Modulational Instability of the Second-order Bright Solitary Wave in Flattened Optical Fiber

ABSTRACT

The criterion of modulational instability of the second order bright solitary wave is studied in this article. The Principle consists initially in seeking all solitary wave solutions of the bright type which verify the nonlinear partial differential equation which governs the dynamics of propagation in flattened optical fibers. When the reference solution to be subjected to a disturbance is identified, the next step consists in establishing the condition of modulational stability/instability.

Keywords: flattened optical fiber, bright solitary wave, modulational instability, iB- function, propagation; nonlinear, dispersive, partial differential equation

1. INTRODUCTION

Modulational instability is defined as a multitude of random frequency signals which accompany a wave or main signal during its propagation. It presents itself as a very negative factor in the propagation of signals, since it facilitates the dissipation of the signal and above all is the source of many disturbances. Over time, the study of modulational instability has become a field of concern for many researchers and authors, in order to ensure the best guarantee of propagation or stability of the signals that one wishes to propagate[1-14]. Many studies have been done to this end. But from a purely analytical angle, the study of the modulational instability of solitary waves is not always easy, no doubt because of the management of the calculations which is not always easy as is often the case for plane waves. Aware of this difficulty, we will want in this work to use an appropriate technique that we have set up during our last investigations in the field of mathematics for nonlinear physics [15-19], to first determine the solitary wave solution of the nonlinear partial differential equation that governs the propagation dynamics in the flattened optical fiber, and then establish its stability/instability criterion. We assume for this purpose that the solitary wave is more robust than the plane wave, which leads us in our calculations to choose a plane wave as the disturber of the solitary wave. Although the final objective of this work is to establish the condition for the solitary wave to be modulationally stable/unstable, we will first find the solitary wave solution worthy of propagation in the flattened single-mode fiber and this in section 2, then establish the criterion of stability / instability in section 3 and quite naturally end our little scientific reflection with a conclusion.

2. SOLITARY WAVE SOLUTIONS

The dynamics of propagation in the flattened optical fiber is governed by the nonlinear partial differential equation of generalized form

$$in_1 \frac{\partial U}{\partial z} + n_2 \frac{\partial^2 U}{\partial t^2} + n_3 |U|^2 U + n_4 \frac{\partial^4 U}{\partial t^4} = 0, \quad (1)$$

where it is assumed that the flattened fiber is immersed in a medium such that each variation is subject to a characteristic coefficient; in particular $n_i (i = 1, 2, 3, 4)$ which have very precise physical meanings. Thus, we propose to determine the solution in the form

$$U(z, \tau) = A(t) \exp ikz, \quad (2)$$

and the equation (1) is transformed into

$$in_1 kA + n_2 \frac{\partial^2 A}{\partial t^2} + n_3 |A|^2 A + n_4 \frac{\partial^4 A}{\partial t^4} = 0. \quad (3)$$

The search for the solitary wave of the bright type requires to choose the ansatz solution of equation (3) in the form

$$A(t) = aJ_{n,0}(\alpha t), \quad (4)$$

where a, n and α are constants to be determined, $J_{n,m}(\alpha t)$ being the iB- function of characteristics, n, m and α [20-24]. The search for the solution in the form (4), is based on the fact that in the case where $n \succ 0$, we have a solitary wave of the bright type. Under these conditions, solving equation (3) amounts to evaluating the terms of equation (3), in particular

$$\frac{\partial^2 U}{\partial t^2} = -an\alpha^2 J_{n,0}(\alpha t) + an(n+1)\alpha^2 J_{n+2,0}(\alpha t), \quad (5)$$

$$\begin{aligned} \frac{\partial^4 U}{\partial \tau^4} &= a\left[n^2 + 2n(n+1)\right]\alpha^4 J_{n,0}(\alpha t) \\ &-\left\{a(n+1)\left[n^2 + 2n(n+1)\right]\alpha^4 + 3an(n+1)(n+2)\alpha^4\right\} J_{n+2,2}(\alpha t) \\ &+an(n+1)(n+2)\alpha^4 J_{n+4,4}(\alpha t), \end{aligned} \quad (6)$$

and

$$|A|^2 = |a|^2 aJ_{3n,0}(\alpha t). \quad (7)$$

Inserting the terms (5), (6) and (7) into (3) gives

$$\begin{aligned} &\left[-n_1 ka - n_2 an\alpha^2 + n_4 a\left(n^2 + 2n(n+1)\right)\alpha^4\right] J_{n,0} \\ &+\left\{ n_2 an(n+1)\alpha^2 - n_4 \left[\begin{matrix} a(n+1)\left[n^2 + 2n(n+1)\right]\alpha^4 \\ +3an(n+1)(n+2)\alpha^4 \end{matrix}\right]\right\} J_{n+2,2} \\ &+n_4 an(n+1)(n+2)\alpha^4 J_{n+4,4} + n_3 |a|^2 aJ_{3n,0} = 0. \end{aligned} \quad (8)$$

Bearing in mind the transformations

$$J_{n+2,2} = J_{n,0} - J_{n+2,0}, \quad (9)$$

and

$$J_{n+4,4} = J_{n,0} - 2J_{n+2,0} + J_{n+4,0}, \quad (10)$$

The range equation (8) becomes

$$\left\{\begin{array}{l}\left[-n_1ka-n_2an\alpha^2+n_4a\left[n^2+2n\left(n+1\right)\right]\alpha^4\right]\\+\left[n_2an\left(n+1\right)\alpha^2-n_4\left[a\left(n+1\right)\left(n^2+2n\left(n+1\right)\right)\right]\alpha^4+3an\left(n+1\right)\left(n+2\right)\alpha^4\right]\\+n_4an\left(n+1\right)\left(n+2\right)\alpha^4\end{array}\right\}J_{n,0}$$

$$+\left\{\begin{array}{l}-\left[n_2an\left(n+1\right)\alpha^2-n_4\left[a\left(n+1\right)\left(n^2+2n\left(n+1\right)\right)\alpha^4\right]+3an\left(n+1\right)\left(n+2\right)\alpha^4\right]\\-2n_4an\left(n+1\right)\left(n+2\right)\alpha^4\end{array}\right\}J_{n+2,0} \qquad (11)$$

$$+n_4an\left(n+1\right)\left(n+2\right)\alpha^4J_{n+4,0}+n_3\left|a\right|^2aJ_{3n,0}=0.$$

The values of n for which certain terms of equation (9) combine are 0, 1 and 2. This leads in what follows to seek the solutions of this equation for these corresponding values of n.

- For $n=0$,

The range coefficients equation (9, gives

$$-n_1k+n_5\left|a\right|^2=0. \qquad (12)$$

Solving equation (12) gives

$$\left|a\right|=\sqrt{\frac{n_1k}{n_3}},n_1n_2k\succ 0, \qquad (13)$$

Relation (13), implies that there exists a real θ such that

$$a=\sqrt{\frac{n_1k}{n_3}}\exp i\theta. \qquad (14)$$

The solution of equation (1) under this condition is a plane wave given by

$$U\left(z,t\right)=\sqrt{\frac{n_1k}{n_3}}\exp i\left(kz+\theta\right). \qquad (15)$$

- For $n=1$,

The range coefficients equation (9) becomes

$$-n_1ka+n_2\alpha^2a-27n_4\alpha^4a+\left(-2n_2\alpha^2+40n_4\alpha^4+n_3\left|a\right|^2\right)aJ_{3,0}+6an_4\alpha^4J_{5,0}=0. \qquad (16)$$

Equation (16) holds if for $a\neq 0$, we have the following relations

$$n_1k-n_2\alpha^2+27n_4\alpha^4=0, \qquad (17)$$

$$-2n_2\alpha^2+40n_4\alpha^4+n_3\left|a\right|^2=0, \qquad (18)$$

and

$$n_4=0. \qquad (19)$$

Equations (17) and (18) respectively give

$$n_1=n_2\alpha^2/k, \qquad (20)$$

and

$$|a|=\alpha\sqrt{\frac{2n_2}{n_3}}\Rightarrow\exists\theta\in R/\,a=\alpha\sqrt{\frac{2n_2}{n_3}}\exp i\theta. \qquad (21)$$

The solution of equation (1) in this case is given by

$$U(z,t)=\alpha\sqrt{\frac{2n_2}{n_3}}J_{1,0}(\alpha t)\exp i(kz+\theta), \qquad (22)$$

by means of the constraint relation (20). This solution modifies the structure of the waveguide and is obtained for very low dispersions of order 4. This would also mean that it is more suitable in non-flattened single mode fiber. The equation that governs the propagation dynamics in this case is given by

$$i\frac{n_2\alpha^2}{k}kA+n_2\frac{\partial^2 A}{\partial t^2}+n_3|A|^2A=0. \qquad (23)$$

- For $n=2,$

The range coefficients equation (9) becomes

$$\begin{aligned}&\left(-n_1ka+4an_2\alpha^2-64n_2a\alpha^4\right)J_{2,0}\\&-\left(6n_2a\alpha^2-56n_4a\alpha^4\right)J_{4,0}+a\left(24n_4\alpha^4+n_3|a|^2\right)J_{6,0}=0.\end{aligned} \qquad (24)$$

Equation (24) is verified if and only if for $a\neq 0$,we have

$$-n_1ka+4an_2\alpha^2-64n_2a\alpha^4=0, \qquad (25)$$

$$3n_2-28n_4\alpha^2=0, \qquad (26)$$

and

$$24n_4\alpha^4+n_3|a|^2=0. \qquad (27)$$

We obtain from equation (27)

$$|a|=\alpha^2\sqrt{\frac{-24n_4}{n_3}},n_3n_4\prec 0\Rightarrow\exists\theta\in R/\,a=\alpha^2\sqrt{\frac{-24n_4}{n_3}}\exp i\theta. \qquad (28)$$

The solution of equation (1) in this case is given by

$$U(z,t)=\alpha^2\sqrt{\frac{-24n_4}{n_3}}J_{2,0}(\alpha t)\exp i(kz+\theta), \qquad (29)$$

by means of the constraint relations $n_2=28n_4\alpha^2/3$ and $n_1=\left(112-1792\alpha^2\right)/3k$.

We note that only the solution (29) is effectively that of the nonlinear partial differential equation which governs the dynamics of propagation in the flattened optical fiber. It is precisely the one that will be at the center of the study of modulational instability because solution (22) works in the non-flattened single-mode optical fiber.

3. CONDITION OF MODULATIONAL INSTABILITY

Let ε be a very small quantity and $\delta(z,t)$ a very small function to serve as a perturber for the solution $U_0(z,t)$ corresponding to the solution (29). We are looking for the relation that $\delta(z,t)$ satisfies when we look for the solution of equation (1) in the form

$$U(x,t)=U_0(x,t)+\varepsilon\delta(x,t). \tag{30}$$

Thus, introducing equation (30) into (1), leads to

$$in_1\frac{\partial\delta}{\partial z}+n_2\frac{\partial^2\delta}{\partial t^2}+2n_3|U_0|^2\delta+n_3U_0^2\delta^*+n_4\frac{\partial^4\delta}{\partial t^4}=0. \tag{31}$$

Neglecting the high spatial frequency term in $U_0(z,t)$ equation (31) reduces to

$$in_1\frac{\partial\delta}{\partial z}+n_2\frac{\partial^2\delta}{\partial t^2}+2n_3|U_0|^2\delta+n_4\frac{\partial^4\delta}{\partial t^4}=0. \tag{32}$$

By choosing the perturbation $\delta(z,t)$ in the form

$$\delta(z,t)=A(z,t)+iB(z,t), i^2=-1, \tag{33}$$

where $A(z,t)$ and $B(z,t)$ are real functions, equation (32) leads to the system of coupled differential equations

$$-n_1\frac{\partial B}{\partial z}+n_2\frac{\partial^2 A}{\partial t^2}+2n_3|U_0|^2A+n_4\frac{\partial^4 A}{\partial t^4}=0, \tag{34}$$

and

$$n_1\frac{\partial A}{\partial z}+n_2\frac{\partial^2 B}{\partial t^2}+2n_3|U_0|^2B+n_4\frac{\partial^4 B}{\partial t^4}=0. \tag{35}$$

We then seek the solutions of equations (34) and (35) in the forms

$$A=aJ_{-1,0}\left[i(k'z-\omega t)\right], \tag{36}$$

and

$$A=-ibJ_{0,1}\left[i(k'z-\omega t)\right], \tag{37}$$

where k' is the spatial frequency of disturbance , ω the angular frequency of disturbance, a and b are reals. The taking into account of the solutions in the forms above, imposes the evaluation of certain terms of the equations (34) and (35). so we have

$$\frac{\partial^2 A}{\partial t^2}=-\omega^2aJ_{-1,0}\left[i(k'z-\omega t)\right], \tag{38}$$

$$\frac{\partial^2 B}{\partial t^2}=i\omega^2bJ_{0,1}\left[i(k'z-\omega t)\right], \tag{39}$$

$$\frac{\partial^4 A}{\partial t^4} = \omega^4 a J_{-1,0}\left[i\left(k'z - \omega t\right)\right], \quad (40)$$

and

$$\frac{\partial^4 B}{\partial t^4} = -i\omega^4 b J_{0,1}\left[i\left(k'z - \omega t\right)\right]. \quad (41)$$

The introduction of terms (38) to (41) in equations (34) and (35) leads to the system of linear equations

$$\left(n_2\omega^2 - 2n_3\left|U_0\right|^2 - \omega^4 n_4\right)a - n_1 k' b = 0, \quad (42)$$

and

$$n_1 k' a - \left(n_2\omega^2 - 2n_3\left|U_0\right|^2 - \omega^4 n_4\right)b = 0. \quad (43)$$

The system of equations (42) and (43) admits non-trivial solutions if and only if we have the following equation

$$\begin{vmatrix} n_2\omega^2 - 2n_3\left|U_0\right|^2 - \omega^4 n_4 & -n_1 k' \\ n_1 k' & -n_2\omega^2 - 2n_3\left|U_0\right|^2 - \omega^4 n_4 \end{vmatrix} = 0. \quad (44)$$

The expansion of equation (44) leads to

$$\left(\omega^2 - \frac{n_2}{2n_4}\right)^2 = \pm\frac{n_1 k'}{n_4} + \left(\frac{n_2}{2n_4}\right)^2 - \frac{2n_3}{n_4}\left|U_0\right|^2. \quad (45)$$

The dispersion relation (45) can still be written

$$\omega^2 = \pm\frac{n_2}{2n_4} + \sqrt{\pm\frac{n_2 k'}{n_4} + \left(\frac{n_2}{2n_4}\right)^2 - \frac{2n_3}{n_4}\left|U_0\right|^2} \quad (46)$$

The validity condition of equation (46) imposes the double condition

$$\left|U_0\right|^2 \le \frac{n_4}{2n_3}\left[\left|\frac{n_1 k'}{n_4}\right| + \left(\frac{n_2}{2n_4}\right)^2\right], \quad (47)$$

and

$$\left|U_0\right|^2 \ge \frac{n_4}{2n_3}\left|\frac{n_1 k'}{n_4}\right|. \quad (48)$$

The inequalities (47) and (48) allow to write

$$\frac{n_4}{2n_3}\left|\frac{n_1 k'}{n_4}\right| \le \left|U_0\right|^2 \le \frac{n_4}{2n_3}\left[\left|\frac{n_1 k'}{n_4}\right| + \left(\frac{n_2}{2n_4}\right)^2\right]. \quad (49)$$

The intensity of the solitary wave must satisfy the criterion above so that there is modulational stability and in the opposite case there is modulational instability.

4. CONCLUSION

We have succeeded in establishing in this article, the criterion of the modulational instability of a solitary wave of the second order pulse type in the flattened optical fiber. What should be noted is that this instability is closely linked to the intensity of the propagating wave, but above all to the characteristic properties of the fiber. We can cite the coefficients of nonlinearity, dispersion of order two and of order four. The importance of this study lies in the fact that it can serve as a base for the experimental work that we want to undertake on the flattened optical fiber.

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