y_m)} \cdot \frac{f(x_m)-2f(y_m)}{f(x_m)-f(y_m)} \cdot \frac{f(y_m)}{f[y_m;w_m]},$$

$$x_{m+1} = z_m - \frac{f(z_m)}{f[z_m;y_m]+f[z_m;y_m;w_m](z_m-y_m)+\cdots+f[z_m;y_m;w_m;x_m;z_{m-1};y_{m-1}](z_m-y_m)\cdots(z_m-w_{m-1})}.$$

we call this scheme AM.3.

### 3.4. Method 4. Consider G as weight function

$$G(t_m) = (1 - t_m)^{\frac{2t_m+1}{t_m+1}};$$

$$y_m = x_m - \frac{\beta_m f(x_m)^2}{f(x_m) - f(w_m)}, \quad w_m = x_m - \beta_m f(x_m), \quad \beta_m = \frac{1}{N'_5(x_m)}$$

$$z_m = y_m - \frac{f(x_m)+f(y_m)}{f(x_m)+(-2)f(y_m)} \cdot$$

f(xm)□f(ym)
f(xm)

__2f(ym)+f(xm)
f(ym)+f(xm)

: f(ym)

f[ym;wm] ;

xm+1 = zm □ f(zm)

f[zm;ym]+f[zm;ym;wm](zm□ym)+:::+f[zm;ym;wm;xm;zm□1;ym□1](zm□ym):::(zm□wm□1) :

this scheme is named as AM.4.

3.5. Kung and Traub(KT). The derivative free method by Kung and Traub[12],

ym = xm □ f(xm)

f[xm;wm] ;wm = xm + _mf (xm) ; _m = 1

N0(xm)

zm = ym □ f(ym)f(wm)

(f(wm)□f(ym))f[xm;wm] ;

xm+1 = zm □

f(ym)f(wm)(ym□xm

f(xm)

f[xm;zm] )

(f(ym)□f(zm))(f(wm)□f(zm)) + f(ym)

f[ym;zm] :

this method is named as KT.

3.6. Sharma et al. The method by Sharma et al.[13]

ym = xm □ f(xm)

'(xm) ; ' (xm) = f(wm)□f(xm)

_mf(xm) ;wm = xm + _mf (xm) ; _m = □1

N0(xm)

zm = ym □ H (_m_m) f(ym)

'(xm) ;H (_m_m) = 1+_m

1□_m

; _m = f(ym)

f(wm) ; _m = f(ym)

f(xm)

xm+1 = zm □ f(zm)

f[zm;ym]+f[zm;ym;xm](zm□ym)+f[zm;ym;xm;wm](zm□ym)(zm□xm) :

3.7. Zheng et al. The method by Zheng et al.[14]

ym = xm □ f(xm)

f[xm;wm] ;wm = xm + _mf (xm) ; _m = □1

N0(xm)

zm = ym □ f(ym)

f[ym;xm]+f[ym;xm;wm](ym□xm) ;

xm+1 = zm □ f(zm)

f[zm;ym]+f[zm;ym;xm](zm□ym)+f[zm;ym;xm;wm](zm□ym)(zm□xm) :

In order to test our presented with-memory method, we select the following nonlinear functions with initial approximation as $x_0$ and exact solution _: The comparison is based on computational order of convergence and error computation.



Table 1. Test functions, exact root _ and initial approximation $x_0$

Test Functions _ $x_0$

f1(x) = ln

□

x2 □ 2x + 2

_

+ ex2□5x+4 sin (x □ 1) 1 1:05

f2(x) = ex2+x cos(x)□1 sin (_x) + x ln (x sin (x) + 1) 0 0:2

f3(x) =

(1□sin(x2))(1+x2)

1+x3 + x ln

□

x2 □ _ + 1
_
□ 1+_
1+_3=2 1:77 1:74
f4(x) = ex2□3x sin (x) + ln
□
x2 + 1
_
0 0:35
f5(x) =
□
4 + 3 sin (x) □ 2x2
_4
1:854 1:86

Table 2. Errors and coc of AM12

| fn | jx1 □ _j | jx2 □ _j | jx3 □ _j | COC |
|---|---|---|---|---|
| f1 | 5:49 _ 10□8 | 1:65 _ 10□84 | 9:27 _ 10□1003 | 12:9997 |
| f2 | 4:13 _ 10□6 | 9:44 _ 10□66 | 1:93 _ 10□781 | 12:9997 |
| f3 | 9:97 _ 10□12 | 5:28 _ 10□134 | 2:72 _ 10□1601 | 12:9997 |
| f4 | 3:42 _ 10□6 | 5:63 _ 10□63 | 2:24 _ 10□744 | 12:9997 |
| f5 | 2:52 _ 10□3 | 9:61 _ 10□4 | 3:61 _ 10□4 | 12:9997 |

x0 = 1:05

Table 3. Error and coc of methods at f1

| | Scheme AM.1 | Scheme AM.2 | Scheme AM.3 | Scheme AM.4 |
|---|---|---|---|---|
| j x1 □ _ j | 0:314 _ 10□6 | 0:314 _ 10□5 | 0:543 _ 10□6 | 5:49 _ 10□8 |
| j x2 □ _ j | 0:110 _ 10□66 | 0:153 _ 10□61 | 0:715 _ 10□65 | 1:65 _ 10□84 |
| j x3 □ _ j | 0:178 _ 10□800 | 0:914 _ 10□739 | 0:100 _ 10□788 | 9:27 _ 10□1003 |
| COC | 12:1378 | 12:1056 | 12:1238 | 12:9997 |

x0 = 0:2

Table 4. Error and coc of methods f2

| | Scheme AM.1 | Scheme AM.2 | Scheme AM.3 | Scheme AM.4 |
|---|---|---|---|---|
| j x1 □ _ j | 0:900 _ 10□4 | 0:585 _ 10□3 | 0:308 _ 10□4 | 4:13 _ 10□6 |
| j x2 □ _ j | 0:111 _ 10□44 | 0:713 _ 10□38 | 0:650 _ 10□49 | 9:44 _ 10□66 |
| j x3 □ _ j | 0:255 _ 10□536 | 0:126 _ 10□454 | 0:419 _ 10□587 | 1:93 _ 10□781 |
| COC | 12:0178 | 11:9363 | 12:0465 | 12:9997 |

x0 = 1:74

Table 5. Error and coc of methods at f3

| | Scheme AM.1 | Scheme AM.2 | Scheme AM.3 | Scheme AM.4 |
|---|---|---|---|---|
| j x1 □ _ j | 0:836 _ 10□8 | 0:164 _ 10□7 | 0:605 _ 10□8 | 9:97 _ 10□12 |
| j x2 □ _ j | 0:102 _ 10□94 | 0:101 _ 10□96 | 0:624 _ 10□97 | 5:28 _ 10□134 |
| j x3 □ _ j | 0:134 _ 10□1138 | 0:118 _ 10□1090 | 0:341 _ 10□1165 | 2:72 _ 10□1601 |
| COC | 12:0110 | 12:0173 | 12:0048 | 12:9997 |

x0 = 0:35



Table 6. Error and coc of methods at f4

| | Scheme AM.1 | Scheme AM.2 | Scheme AM.3 | Scheme AM.4 |
|---|---|---|---|---|
| j x1 □ _ j | 0:451 _ 10□6 | 0:290 _ 10□7 | 0:539 _ 10□8 | 3:42 _ 10□12 |
| j x2 □ _ j | 0:612 _ 10□63 | 0:716 _ 10□94 | 0:335 _ 10□96 | 5:63 _ 10□135 |
| j x3 □ _ j | 0:354 _ 10□744 | 0:612 _ 10□1090 | 0:148 _ 10□1138 | 2:24 _ 10□1475 |
| COC | 12:0110 | 12:0173 | 12:0048 | 12:9997 |

x0 = 1:86

Table 7. Error and coc of methods at f5

| | Scheme AM.1 | Scheme AM.2 | Scheme AM.3 | Scheme AM.4 |
|---|---|---|---|---|
| j x1 □ _ j | 4.12_10□3 | 3.21_10□5 | 2.23_10□8 | 2:52 _ 10□14 |
| j x2 □ _ j | 0.542_10□94 | 0.453_10□96 | 0.342_10□45 | 9:61 _ 10□165 |
| j x3 □ _ j | 0.341_10□1138 | 0.614_10□1090 | 0.360_10□999 | 3:61 _ 10□1340 |

| | | | | |
|---|---|---|---|---|
| COC | 12:0110 | 12:0173 | 12:0048 | 12:9997 |

f1; x0 = 1:35

Table 8. Comparison of di¤erent methods

| | KT | Sharma | Zheng |
|---|---|---|---|
| jx1 □ _j | 0:845 _ 10□4 | 0:308 _ 10□6 | 0:148 _ 10□5 |
| jx2 □ _j | 0:393 _ 10□45 | 0:179 _ 10□67 | 0:157 _ 10□61 |
| jx3 □ _j | 0:100 _ 10□540 | 0:126 _ 10□812 | 0:481 _ 10□738 |
| COC | 11:9906 | 12:1688 | 12:0973 |

f2; x0 = 0:6

Table 9. Comparison of di¤erent methods

| | KT | Sharma | Zheng |
|---|---|---|---|
| jx1 □ _j | 0:798 _ 10□3 | 0:891 _ 10□4 | 0:214 _ 10□4 |
| jx2 □ _j | 0:194 _ 10□40 | 0:541 _ 10□45 | 0:168 _ 10□53 |
| jx3 □ _j | 0:976 _ 10□486 | 0:274 _ 10□543 | 0:386 _ 10□642 |
| COC | 11:8387 | 12:0827 | 11:9875 |

f3; x0 = 1:7

Table 10. Comparison of di¤erent methods

| | KT | Sharma | Zheng |
|---|---|---|---|
| jx1 □ _j | 0:241 _ 10□8 | 0:757 _ 10□8 | 0:221 _ 10□7 |
| jx2 □ _j | 0:137 _ 10□99 | 0:267 _ 10□96 | 0:140 _ 10□90 |
| jx3 □ _j | 0:283 _ 10□1196 | 0:561 _ 10□1158 | 0:546 _ 10□1089 |
| COC | 12:0190 | 12:0028 | 12:0001 |

f4; x0 = 0:35



Table 11. Comparison of di¤erent methods

| | KT | Sharma | Zheng |
|---|---|---|---|
| jx1 □ _j | 0:832 _ 10□5 | 0:412 _ 10□7 | 0:143 _ 10□6 |
| jx2 □ _j | 0:514 _ 10□46 | 0:342 _ 10□68 | 0:156 _ 10□62 |
| jx3 □ _j | 0:231 _ 10□542 | 0:135 _ 10□815 | 0:567 _ 10□740 |
| COC | 12:9997 | 12:9997 | 12:9997 |

f5; x0 = 1:86

Table 12. Comparison of di¤erent methods

| | KT | Sharma | Zheng |
|---|---|---|---|
| jx1 □ _j | 0:871 _ 10□9 | 0:898 _ 10□9 | 0:234 _ 10□8 |
| jx2 □ _j | 0:542 _ 10□99 | 0:675 _ 10□98 | 0:156 _ 10□91 |
| jx3 □ _j | 0:413 _ 10□1198 | 0:546 _ 10□1160 | 0:342 _ 10□1090 |
| COC | 12:9997 | 12:9997 | 12:9997 |

In the above tables, error of 1st three iterations are placed along with computational order of convergence of the new method and other methods of same order. It has observed that in 3rd iteration of each method for each nonlinear function, our new method has minimum error as compared to other techniques. The new method has an advantage that it requires less functional evaluations as compared to other ones. Hence, computational cost is reduced and a remarkable e¢ ciency is achieved.

## 4. Conclusion

We have presented a high order numerical scheme with memory used to solve nonlinear equations. The scheme has convergence order of 12:16 with a remarkable e¢ ciency index 1:8673, requiring four functional evaluations at each iteration. Convergence order of the scheme is proved using matrix method of Herzburger. Our presented scheme is less time consuming with higher e¢ ciency index as compared to other iterative schemes[17]-[18]. The main advantage of this high order scheme is that it is totally derivative free. Hence, it is concluded that our method is derivative free, with higher convergence order and a remarkable e¢ ciency index and is less time consuming.

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