

Characterization of a new class of stochastic processes including all known extensions of the class (\underline{L})

Abstract

The class (\underline{L}) is an important family of semimartingales defined by Yor. These processes play a key role in the theory of probability and their applications. For instance, such processes are used to resolve the Skorokhod Imbedding Problem and to construct solutions for homogeneous and inhomogeneous skew Brownian Motion equations. This paper contributes to the study of classes (\underline{L}) and (\underline{L}_r) . But, instead of considering as it is customary, the semi-martingales whose finite variational part is continuous, we will consider those whose finite variational part is càdlàg. The two main contributions of this paper are as follows. First, we present a new characterization result for the stochastic processes of class (\underline{L}_r) . Second, we provide a framework for unifying the studies of classes (\underline{L}) and (\underline{L}_r) . More precisely, we define and study a new larger class that we call class (\underline{L}_g) and for which we give characterization results. In addition, we derive some structural properties inspired of those obtained for classes (\underline{L}) and (\underline{L}_r) . Finally, we show that some processes of this new class can take the form of relative martingales. More precisely, we derive a formula allowing to recover some processes of the class (\underline{L}_g) from an honest time and their final value.

Keywords: class (\underline{L}) ; class (\underline{L}_r) ; Balayage formula; Honest time; Relative martingales
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1 Introduction

This study investigates càdlàg (right continuous and admits a left limite) semimartingales of classes (\underline{L}) and (\underline{L}_r) . These are stochastic processes X of the following form:

$$X = M + A; \quad (1.1)$$

where M is a càdlàg local martingale with $M_0 = 0$ and A is an adapted predictable process of finite variation with $A_0 = 0$, such that the signed measure induced by A is carried by an optional random set H , where Z_t

$$\int_{H_t} dA_s = 0, \quad \forall t \geq 0; \quad (1.2)$$

Such processes are strongly related to many probabilistic studies. Well-known examples of studies where the use of such processes is capitalized include the theory of Azéma–Yor martingales, the study of zeros of continuous martingales [2], the study of Brownian local times, the balayage formulas in the progressive case [14], the construction of solutions for skew Brownian motion equations [10], and the resolution of Skorokhod’s reflection equation and embedding problem [3]. These classes are represented in the form of H . More precisely, for the processes of class (\underline{L}) , we have

$$H = \{t \geq 0 : X_t = 0\};$$

By contrast, for class (\underline{L}_r) , the random set H takes the following form

$$H = \{t \geq 0 : X_{t-} = 0\};$$

The stochastic processes of class (\underline{L}) , whose finite variational part is continuous, have been studied extensively by several authors, including Yor, Najnudel, Nikeghbali, Cheridito, Platen, Ouknine, Bouhadou, Eyi Obiang, Moutsinga, and Trutnau (see [5, 6, 9, 10, 15, 16, 17, 18, 19, 20, 23]). Recently, this notion of class (\underline{L}) has been extended to other areas. For instance, there has been a series of articles in the field of stochastic Calculus for Signed Measures (see [7, 8, 9, 10, 21, 22]) and in the field of Stochastic Calculus for càdlàg semimartingales (see [12]). The authors of all above mentioned references studied the main properties of these processes, presented their applications, and relaxed the original hypotheses. The notion of stochastic processes of class (\underline{L}) has evolved over time, and the present study considers the most general definition presented by Eyi Obiang et al. in [10], which extends the notion of class (\underline{L}) to càdlàg semimartingales, whose finite variational part is considered càdlàg instead of continuous. We consider the following definition:

Definition 1.1. We say that a semimartingale X is of class (\underline{L}) if it decomposes as $X = M + A$, where

1. M is a càdlàg local martingale, with $M_0 = 0$;
2. A is an adapted càdlàg predictable process with finite variations such that $A_{0-} = A_0 = 0$;
- 3.

R_t

$$\int_{R_t} dA_s = 0 \text{ for all } t \geq 0.$$

By contrast, the study of class (\underline{L}_r) is quite recent. In 2018, Akdim et al. [1] first characterized and studied the structural properties of the positive submartingales of the said class. However, it should be noted that the use of the processes of class (\underline{L}_r) has a longer history. For instance, in 1981, Barlow [4] used these processes to show that any positive submartingale is equal to the absolute value of a martingale. More precisely, we consider the following definition:

Definition 1.2. We say that a semimartingale X is of class $(\underline{\quad})$ if it decomposes as $X = M + A$, where

1. M is a càdlàg local martingale, with $M_0 = 0$;
2. A is an adapted càdlàg predictable process with finite variations such that $A_{0^+} = A_0 = 0$;
- 3.

\mathbb{R}_+

$$\int_0^t \mathbb{1}_{\{X_s \neq 0\}} dA_s = 0 \text{ for all } t \geq 0.$$

Notably, the two above-mentioned classes coincide for the processes X , whose finite variational part A is considered continuous (i.e., class $(\underline{\quad})$ under the hypotheses considered by Nikeghbali [19] and Cheridito et al. [6]). However, it is possible to determine processes belonging to at least one of these classes that are not present in another class.

This study contributes toward existing literature by enriching the general framework and developing techniques for dealing with stochastic processes of class $(\underline{\quad})$ and the càdlàg semimartingales of

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class $(\underline{\quad})$, whose finite variational part is càdlàg instead of continuous. First, we study the processes of class $(\underline{\quad})$ by proposing a new method to characterize such stochastic processes.

Second, we present a general framework that unifies the study of the two above-mentioned classes. More precisely, we propose a new larger class that includes all the processes of the classes $(\underline{\quad})$ and $(\underline{\quad})$. We term this class as $(\underline{\quad})$ and define it as follows:

Definition 1.3. We say that a stochastic process X is of the class $(\underline{\quad})$ if it decomposes as $X = M + A$, where

1. M is a càdlàg local martingale, with $M_0 = 0$;
2. A is an adapted càdlàg predictable process with finite variations such that $A_{0^+} = A_0 = 0$;
- 3.

\mathbb{R}_+

$$\int_0^t \mathbb{1}_{\{X_s \neq 0\}} dA_s = 0 \text{ for all } t \geq 0.$$

Hence, we explore and extend the general properties obtained for classes $(\underline{\quad})$ and $(\underline{\quad})$ in [19, 6, 20, 1]. For instance, we study the positive and negative parts of the processes of class $(\underline{\quad})$ and show that the product of the processes of class $(\underline{\quad})$ with vanishing quadratic covariation also belongs to class $(\underline{\quad})$. Further, we show that every positive process X of class $(\underline{\quad})$ has a multiplicative decomposition. In other words, it can be decomposed as

$$X = CW \square 1;$$

where W is a positive local martingale with $W_0 = 1$, and C is a non-decreasing process. This result is an extension of that obtained by Nikeghbali for positive and continuous submartingales [20]. We also present a result that enables the recovery of any process of class $(\underline{\quad})$ from its final value X_1 and of an honest time g , which is the last time $(X_t : t \geq 0)$ or $(X_{t^+} : t \geq 0)$ visited the origin. More precisely, this formula has the following form:

$$X_t = E$$

$$\overline{X}_{1 \wedge t \wedge g} | \mathcal{F}_t$$

;

where X is the process of class $(\underline{\quad})$, $X_1 = \lim_{t \rightarrow 1^-} X_t$, and $g = \sup\{t \geq 0 : X_t X_{t^+} = 0\}$. **This result generalizes both the representation formulas of the class $(\underline{\quad})$ in [6] and those obtained for processes of class $(\underline{\quad})$ [1]. A particular case of such formulas appears in works of Madan, Roynette and Yor [13]. Specifically, they proved that**

$$(k \square M_T)_+ = E$$

$$\overline{(k \square M_1)_+ 1_{\{g_k \leq t\}}} | \mathcal{F}_t$$

;

where k is a constant, M is a local martingale with no positive jumps and $g_k = \sup\{t \geq 0 : M_t \leq kg\}$. We extend this formula to all càdlàg local martingales in Section 3. Finally, we derive a series of characterization results for $(\underline{\quad})$. One of these results generalizes the above mentioned characterization result of the class $(\underline{\quad})$ and a result characterizing positive sub-martingales of class $(\underline{\quad})$ known in the literature as the martingale characterization theorem. This last mentioned result is initially established by Nikeghbali (Theorem 2.1 of [19]) for sub-martingales, whose finite variational part is continuous and extended to positive special semimartingales of class $(\underline{\quad})$, whose finite variational part is càdlàg (see Theorem 2 of [10]).

The remainder of this paper is organized as follows. In Section 2, we present some useful preliminaries and introduce new characterization of class $(\underline{\quad})$. Section 3 is devoted to the study of the new class $(\underline{\quad})$. Finally, Section 4 summarizes the related approaches and methods.

2 Preliminaries and new characterization of the class (\underline{r})

The main purpose of this section is to contribute toward the framework for studying the processes of class (\underline{r}) . More precisely, we propose a new method for characterizing the positive processes of class (\underline{r}) . However, we first recall some results and notations that will be useful for understanding this work.

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2.1 Notations and Preliminaries

In this work, we fix a filtered probability space $(\Omega, (\mathcal{F}_t)_{t \geq 0}, \mathcal{F}_\infty, P)$ that satisfies the usual conditions. Throughout this work, for any càdlàg stochastic process X , we consider that X_c is its continuous part, and $(X_{t-})_{t \geq 0}$ denotes the process defined by $\forall t > 0$, where X_{t-} is the left limit of X in t and $X_{0-} = X_0$.

Now, let us recall the version of class (\underline{r}) studied by Nikeghbali [19] and Cheridito et al. [6]

Definition 2.1. We say that a semi-martingale X is of class (\underline{r}) if it decomposes as $X = M + A$, where

1. M is a càdlàg local martingale, with $M_0 = 0$;
2. A is an adapted continuous process with finite variations such that $A_0 = 0$;
- 3.

R_t

$$\int_0^t \mathbf{1}_{\{X_{s-} \leq 0\}} dA_s = 0 \text{ for all } t \geq 0.$$

Nikeghbali's Theorem 2.1 [19] serves as a method to characterize the non-negative processes that satisfy the assumptions of Definition 2.1. This result is called the characterization martingale theorem. We recall it as follows:

Theorem 2.1. Let $X = M + A$ be a non-negative semi-martingale. Then, the following are equivalent:

1. $X \in (\underline{r})$;
2. There exists a non-decreasing continuous process V with $V_0 = 0$ such that, for any locally bounded Borel function f with $F(x) =$

R_x

$\int_0^x f(z) dz$, the process

$$(F(V_t) - \int_0^t f(V_s) X_s; t \geq 0)$$

is a càdlàg local martingale and $V \in A$.

This result was extended by Eyi Obiang et al. [10] for càdlàg non-negative processes satisfying Definition 1.3, as follows:

Theorem 2.2. Let $X = M + A$ be a non-negative and càdlàg semi-martingale. Then, the following are equivalent:

1. $X \in (\underline{r})$;
2. There exists a càdlàg non-decreasing predictable process with finite variations V such that, for any $f \in C_1$ with $F(x) =$

R_x

$\int_0^x f(z) dz$, the process

0

$$\int_0^t f(V_s) d[V_c]$$

$$- \int_0^t f(V_s) X_s dA_s$$

$$+ \int_0^t f(V_s) X_{s-} dA_s$$

$$+ \int_0^t f(V_s) d[V_c]$$

$$- \int_0^t f(V_s) X_{s-} dA_s$$

$$+ \int_0^t f(V_s) d[V_c]$$

$$- \int_0^t f(V_s) X_{s-} dA_s$$

$$+ \int_0^t f(V_s) d[V_c]$$

$$- \int_0^t f(V_s) X_{s-} dA_s; t \geq 0$$

1

A

is a càdlàg local martingale and $V \in A$.

Now, recall that the family of processes of class (\underline{r}) considered by Nikeghbali (Definition 2.1) is also included in class (\underline{r}) . Indeed, it suffices to say that if A is a continuous process, we have

Z_t

0

$$\int_0^t \mathbf{1}_{\{X_{s-} \leq 0\}} dA_s =$$

Z_t

0

$$\int_0^t \mathbf{1}_{\{X_{s-} \leq 0\}} dA_s = 0:$$

Hence, in the next subsection, we present an extension of the characterization martingale theorem for the processes of class (\underline{r}) .

2.2 New characterization result for the class (\underline{r})

Let us begin with an extension of Lemma 2.3 of [6].

Lemma 2.3. Let $X = M + A$ be a process of the class (\underline{r}) and A_c be the continuous part of A . For every C_1 function f and a function F defined by $F(x) =$

$\int_0^x f(z) dz$, the process

$(F(A_{ct}) - \int_0^t f(A_{ct}) dA_{ct})_{t \geq 0}$

is a càdlàg local martingale.

Proof. Through integration by parts, we get

$$F(A_{ct}) - \int_0^t f(A_{ct}) dA_{ct} = \int_0^t f(A_c) dX_s + \int_0^t f(A_c) dA_c$$

Hence, we have

$$F(A_{ct}) - \int_0^t f(A_{ct}) dA_{ct} = \int_0^t f(A_c) dX_s + \int_0^t f(A_c) dA_c$$

X_{s-}

f

(A_{ct})

$(X_{s-} - A_s)$

because $A = A_c +$

P

$(X_{s-} - A_s)$. Furthermore, we have

R_t

\int_0^t

(A_c)

$\int_0^t f(A_c) dA_s = 0$ since dA is carried by

$\{t \geq 0 : X_t = 0\}$. Therefore, it follows that

$(F(A_{ct}) - \int_0^t f(A_{ct}) dA_{ct})_{t \geq 0}$

$(X_t - \int_0^t f(A_{ct}) dA_{ct})_{t \geq 0}$

$$f(A_{ct}) X_t = \int_0^t f(A_{cs}) dX_s + \int_0^t f(A_{cs}) dM_s + \int_0^t f(A_{cs}) dA_s$$

$$= \int_0^t f(A_{cs}) dM_s + \int_0^t f(A_{cs}) dA_s + \int_0^t [f(A_{cs}) - f(A_{ct})] X_s dA_s$$

Consequently,

$$f(A_{ct}) X_t = \int_0^t f(A_{cs}) dM_s + F(A_{ct}) + \int_0^t [f(A_{cs}) - f(A_{ct})] X_s dA_s$$

This implies that

$$F(A_{ct}) + \int_0^t [f(A_{cs}) - f(A_{ct})] X_s dA_s = \int_0^t f(A_{cs}) dM_s$$

This completes the proof.

Now, we shall present our martingale characterization theorem for the class (\underline{L}) .

Theorem 2.4. Let $X = M + A$ be a positive semi-martingale. Then, the following are equivalent:

1. $X \in (\underline{L})$;
2. There exists a non-decreasing predictable process V such that, for any $F \in C_2$, the process

$$\int_0^t F(V_{cs}) dX_s + \int_0^t F(V_{cs}) dM_s$$

$$X_{s,t} = \int_0^t (V_c)_s \square F_{00}(V_c)_s X_{s-}]_V s; t _ 0$$

1
A
is a càdlàg local martingale and $V _ A$.

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Proof. (1)) (2) Let us consider $V = A$. Hence, from Lemma 2.3, we determine that

$$0 \text{ @ } F(A_{ct}) \square F_{00}(A_{ct}) X_{t+} X_{s,t} = \int_0^t (A_c)_s \square F_{00}(A_c)_s X_{s-}]_A s; t _ 0$$

1
A
is a càdlàg local martingale.

(2)) (1) First, let $F(x) = x$. Then, the process W defined by

$$W_t = V_c \int_0^t X_{s-} dV_c = V_t \square X_t$$

is a local martingale. Hence, owing to the uniqueness of the special semi-martingale decomposition, we obtain $V = A$. Next, we take $F(x) = x^2$. Thus, process B defined by

$$B_t = (V_c)_t^2 \square 2 \int_0^t X_{s-} dV_c = X_{s,t}^2 \square 2 \int_0^t X_{s-} dV_c$$

is a local martingale. However, through integration by parts, it follows that

$$B_t = 2 \int_0^t V_c dV_c \square 2 \int_0^t X_{s-} dV_c$$

$$V_c$$

$$s _ V_s \square 2$$

$$X$$

$$s _ t$$

$$X_{s \square} _ V_s$$

$$= 2$$

$$Z_t$$

$$0$$

$$V_c$$

$$s d$$

$$0$$

$$@V_c$$

$$s +$$

$$X$$

$$u _ s$$

$$_ V_u \square X_s$$

$$1$$

$$A \square 2$$

$$Z_t$$

$$0$$

$$X_{s \square} d$$

$$0$$

$$@V_c$$

$$s +$$

$$X$$

$$u _ s$$

$$_ V_u$$

$$1$$

$$A$$

$$= 2$$

$$Z_t$$

$$0$$

$$V_c$$

$$s dW_s \square 2$$

$$Z_t$$

$$0$$

$$X_{s \square} dV_s:$$

$$\text{Consequently, we must have } Z_t$$

$$0$$

$$X_{s \square} dV_s = 0:$$

$$\text{In other words, } dA \text{ is carried by the set } ft _ 0 : X_{t \square} = 0g.$$

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$$X_{s \square} dV_s = 0:$$

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$$X_{s \square} dV_s = 0:$$

$$\text{In other words, } dA \text{ is carried by the set } ft _ 0 : X_{t \square} = 0g.$$

3 Characterization of the new class of stochastic processes

We propose unifying the study of the stochastic processes of classes $(_)$ and $(_r)$. More precisely, we provide a general framework to study a larger class that we term as class $(_g)$.

3.1 First characterization and some properties

As is evident from the above definition, classes $(_)$ and $(_r)$ are included in class $(_g)$. Indeed, we can see that $fX_t = 0g _ fX_t X_{t \square} = 0g$ and $fX_{t \square} = 0g _ fX_t X_{t \square} = 0g$. However, there exist processes of class $(_g)$ that do not belong to classes $(_)$ and $(_r)$. For instance, if M is a càdlàg local martingale, M_+ and M_- are elements of the class $(_g)$ (see Lemma 3.8). However, M_+^2 $(_r)$ only if, M has no negative jump and M_+^2 $(_)$ only if, M has no positive jump. Next, we present the first method to characterize the stochastic processes of class $(_g)$.

Theorem 3.1. Let $X = M + A$ be a càdlàg semi-martingale. Then, the following are equivalent:

1. X^2 $(_g)$;
2. there exist two predictable processes C and V such that $A = C + V$ and

$$Z_t$$

$$0$$

$$1_{fX_{s \square} \neq 0g} dC_s =$$

$$Z_t$$

$$0$$

$$1_{fX_{s \square} \neq 0g} dV_s = 0:$$

Proof. (1) \implies (2) We can see that, for all $t _ 0$,

$$A_t =$$

$$Z_t$$

$$0$$

$$\int_0^t f_{X_s=0} g dA_s +$$

$$\int_0^t$$

$$\int_0^t$$

$$\int_0^t f_{X_s=0} g dA_s =$$

$$\int_0^t$$

$$\int_0^t$$

$$\int_0^t f_{X_s=0} g dA_s +$$

$$\int_0^t$$

$$\int_0^t$$

$$\int_0^t f_{X_s \square 6=0} g dA_s +$$

$$\int_0^t$$

$$\int_0^t$$

$$\int_0^t f_{X_s=0; X_s \square =0} g dA_s:$$

However,

$$\int_0^t$$

$\int_0^t f_{X_s \square 6=0} g dA_s = 0$ as dA_s is carried by $f_{X_s \square =0}$. Hence, it entails the following:

$$\int_0^t$$

$$\int_0^t$$

$$\int_0^t$$

$$\int_0^t f_{X_s=0} g dA_s +$$

$$\int_0^t$$

$$\int_0^t$$

$$\int_0^t f_{X_s=0; X_s \square =0} g dA_s:$$

Now, let us substitute $C_t =$

$$\int_0^t$$

$$\int_0^t f_{X_s=0} g dA_s \text{ and } V_t =$$

$$\int_0^t$$

$$\int_0^t f_{X_s=0; X_s \square =0} g dA_s. \text{ Thus, we obtain } \int_0^t _ 0,$$

$$\int_0^t$$

$$\int_0^t$$

$$\int_0^t f_{X_s=0} g dC_s =$$

$$\int_0^t$$

$$\int_0^t$$

$$\int_0^t f_{X_s=0} g \int_0^t f_{X_s=0} g dA_s = 0 \text{ and}$$

$$\int_0^t$$

$$\int_0^t$$

$$\int_0^t f_{X_s \square 6=0} g dV_s =$$

$$\int_0^t$$

$$\int_0^t$$

$$\int_0^t f_{X_s \square 6=0} g \int_0^t f_{X_s=0; X_s \square =0} g dA_s = 0:$$

(2)) (1)

Now, assume that $A = C + V$ with

$$\int_0^t$$

$$\int_0^t f_{X_s \square 6=0} g dV_s =$$

$$\int_0^t$$

$$\int_0^t f_{X_s=0} g dC_s = 0. \text{ One has } \int_0^t _ 0,$$

$$\int_0^t$$

$$\int_0^t$$

$$\int_0^t f_{X_s \square 6=0} g dA_s =$$

$$\int_0^t$$

$$\int_0^t$$

$$\int_0^t f_{X_s \square 6=0} g dC_s +$$

$$\int_0^t$$

$$\int_0^t$$

$$\int_0^t f_{X_s \square 6=0} g dV_s:$$

However,

$$\int_0^t$$

$$\int_0^t$$

$$\int_0^t f_{X_s \square 6=0} g dC_s =$$

$$\int_0^t$$

$$\int_0^t$$

$$\int_0^t f_{X_s \square 6=0} g \int_0^t f_{X_s=0} g dC_s = 0, \text{ since } \int_0^t f_{X_s=0} g dC_s _ 0$$

and \int_0^t

$$\int_0^t$$

$$\int_0^t$$

$$\int_0^t f_{X_s \square 6=0} g dV_s =$$

$$\int_0^t$$

$$\int_0^t$$

$$\int_0^t f_{X_s=0} g \int_0^t f_{X_s \square 6=0} g dV_s = 0, \text{ since } \int_0^t f_{X_s \square 6=0} g dV_s _ 0:$$

This completes the proof.

As an application of Theorem 3.1, we present two corollaries that provide a new approach to characterize the classes $(_)$ and $(_)$.

Corollary 3.2. Let $X = M + A$ be a càdlàg semi-martingale. Then, the following are equivalent:

1. $X \in (_)$;
2. there exist a continuous finite variation process V and a càdlàg predictable process C such that $A = C + V$ and

$$\int_0^t f_{X_s} dC_s = 0$$

$$\int_0^t f_{X_s} dV_s = 0:$$

Proof. (1) \implies (2) Assume that X is an element of the class $(_)$. Hence, it follows from Definition 1.3 that there exists a local martingale M and a càdlàg, predictable process A such that $\int_0^t dA_s$ is carried by $f_{X_t} = 0$ and $X = M + A$. It is evident that (2) yields by taking $C = A$ and $V = 0$.

(2) \implies (1) Now, assume that Assertion (2) is true. We have $\int_0^t dC_s = 0$,

$$\int_0^t f_{X_s} dA_s =$$

$$\int_0^t f_{X_s} dC_s +$$

$$\int_0^t f_{X_s} dV_s:$$

However,

$$\int_0^t f_{X_s} dC_s = 0 \text{ as } dC_s \text{ is carried by } f_{X_s} = 0. \text{ Hence,}$$

$$\int_0^t f_{X_s} dA_s =$$

$$\int_0^t f_{X_s} dV_s:$$

Furthermore,

$$\int_0^t f_{X_s} dV_s =$$

$$\int_0^t f_{X_s} dV_s \text{ because } V \text{ is continuous. Therefore,}$$

$$\int_0^t f_{X_s} dA_s =$$

$$\int_0^t f_{X_s} dV_s = 0:$$

Consequently, X is an element of the class $(_)$. This completes the proof.

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Corollary 3.3. Let $X = M + A$ be a càdlàg stochastic process. Then, the following are equivalent:

1. $X \in (_)$;
2. there exist a continuous finite variation process C and a càdlàg predictable process V such that $A = C + V$ and

$$\int_0^t f_{X_s} dC_s = 0$$

$$\int_0^t f_{X_s} dV_s = 0.$$

Proof. (1) \implies (2) Assume that X is an element of class $(_)$. Hence, there exist a local martingale M and a càdlàg predictable process A such that $\int_0^t dA_s$ is carried by $f_{X_t} = 0$ and $X = M + A$. It is clear that (2) yields by taking $V = A$ and $C = 0$.

(2) \implies (1) Now, assume that Assertion (2) is true. We have $\int_0^t dC_s = 0$,

$$\int_0^t f_{X_s} dA_s =$$

$$\int_0^t f_{X_s} dC_s +$$

$$\int_0^t f_{X_s} dV_s:$$

However,

R_t
 $\int_0^t 1_{f(X_s)=0} g dV_s = 0$ as dV_s is carried by $f(X_s) \neq 0$. Hence,

Z_t
 $\int_0^t 1_{f(X_s)=0} g dA_s =$

Z_t
 $\int_0^t 1_{f(X_s)=0} g dC_s:$

Furthermore,

R_t
 $\int_0^t 1_{f(X_s)=0} g dC_s =$

R_t
 $\int_0^t 1_{f(X_s)=0} g dC_s$ because C is continuous. Therefore,

Z_t
 $\int_0^t 1_{f(X_s)=0} g dA_s =$

Z_t
 $\int_0^t 1_{f(X_s)=0} g dC_s = 0:$

Consequently, X is an element of class (\underline{r}) . This completes the proof.

Now, we explore some general properties of the stochastic processes of the class (\underline{g}) . Hence, we begin by deriving the properties using the balayage formulas:

Lemma 3.4. Let X be a process of class (\underline{g}) , and let $t = \sup\{s : X_s = 0\}$. Then, for any bounded predictable process k , $k \cdot X$ is also an element of class (\underline{g}) .

Proof. By applying the balayage formula for the càdlàg case, we obtain the following:

$k_t X_t = k_0 X_0 +$

Z_t
 $\int_0^t k_s dX_s =$

Z_t
 $\int_0^t k_s dM_s +$

Z_t
 $\int_0^t k_s dC_s +$

Z_t
 $\int_0^t k_s dV_s:$

It is clear that

R_{\cdot}
 $\int_0^{\cdot} k_s dM_s$ is a local martingale; furthermore, $k_t dC_t$ is carried by $f(t, X_t) = 0$ and $k_t dV_t$ is carried by $f(t, X_t) = 0$. This completes the proof.

Corollary 3.5. Let $X = M + C + V = M + A$ be a process of class (\underline{g}) , and let f be a bounded Borel function. Then, the process $(f(C_t)X_t : t \leq 0)$ is an element of the class (\underline{g}) , and its finite variation part is defined by $\int_0^t f(C_s) dA_s$.

$\int_0^t f(C_s) dA_s$

$t =$
 R_t
 $\int_0^t f(C_s) d(C_s + V_s).$

Proof. According to Lemma 3.4, $(f(C_t)X_t : t \leq 0)$ is an element of the class (\underline{g}) . Furthermore, we have $\int_0^t f(C_s) dM_s =$

R_t
 $\int_0^t f(C_s) dM_s +$

R_t
 $\int_0^t f(C_s) dA_s$. As dC_t is carried by $f(X_t) = 0$, we have $\int_0^t f(C_s) dC_s = 0$. Consequently, $\int_0^t f(C_s) dA_s =$

R_t
 $\int_0^t f(C_s) dM_s +$

R_t
 $\int_0^t f(C_s) dA_s$. This completes the proof.

Corollary 3.6. Let $X = M+C+V$ be a positive process of the class (\underline{g}) . Then, there exist a càdlàg non-decreasing predictable process \square satisfying $\text{Supp}(d\square_t) \subseteq f(X_t) = 0$ and a positive submartingale $W = m + I$ with $W_0 = 1$; the measure dI_t is carried by $f(X_t) = 0$ such that $\int_0^t \square_s dW_s = 1$.

Proof. As the function f , defined by $f(x) = e^{-\lambda x}$, is a bounded Borel function on $[0; +1[$, it follows from Corollary 3.5 that $f(C_t)X_t \square$

$$\begin{aligned} & \int_0^t f(C_s) dC_s = \\ & \int_0^t f(C_s) dM_s + \\ & \int_0^t f(C_s) dV_s. \end{aligned}$$

Hence, we obtain that, $\forall t \geq 0$,

$$\begin{aligned} & e^{-\lambda C_t} (X_t + 1) \square 1 = \\ & \int_0^t e^{-\lambda C_s} dM_s + \\ & \int_0^t e^{-\lambda C_s} dV_s. \end{aligned}$$

Therefore, considering $W_t = 1 +$

$$\begin{aligned} & \int_0^t e^{-\lambda C_s} dM_s + \\ & \int_0^t e^{-\lambda C_s} dV_s, \end{aligned}$$

we get $e^{-\lambda C_t} (X_t + 1) = W_t$: (3.1)

Consequently, $X_t = \square_t W_t \square 1$, where $\square_t = e^{C_t}$. It is evident from (3.1) that W is a positive submartingale with $W_0 = 1$, and its non-decreasing part $L_t =$

$$\begin{aligned} & \int_0^t e^{-\lambda C_s} dV_s \text{ is such that } \text{Supp}(dL_t) \square \\ & fX_t \square = 0g. \end{aligned}$$

Now, we study the negative and positive parts of the stochastic processes of the class (\underline{g}) .

Lemma 3.7. Let $X = M + A = M + C + V$ be a process of class (\underline{g}) . The following hold:

1. If C is a non-decreasing process, then X_+ is a local submartingale.
2. If C is a decreasing process, then X_- is a local submartingale.
3. If C has no negative jump and

$$\int_0^t 1_{fX_s \leq 0} g dC_s = 0,$$

then X_+ is a local submartingale.

4. If C has no positive jump and

$$\int_0^t 1_{fX_s \leq 0} g dC_s = 0,$$

then X_- is a local submartingale.

Proof. From Tanaka's formula, we have

$$\begin{aligned} & X_+ \\ & = \\ & \int_0^t 1_{fX_s > 0} g dX_s + \\ & X \\ & \int_0^{s-} 1_{fX_s \leq 0} g dX_+ \\ & + \\ & X \\ & \int_0^{s-} 1_{fX_s > 0} g dX_- \\ & + \\ & 1 \\ & 2 \\ & L_{0t} \\ & : \end{aligned}$$

However,

$$\begin{aligned} & \int_0^t 1_{fX_s > 0} g dX_s = \\ & \int_0^t 1_{fX_s \leq 0} g dM_s + \end{aligned}$$

$$\begin{aligned}
& R_t \\
& \int_0^t 1_{fX_s \geq 0} g dC_s + \\
& R_t \\
& \int_0^t 1_{fX_s \geq 0} g dV_s. \text{ Hence,} \\
& Z_t \\
& \int_0^t 1_{fX_s \geq 0} g dX_s = \\
& Z_t \\
& \int_0^t 1_{fX_s \geq 0} g dM_s + \\
& Z_t \\
& \int_0^t 1_{fX_s \geq 0} g dC_s
\end{aligned}$$

as
 $R_t \int_0^t 1_{fX_s \geq 0} g dV_s = 0$; this is because dV_t is carried by $fX_t = 0g$. Then,

$$\begin{aligned}
& X_+ \\
& t = \\
& Z_t \\
& \int_0^t 1_{fX_s \geq 0} g dM_s + \\
& Z_t \\
& \int_0^t 1_{fX_s \geq 0} g dC_s + \\
& X \\
& \int_{0 \leq s < t} 1_{fX_s < 0} g X_+ \\
& s + \\
& X \\
& \int_{0 \leq s < t} 1_{fX_s > 0} g X_- \\
& s + \\
& 1 \\
& 2 \\
& L_{0t} \\
& : (3.2)
\end{aligned}$$

Thus, we have the following:

1. We first remark that

$$\begin{aligned}
& 0 \\
& @ \\
& Z_t \\
& \int_0^t 1_{fX_s \geq 0} g dC_s + \\
& X \\
& \int_{0 \leq s < t} 1_{fX_s < 0} g X_+ \\
& s + \\
& X \\
& \int_{0 \leq s < t} 1_{fX_s > 0} g X_- \\
& s + \\
& 1 \\
& 2 \\
& L_{0t} \\
& ; t_0 \\
& 1 \\
& A
\end{aligned}$$

is an increasing process that vanishes at zero, as C is a non-decreasing process. Furthermore, M and

$R_t \int_0^t 1_{fX_s \geq 0} g dM_s$ are local martingales. Then, X_+ is a local submartingale.

2. Now, for any process of the class (\underline{g}) , $\int_0^t X_-$ is again an element of the class (\underline{g}) . Therefore, it follows that $X_- = (\int_0^t X_-)_+$ is a local submartingale when the process C decreases.

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 3. We obtain the following from identity (3.2):

X_+
 $t =$
 Z_t
 0
 $1_{X_{s-} > 0} dM_s +$
 X
 $0 < s < t$
 $1_{X_{s-} > 0} dC_s +$
 X
 $0 < s < t$
 $1_{X_{s-} > 0} dX_s +$
 $s +$
 X
 $0 < s < t$
 $1_{X_{s-} > 0} dX_s -$
 $s +$
 1
 2
 $L_0 t$
 $;$
 as
 Z_t
 0
 $1_{X_{s-} > 0} dC_s =$
 Z_t
 0
 $1_{X_{s-} > 0} dC_s -$
 $s +$
 X
 $0 < s < t$
 $1_{X_{s-} > 0} dC_s$ and
 Z_t
 0
 $1_{X_{s-} > 0} dC_s -$
 $s =$
 Z_t
 0
 $1_{X_{s-} > 0} dC_s$
 $s = 0:$
Hence, 0
@
 X
 $0 < s < t$
 $1_{X_{s-} > 0} dC_s +$
 X
 $0 < s < t$
 $1_{X_{s-} > 0} dX_s +$
 $s +$
 X
 $0 < s < t$
 $1_{X_{s-} > 0} dX_s -$
 $s +$
 1
 2
 $L_0 t$
 $;$ $t = 0$
 1
A

is an increasing process because C has no negative jump. Consequently, X_+ is a local submartingale.

Remark 3.1. A direct consequence is that any non-negative stochastic process of class (\underline{g}) satisfying the assumptions of Lemma 3.7 is a submartingale.

Lemma 3.8. Let X be a process of class (\underline{g}) . Hence, X_+ and X_- are stochastic processes of class (\underline{g}) .

Proof. Based on Tanaka's formula, we have

X_+

$t =$

Z_t

0

$1_{f_{X_{s \leq t}} > 0} g dX_s +$

X

$0 < s \leq t$

$1_{f_{X_{s \leq t}} > 0} g X_s +$

$s +$

X

$0 < s \leq t$

$1_{f_{X_{s \leq t}} > 0} g X_s$

$s +$

1

2

L_{0t}

$:$

However,

Z_t

0

$1_{f_{X_{s \leq t}} > 0} g dX_s =$

Z_t

0

$1_{f_{X_{s \leq t}} > 0} g dM_s +$

Z_t

0

$1_{f_{X_{s \leq t}} > 0} g dC_s +$

Z_t

0

$1_{f_{X_{s \leq t}} > 0} g dV_s =$

Z_t

0

$1_{f_{X_{s \leq t}} > 0} g dM_s +$

Z_t

0

$1_{f_{X_{s \leq t}} > 0} g dC_s;$

as dV_t is carried by $f_{X_{t \leq t}} = 0g$. Hence,

X_+

$t =$

Z_t

0

$1_{f_{X_{s \leq t}} > 0} g dM_s +$

Z_t

0

$1_{f_{X_{s \leq t}} > 0} g dC_s +$

X

$0 < s \leq t$

$1_{f_{X_{s \leq t}} > 0} g X_s +$

$s +$

X

$0 < s \leq t$

$1_{f_{X_{s \leq t}} > 0} g X_s$

$s +$

1

2

L_{0t}

$:$

(3.3)

Now, let us set $Y_t =$

P

$0 < s \leq t$

$1_{f_{X_{s \leq t}} > 0} g X_s +$

s and $Z_t =$

P

$0 < s \leq t$

$1_{f_{X_{s \leq t}} > 0} g X_s$

s . As M and

R_+

0

$1_{f_{X_{s \leq t}} > 0} g dM_s$

are local martingales and $C + V$ is a càdlàg, there exists a sequence of stopping times $(T_n; n \geq 2, N)$

increasing to 1, such that

$$E[(X_{T_n})_+] = E[(M_{T_n} + C_{T_n} + V_{T_n})_+] < 1 \text{ and } E$$

$$\int_0^{T_n} 1_{X_{s-} > 0} g dM_s$$

$n \in \mathbb{N}$:

It follows from Equation (3.3) that $E[Y_{T_n}] = E$

$$\int_0^{T_n} 1_{X_{s-} > 0} g dC_s$$

$$\int_0^{T_n} 1_{X_{s-} > 0} g dC_s$$

< 1 and

$$E[Z_{T_n}] = E$$

$$\int_0^{T_n} 1_{X_{s-} > 0} g dC_s$$

$$\int_0^{T_n} 1_{X_{s-} > 0} g dC_s$$

< 1

for all $n \in \mathbb{N}$. Thus, based on Theorem VI.80 of [11], there exist right continuous increasing predictable processes V_Y and V_Z such that $Y = V_Y$ and $Z = V_Z$ are local martingales vanishing at zero.

Moreover, there exists a sequence of stopping times $(R_n; n \in \mathbb{N})$ increasing to 1, such that

$$E \int_0^{R_n} 1_{X_{s-} > 0} g dV_Y$$

$= E$

$$\int_0^{R_n} 1_{X_{s-} > 0} g d(V_Y + Y_s) +$$

$$\int_0^{R_n} 1_{X_{s-} > 0} g dY_s$$

\vdots

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As

$$\int_0^{R_n} 1_{X_{s-} > 0} g d(V_Y + Y_s)$$

$$E \int_0^{R_n} 1_{X_{s-} > 0} g dV_Y$$

$= E$

$$\int_0^{R_n} 1_{X_{s-} > 0} g dY_s$$

\vdots

Therefore,

E

$$\int_0^Z t^{Rn}$$

$$1fX+$$

$$s=6=0gdV_Y$$

$$= E$$

2

4

X

$$0<s_t^{Rn}$$

$$1fX+$$

$$s=6=0g1fXs=0gX+$$

3

$$5 = E$$

2

4

X

$$0<s_t^{Rn}$$

$$1fX+$$

$$s=6=0g1fX+$$

$$s=0gX+$$

3

$$5 = 0:$$

In other words,

R t

$$0 1fX+$$

$$s=6=0gdV_Y$$

s = 0. Then, dV_Y

t is carried by fX+

t = 0g. However, we have

E

$$\int_0^Z t^{Rn}$$

$$1fX+$$

$$s=6=0gdV_Z$$

s

$$= E$$

$$\int_0^Z t^{Rn}$$

$$1fX+$$

$$s=6=0gd(V_Z$$

$$s \square Z_s) +$$

$$\int_0^Z t^{Rn}$$

$$1fX+$$

$$s=6=0gdBs$$

$$= E$$

$$\int_0^Z t^{Rn}$$

$$1fX+$$

$$s=6=0gadzs$$

$$= E$$

$$\int_0^Z t^{Rn}$$

$$1fX+$$

$$s=6=0gadzs$$

Hence,

E

$$\int_0^Z t^{Rn}$$

$$1fX+$$

$$s=6=0gdV_Z$$

s

$$\begin{aligned}
&= E \\
&2 \\
&4 \\
&X \\
&0 < s < t \wedge R_n \\
&1_{fX+} \\
&s \int_0^s 1_{fX_s > 0} g X_s \\
&3 \\
&5 = E \\
&2 \\
&4 \\
&X \\
&0 < s < t \wedge R_n \\
&1_{fX_s > 0} g 1_{fX_s > 0} X_s \\
&3
\end{aligned}$$

5 :
This entails that

$$\begin{aligned}
&E \\
&\int_0^t Z_s \\
&1_{fX+} \\
&s \int_0^s dV Z \\
&3
\end{aligned}$$

= 0;
since $1_{fX_s > 0} X_s$
= 0. This shows that

$$\begin{aligned}
&R_t \\
&0 1_{fX+} \\
&s \int_0^s dV Z \\
&s = 0. \text{ Therefore, } dV Z \\
&t \text{ is carried by } f_t
\end{aligned}$$

0; X_+
 $t = 0g$. Consequently, we determine that

$$\begin{aligned}
&X_+ \\
&t = \\
&\int_0^t Z_s \\
&1_{fX_s > 0} g dM_s + (Y_t \square V Y \\
&t) + (Z_t \square V Z \\
&t)
\end{aligned}$$

$$\begin{aligned}
&+ V Y \\
&t +
\end{aligned}$$

$$\begin{aligned}
&\int_0^t Z_s \\
&t + \\
&Z_t \\
&0 \\
&1_{fX_s > 0} g dC_s + \\
&1 \\
&2 \\
&L_{0t}
\end{aligned}$$

is a stochastic process of the class (\underline{g}) . This is also true for X_+ as $(\square X)$ is also from class (\underline{g}) . It is well known that M_+ and M_+ are stochastic processes of class (\underline{g}) when M is a continuous local martingale. The next corollary of Lemma 3.8 shows that M_+ and M_+ are elements of class (\underline{g}) when M is a càdlàg local martingale.

Corollary 3.9. Let M be a càdlàg local martingale vanishing at zero. Then, the processes M_+ and M_+ are elements of class (\underline{g}) .

Now, we show that the product of the processes of class (\underline{g}) with vanishing quadratic covariations is again of class (\underline{g}) .

Lemma 3.10. Let $(X_1$

$t)_{t=0}; \dots; (X_n$

$t)_{t=0}$ be processes of class (\mathcal{G}) , such that $[X_i; X_j] = 0$ for $i \neq j$.

Then, $(\sum_{i=1}^n X_i$

$t)_{t=0}$ is also of class (\mathcal{G}) .

Proof. As $[X_1; X_2] = 0$, integration by parts yields

X_1

$t)_{t=0}$

$=$

\int_0^t

X_1

$\int_0^t dX_2$

$+$

\int_0^t

X_2

$\int_0^t dX_1$

$;$

In other words,

X_1

$t)_{t=0}$

$=$

\int_0^t

X_1

$\int_0^t dM_2$

$+$

\int_0^t

X_2

$\int_0^t dM_1$

$;$

$+$

\int_0^t

X_1

$\int_0^t dC_2$

$+$

\int_0^t

X_2

$\int_0^t dC_1$

$;$

$+$

\int_0^t

X_1

$\int_0^t dV_2$

$+$

\int_0^t

X_2

$\int_0^t dV_1$

$;$

$+$

\int_0^t

X_1

$\int_0^t dV_1$

$;$

$+$

\int_0^t

X_1

$\int_0^t dM_2$

$;$

$+$

\int_0^t

X_1

$;$

$+$

\int_0^t

X_1

$\int_0^t dM_2$

$;$

$+$

\int_0^t

X_1

It can be observed that $M_t =$

R_t

\int_0^t

X_1

$\int_0^t dM_2$

$;$

$+$

\int_0^t

X_1

$\int_0^t dM_2$

$;$

$+$

\int_0^t

X_1

R_t

$\int_0^t X_2$

$\int_0^t dM_1$

s is a càdlàg local martingale. Furthermore,

the process $C_t =$

R_t

$\int_0^t X_1$

$\int_0^t dC_2$

$s +$

R_t

$\int_0^t X_2$

$\int_0^t dC_1$

s is a finite variation process, such that

$dC_t = X_1$

$\int_0^t dC_2$

$\int_0^t X_2$

$\int_0^t dC_1$

t

is carried by $\mathcal{F}_{t-0} : X_1$

$\int_0^t X_2$

$\int_0^t = 0$ g. By contrast, $V_t =$

R_t

$\int_0^t X_1$

$\int_0^t dV_2$

$s +$

R_t

$\int_0^t X_2$

$\int_0^t dV_1$

s is a finite variation

process, such that

$dV_t = X_1$

$\int_0^t dV_2$

$\int_0^t X_2$

$\int_0^t dV_1$

t

is carried by $\mathcal{F}_{t-0} : X_1$

$\int_0^t X_2$

$\int_0^t = 0$ g. Therefore, $X_1 X_2$ is of class (\underline{g}) . If $n \geq 3$, and $[X_1 X_2; X_3] = 0$.

Thus, we obtain the result by induction.

Definition 3.1. A stochastic process X is said of class D if

$\mathcal{F}_{\tau-0} : \tau < 1$ is a stopping timeg is uniformly integrable.

Theorem 3.11. Let $X = M + C + V$ be a process of class (\underline{g}) and an element of class D. Then,

there exists a random variable X_1 such that

\lim

$\int_0^T X_t = X_1$

, and for every stopping time $T < 1$, we have

$X_T = E$

$\int_0^T X_{1 \mathcal{F}_{t-0} < T} \mathcal{F}_T$

; (3.4)

where $g = \sup \mathcal{F}_{t-0} : X_t X_{t-0} = 0$ g.

Proof. Let us substitute $t = \inf \{s > t-0 : X_s X_{s-0} = 0$ g. It is evident that t is the stopping time.

Furthermore,

$X_{1 \mathcal{F}_{t-0} < T} = X_T = M_T + C_T + V_T$:

However, $C_T = C_T$ and $V_T = V_T$ as dC and dV are carried by $\mathcal{F}_{t-0} : X_t = 0$ g and $\mathcal{F}_{t-0} :$

$X_{t-0} = 0$ g, respectively; further, $g = \sup \mathcal{F}_{t-0} : X_t = 0$ g $\sup \mathcal{F}_{t-0} : X_{t-0} = 0$ g. This entails that

$X_{1 \mathcal{F}_{t-0} < T} = X_T = M_T + C_T + V_T$:

Hence,

E

$\int_0^T X_{1 \mathcal{F}_{t-0} < T} \mathcal{F}_T$

$$E[M_T | \mathcal{F}_T] + C_T + V_T$$

Therefore,

$$X_T = E$$

$$E[1_{fg < Tg} | \mathcal{F}_T]$$

as M is a uniformly integrable martingale.

Corollary 3.12. Let M be a non-negative càdlàg uniformly integrable martingale such that $M_0 > 0$ and $\lim_{t \rightarrow \infty} M_t = 0$. Let us consider $k > 0$. Then,

$$P(g_k < T | \mathcal{F}_t) = 1 - \frac{M_t}{k}$$

$$\frac{M_t}{k}$$

k

;

where $g_k = \sup_{t \geq 0} : M_t \leq k$ or $M_{t \wedge g_k}$

Proof. It follows from Theorem 2.4 that

$$(k - M_t)^+ = E$$

$$E[1_{fg_k < Tg} | \mathcal{F}_t]$$

$$= kE$$

$$E[1_{fg_k < Tg} | \mathcal{F}_t]$$

;

Hence,

$$(k - M_t)^+ = kP(g_k < T | \mathcal{F}_t)$$

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Consequently,

$$P(g_k < T | \mathcal{F}_t) =$$

$$1 - \frac{M_t}{k}$$

$$\frac{M_t}{k}$$

k

;

;

Therefore,

$$P(g_k < T | \mathcal{F}_t) = 1 - \frac{M_t}{k}$$

$$1 - \frac{M_t}{k}$$

$$\frac{M_t}{k}$$

k

;

$$= 1 - \frac{M_t}{k}$$

$$\frac{M_t}{k}$$

k

;

;

This completes the proof.

3.2 Extension of characterization martingale

Lemma 3.13. Let $X = M + A$ be a process of class (\underline{g}) , where $A = C + V$ and A_c denote the continuous part of A . Then, for every C^1 function f and

$$F(x) =$$

$$R_x$$

of $f(z)dz$, the process

$$0$$

$$@F(A_{ct})$$

$$) - f(A_{ct})$$

$$)X_t +$$

$$X_{s,t} = \int_0^t f(A_c) dX_s + \int_0^t f(A_c) dC_s +$$

$$X_{s,t} = \int_0^t f(A_c) dX_s + \int_0^t f(A_c) dV_s; t \geq 0$$

1
A
is a local martingale.

Proof. Integration by parts yields

$$f(A_{ct}) X_t = \int_0^t f(A_c) dX_s + \int_0^t f(A_c) X_{s-} dA_c$$

s:
In other words,

$$f(A_{ct}) X_t = \int_0^t f(A_c) dX_s + \int_0^t f(A_c) X_{s-} dC_c$$

$$+ \int_0^t f(A_c) X_{s-} dV_c$$

since $A_c = C_c + V_c$. Furthermore,

$$\int_0^t f(A_c) X_{s-} dC_c = \int_0^t f(A_c) X_s dC_c$$

because C_c is continuous. Therefore, we obtain
 $f(A_{ct})$

$$\begin{aligned}
 & \int_0^t X_t = \\
 & \int_0^t f(A_c) \\
 & s) dX_s + \\
 & \int_0^t f \\
 & 0 \\
 & (A_c) \\
 & s) X_s dC_c \\
 & s + \\
 & \int_0^t f \\
 & 0 \\
 & (A_c) \\
 & s) X_s dV_c \\
 & s :
 \end{aligned}$$

This entails the following:

$$\begin{aligned}
 & f(A_{ct}) \\
 & \int_0^t X_t = \\
 & \int_0^t f(A_c) \\
 & s) dX_s + \\
 & 2 \\
 & 4 \\
 & \int_0^t f \\
 & 0 \\
 & (A_c) \\
 & s) X_s dC_s \square \\
 & X \\
 & s_t \\
 & f \\
 & 0 \\
 & (A_c) \\
 & s) X_s dC_s \\
 & 3 \\
 & 5+ \\
 & 2 \\
 & 4 \\
 & \int_0^t f \\
 & 0 \\
 & (A_c) \\
 & s) X_s dV_s \square \\
 & X \\
 & s_t \\
 & f \\
 & 0 \\
 & (A_c) \\
 & s) X_s dV_s \square \\
 & 3
 \end{aligned}$$

5:
Thus, it follows that

$$\begin{aligned}
 & f(A_{ct}) X_t = \\
 & \int_0^t f(A_c) \\
 & s) dX_s \square \\
 & X \\
 & s_t \\
 & f \\
 & 0 \\
 & (A_c)
 \end{aligned}$$

$$s)X_{s,C_s} \square$$

X

s_t

f

0

(A_c

$$s)X_{s,V_s}$$

as dC and dV are carried by $f_{t,0} : X_t = 0g$ and $f_{t,0} : X_{t,0} = 0g$, respectively. This entails that

$$f(A_{ct})$$

$$)X_t =$$

Z_t

0

$$f(A_c$$

$$s)dM_{s+}$$

Z_t

0

$$f(A_c$$

$$s)dA_c$$

s+

X

s_t

h

$$f(A_c$$

$$s) \square f$$

0

$$(A_c$$

$$s)X_s$$

i

$$_C_{s+}$$

X

s_t

h

$$f(A_c$$

$$s) \square f$$

0

$$(A_c$$

$$s)X_{s,0}$$

i

$$_V_{s,0}$$

$$13$$

Consequently,

$$F(A_{ct})$$

$$\square f(A_{ct})$$

$$)X_{t+}$$

X

s_t

h

$$f(A_c$$

$$s) \square f$$

0

$$(A_c$$

$$s)X_s$$

i

$$_C_{s+}$$

X

s_t

h

$$f(A_c$$

$$s) \square f$$

0

$$(A_c$$

$$s)X_{s,0}$$

i

$$_V_{s,0} = \square$$

Z_t

0

$$f(A_c$$

s) dMs:

In other words,

0

@F(A_{ct}

) □ f(A_{ct}

)X_t +

X

s,t

[f(A_c

s) □ f

0

(A_c

s)X_s]_C_s +

X

s,t

[f(A_c

s) □ f

0

(A_c

s)X_s]_V_s; t _ 0

1

A

is a local martingale.

Theorem 3.14. Let $X = M + A$ be a positive semi-martingale. Then, the following are equivalent:

1. $X \in \mathcal{C}_2$;

2. There exists two càdlàg and non-decreasing predictable processes V and C such that, for

$W = C + V$ and for any $F \in \mathcal{C}_2$, the process

0

@F(W_c

t) □ F

0

(W_c

t)X_t +

X

s,t

[F

0

(W_c

s) □ F

00

(W_c

s)X_s]_C_s +

X

s,t

[F

0

(W_c

s) □ F

00

(W_c

s)X_s]_V_s; t _ 0

1

A

is a càdlàg local martingale and $W = A$.

Proof. (1) (2) Let us consider $W = A$. Hence, from Lemma 3.13, we obtain

0

@F(A_{ct}

) □ F

0

(A_{ct}

)X_t +

X

s,t

[F

0

(A_c

s) □ F

00

$$\begin{aligned}
 & (A_c) \\
 & \int_0^t X_s dC_s + \\
 & X_{s,t} \\
 & [F \\
 & 0 \\
 & (A_c) \\
 & \int_0^t F \\
 & 0 \\
 & (A_c) \\
 & \int_0^t X_s dV_s; t \geq 0 \\
 & 1 \\
 & A
 \end{aligned}$$

is a càdlàg local martingale.

(2) (1) First, let $F(x) = x$. Then, the process W defined by

$$\begin{aligned}
 & W \\
 & 0 \\
 & t = W_c \\
 & t + \\
 & X \\
 & \int_0^t C_s + \\
 & X \\
 & \int_0^t V_s dX_t = W_t dX_t
 \end{aligned}$$

is a local martingale. Hence, owing to the uniqueness of the Doob–Meyer decomposition, we obtain

$W = A$. Next, we consider $F(x) = x^2$. Then, the process B defined by

$$\begin{aligned}
 & B_t = (W_c \\
 & t) d \int_0^t 2W_c \\
 & t X_t + 2 \\
 & X \\
 & \int_0^t W_c \\
 & \int_0^t C_s + 2 \\
 & X \\
 & \int_0^t W_c \\
 & \int_0^t V_s dX_t + 2 \\
 & X \\
 & \int_0^t X_s dC_s + 2 \\
 & X \\
 & \int_0^t X_s dV_s
 \end{aligned}$$

is a local martingale. However, through integration by part, it follows that

$$\begin{aligned}
 & B_t = 2 \\
 & Z_t \\
 & 0 \\
 & W_c \\
 & \int_0^t dW_c \\
 & \int_0^t 2 \\
 & Z_t \\
 & 0 \\
 & W_c \\
 & \int_0^t dX_s + 2 \\
 & Z_t \\
 & 0 \\
 & X_s dW_c \\
 & \int_0^t s + 2 \\
 & X \\
 & \int_0^t W_c \\
 & \int_0^t C_s + 2 \\
 & X \\
 & \int_0^t W_c
 \end{aligned}$$

$s_{-}V_s \square 2$
 X
 $s_{-}t$
 $X_s_{-}C_s \square 2$
 X
 $s_{-}t$
 $X_s \square_{-}V_s$
 $= 2$
 Z^t
 0
 W_c
 $s dW_c$
 $s \square 2$
 Z^t
 0
 W_c
 $s dX_s \square 2$
 Z^t
 0
 $X_s dC_c$
 $s \square 2$
 Z^t
 0
 $X_s dV_c$
 $s + 2$
 X
 $s_{-}t$
 W_c
 $s_{-}C_s + 2$
 X
 $s_{-}t$
 W_c
 $s_{-}V_s \square 2$
 X
 $s_{-}t$
 $X_s_{-}C_s \square 2$
 X
 $s_{-}t$
 $X_s \square_{-}V_s$
 $= 2$
 Z^t
 0
 W_c
 $s dW_c$
 $s \square 2$
 Z^t
 0
 W_c
 $s dX_s \square 2$
 Z^t
 0
 $X_s dC_c$
 $s \square 2$
 Z^t
 0
 $X_s dV_c$
 $s + 2$
 X
 $s_{-}t$
 W_c
 $s_{-}C_s + 2$
 X
 $s_{-}t$
 W_c
 $s_{-}V_s \square 2$
 X
 $s_{-}t$
 $X_s_{-}C_s \square 2$

$X_{s,t}$
 $X_{s,t} - V_s$
 $= 2$
 Z_t
 0
 W_c
 $s d$
 0
 $@W_c$
 $s +$
 $X_{u,s}$
 $- C_u +$
 $X_{u,s}$
 $- V_u \square X_s$
 1
 $A \square 2$
 Z_t
 0
 $X_{s,d}$
 0
 $@C_c$
 $s +$
 $X_{u,s}$
 $- C_u$
 1
 $A \square 2$
 Z_t
 0
 $X_{s,d}$
 0
 $@V_c$
 $s +$
 $X_{u,s}$
 $- V_u$
 1
 A
 14
 $= 2$
 Z_t
 0
 W_c
 $s d W$
 0
 $s \square 2$
 Z_t
 0
 $X_{s,d} C_s \square 2$
 Z_t
 0
 $X_{s,d} V_s:$
 Consequently, we must have
 R_t
 $0 X_{s,d} C_s +$
 R_t
 $0 X_{s,d} V_s = 0$. Hence, we determine that
 Z_t
 0
 $X_{s,d} C_s =$
 Z_t
 0
 $X_{s,d} V_s = 0;$
 as

R_t
 $\int_0^t X_s dC_s$ and

R_t
 $\int_0^t X_s dV_s$ are non-negative. In other words, dA is carried by the set τ_0 :
 $X_t X_{\tau_0} = 0$.

Corollary 3.15. Let $X = M + A$ be a positive semi-martingale. Then, the following are equivalent:

1. $X \in \mathcal{C}_2$;
2. There exists a non-decreasing predictable process V such that, for any $F \in \mathcal{C}_2$, the process

$$\int_0^t F(V_c) d(V_c + X_t) - \int_0^t F(V_c) dV_s; t \geq 0$$

is a càdlàg local martingale and $V \leq A$.

Corollary 3.16. Let $X = M + A$ be a positive semi-martingale. Then, the following are equivalent:

1. $X \in \mathcal{C}_1$;
2. There exists a non-decreasing predictable process V such that, for any $F \in \mathcal{C}_2$, the process

$$\int_0^t F(V_c) d(V_c + X_t) - \int_0^t F(V_c) dV_s; t \geq 0$$

is a càdlàg local martingale and $V \leq A$.

Corollary 3.17. Let $X = M + A$ be a positive semi-martingale. Then, the following are equivalent:

1. $X \in \mathcal{C}_1$ (in sense of Nikeghbali [19]);
2. There exists a non-decreasing predictable process V such that for any $F \in \mathcal{C}_2$, the process

$$\int_0^t F(V_t) d(V_t + X_t); t \geq 0$$

is a càdlàg local martingale and $V \leq A$.

4 Conclusion

This study contributes to the development of a general framework and techniques for dealing with stochastic processes of classes which extend the notion of class (\mathcal{C}_1) to càdlàg semimartingales, whose the finite variational part is also càdlàg instead of continuous. First, we have proposed a new method to characterize stochastic processes of the class (\mathcal{C}_1) . Second, we have unified the study of the class (\mathcal{C}_1) and semimartingales of class (\mathcal{C}_1) , whose the finite variational part is càdlàg by studying a new larger family that we term class (\mathcal{C}_g) . Hence, we have explored some general and structural properties. We have also presented a result that enables the recovery of any process of class (\mathcal{C}_g) from its final value X_1 and of an honest time g , which is the last time $(X_t; t \leq 0)$ or

$(X_{t_0} : t_0 = 0)$ visited the origin. More precisely, this formula has the following form:
 $X_t = E$

$\bar{X}_{11} f_{g_{t_0}} F_t$

;

where X is the process of class (\underline{g}) , $X_1 = \lim_{t \rightarrow 1} X_t$, and $g = \sup_{t \geq 0} X_t X_{t_0} = 0g$. Finally, we derived some characterization results which generalize those already existing for the classes (\underline{g}) and (\underline{r}) .

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Declaration of competing interest

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