Original Research Article

Juchez Probability Distribution: Properties and Applications

ABSTRACT

A novel distribution called the Juchez distribution is proposed and studied. This distribution is composite of both exponential and gamma distributions. The properties and features of this distribution are studied, with empirical emphasis: on the inequality relationship within the measures of central tendency, and the coefficient of variation. The model parameter was estimated using the method of maximum likelihood, where the asymptotic and consistent properties are numerically studied as well. The flexibility of this distribution is shown, through an application to a facebook Live-Streaming data set. This distribution shows a high efficiency when compared with other one parameter distributions.

Keywords: Juchez distribution, Coefficient of variation, Live-streaming data, Mixture distribution, Performance comparison.

1. INTRODUCTION

The modeling and analysis of lifetime data is an important aspect of statistical work in a wide variety of scientific and technological fields. The field of lifetime data analysis has grown and expanded rapidly with respect to methodology, theory and application. In the context of modeling the real life phenomena, continuous probability distributions and transformation methods have been proposed. Data help us to keep track of the events happening around us; and they reveal patterns and behaviors of outcomes. Lifetime data simply refers to such data whose events exhibit some propensity of failure after a measurable amount of valid cycle or success and vice versa. It is generally accepted that lifetimes of individuals, components, systems, etc. are not deterministic, and hence amenable only to probabilistic and statistical laws (Deshpande, 2015). The conception of models to handle such random variables, took place in twentieth century, and it developed into two main similar subjects: the *reliability theory*, which is concerned with modeling lifetimes for components and systems, in the engineering and industrial fields. At the other hand, the *survival analysis* is the term used in biological field.

In probability distribution theory, flexibility and tractability are given a great deal of preference in modeling lifetime data. The tractability of a probability distribution may be useful in theory because such distribution would be easy to work with; especially when it comes to simulation of random samples, but its flexibility could be of interest industrially (Oguntunde et al 2016). The implication of this is that while we appreciate the existence of tractable distributions for class use, more complex ones would also be developed for field application.

Statistically, data are transformed to satisfy some assumptions, before we can model them. However it is preferable to use probability distributions that best fit the available data set than to transform them; as this may affect the originality of the data set. Consequently, several efforts have been made in recent years to ensure development of new distributions and their extensions; owing to the increasing number of data from different fields of interest (Merovci, 2013).

Lindsay (1995) provided the model which can be employed for development of new distributions from the parent k-distributions. He stated that these k-distributions can be from different families of distributions to form the new. However, this makes the problem very complex and sometimes useless; therefore, most mixtures are preferably taken from one family of distribution (e.g., all normal distributions) but with different parameters and/or distributions that share same support or range.

In probability and statistics, a compound distribution is the probability distribution that results from assuming that a random variable is distributed according to some parameterized distribution, with (some of) the parameters of that distribution themselves being random variables. Sankaran (1970) introduced compounding in poison-lindley distribution. Mc Nolty et al. (1980) used compound method to derive a mixture distribution named type I exponential mixture.

Jasiulewicz and Kordecki (2003) explored the convolutions of Erlang and of Pascal distributions; Cohen and Balakrishnan (1991) also did for geometric distribution. Among many, Ghitany (2007) derived the sums of independent Lindley distributions. In our view, compounding, unlike mixture method and convolution, can only take two distributions employed at once, whereas the others can take k number of distributions. This paper aims at proposing a probability distribution that will be used to model live-streaming data.

2 MODEL PROPOSITION

Juchez distribution is derived from the composition of exponential and gamma distributions with suitable mixing probabilities; where the gamma distribution is characterized by a constant scale parameter θ and two different shape parameters: $\alpha = 2$ and 4.

2.1 Exponential distribution

In probability theory and statistics, the exponential distribution is the model of the time between events in a poison process. The event is always independent and continuous at a constant average rate. It has a probability density function (pdf) defined as

$$f(x;\theta) = \begin{cases} \theta e^{-\theta x} & x \geq 0 \\ 0, & x < 0 \end{cases} \quad (1)$$

where the rate parameter $\theta > 0$, and the cumulative distribution function is given as

$$F(x;\ \theta) = \begin{cases} 1 - e^{-\theta x} & x \geq 0 \\ 0, & x < 0 \end{cases} \qquad (2)$$

2.2 Gamma Distribution

This is a two parameter family of continuous probability model: a scale parameter $\theta, (\theta > 0)$ and shape parameter $\alpha = k, k > 0$. It has a probability density function, pdf defined as

$$f(x; \theta, \alpha) = \begin{cases} \frac{x^{\alpha-1}\theta^{\alpha}e^{-\theta x}}{\Gamma(\alpha)} & x > 0 \\ 0, & x < 0 \end{cases} \qquad (3)$$

2.3 Mixture Distributions

A probability distribution function $f(x)$ is a mixture of k-component distributions, if

$$f(x) = \sum_{i=1}^{k} d_i g_i\ ;\ \text{where}\ \sum_{i=1}^{k} d_i = 1, \quad d_i > 0 \qquad (4)$$

Given that the mixture components are

$$g_1 = \text{gamma}\ (x, \theta, 1) = \theta e^{-\theta x} \qquad (5)$$

$$g_2 = \text{gamma}\ (x, \theta, 2) = \theta^2 x e^{-\theta x} \qquad (6)$$

$$g_3 = \text{gamma}\ (x, \theta, 4) = \frac{\theta^4 x^3\ e^{-\theta x}}{6} \qquad (7)$$

where d_i is the mixing probability (Friedman et al., 2009) and derived as the only weight that complements each of the mixing component for the validity of the new probability density function. However, the weights as would be used are given as

$$d_1 = \frac{\theta^3}{\theta^3+\theta^2+6}\ ,\ d_2 = \frac{\theta^2}{\theta^3+\theta^2+6}\ ,\ d_3 = \frac{6}{\theta^3+\theta^2+6} \qquad (8)$$

2.4 JUCHEZ Distribution

The JUCHEZ distribution denoted as $j(x)$ is derived by employing this mixture model for three component mixing probabilities, i.e.$k = 3$.

$$j(x, \theta) = d_1 g_1(\ x, \theta, 1) +\ d_2\ g_2(x, \theta, 2) + d_3\ g_3(x, \theta, 4) \qquad (9)$$

$$= \frac{\theta^3}{\theta^3+\theta^2+6}\left\{\theta e^{-\theta x}\right\} + \frac{\theta^2}{\theta^3+\theta^2+6}\left\{\theta^2 x\ e^{-\theta x}\right\} + \frac{6}{\theta^3+\theta^2+6}\left\{\frac{\theta^4 x^3\ e^{-\theta x}}{6}\right\}$$

$$j(x, \theta) = \frac{\theta^4}{\theta^3+\theta^2+6}\ (\ 1 + x +\ x^3\ ) e^{-\theta x}\ ,\ x > 0,\ \theta > 0 \qquad (10)$$

The Juchez Distribution is a valid distribution (properness of probability density function, pdf), that is $\int_{-\infty}^{\infty} f(x)dx = 1$, Proof

$$\int_0^{\infty} f(x)dx = \int_0^{\infty} \frac{\theta^4}{\theta^3+\theta^2+6}\ (\ 1 + x +\ x^3\ )\ e^{-\theta x}\ dx \qquad (11)$$

Given that $\int_0^\infty x^C e^{-\theta x}\, dx = \frac{\Gamma(C+1)}{\theta^{C+1}}$

$$= \int_0^\infty \frac{\theta^4}{\theta^3+\theta^2+6}\,(1 + x + x^3)e^{-\theta x}\, dx$$

$$\int_0^\infty f(x)dx = \frac{\theta^4}{\theta^3+\theta^2+6}\left[\frac{\theta^3+\theta^2+6}{\theta^4}\right] = 1 \qquad (12)$$

3 PROPERTIES OF JUCHEZ DISTRIBUTION

3.1 Cumulative Distribution Function (CDF)

The Cumulative Distribution Function (CDF) for Juchez distribution is an integral derivation of the proposed pdf in equation (10),

$$F(x) = \int_0^x f(t,\theta)dt = \int_0^x \frac{\theta^4}{\theta^3+\theta^2+6}\,(1 + x + x^3)\, e^{-\theta x}\, dt \qquad (13)$$

$$= \frac{\theta^4}{\theta^3+\theta^2+6}\left[\int_0^x e^{-\theta x}dx + \int_0^x x\, e^{-\theta x}dx + \int_0^x x^3\, e^{-\theta x}dx\right]$$

$$= \frac{\theta^4}{\theta^3+\theta^2+6}\left(\frac{\left[\theta^3+\theta^2+6-(\theta^3+\theta^2+6)e^{-\theta x}-\theta x\left(6+3\theta x+\theta^2 x^2\right)e^{-\theta x}\right]}{\theta^4}\right)$$

$$F(x,\theta) = 1 - \left(1 + \frac{\theta x\,[\,\theta^2 + \theta^2 x^2 + 3\theta x + 6\,]}{\theta^3 + \theta^2 + 6}\right) e^{-\theta x} \qquad (14)$$

3.2 Mode

The Mode of Juchez distribution is obtain through the first derivative of equation (10), and equating to zero

$$Mode = \frac{d}{dx} f(x,\theta) = 0 \qquad (15)$$

$$\frac{d}{dx}\left[\frac{\theta^4}{\theta^3+\theta^2+6}\,(1 + x + x^3)e^{-\theta x}\right] = 0$$

$$\frac{\theta^4}{\theta^3+\theta^2+6}\,(\theta x^3 - 3x^2 + \theta x + \theta - 1)e^{-\theta x} = 0 \qquad (16)$$

Solving equation (16) completely, and obtaining the positive root

$$\theta x^3 - 3x^2 + \theta x + \theta - 1 = 0$$

$$Mode = \frac{1}{\theta} + \frac{A}{3\theta} + \frac{3\sqrt[3]{2}}{\theta B} = 0 \qquad (17)$$

where $A = \sqrt[3]{\left((27 + \frac{27}{2}\theta^2 - \frac{27}{2}\theta^3 + \frac{\sqrt{(2916\theta^2 - 2916\theta^3 + 729\theta^4 - 1458\theta^5 + 729\theta^6)}}{2}\right)}$ and

$$B = \sqrt[3]{\left((54 + 27\theta^2 - 27\theta^3 + \frac{\sqrt{(2916\theta^2 - 2916\theta^3 + 729\theta^4 - 1458\theta^5 + 729\theta^6)}}{2}\right)}$$

3.3 Median

The Median of Juchez distribution is obtained by integrating equation (11),

$$\text{Median} = \int_{-\infty}^{m} f(x)dx = \frac{1}{2} \ \ or \ \ \int_{m}^{\infty} f(x)dx = \frac{1}{2} \tag{18}$$

$$\int_{0}^{m} \frac{\theta^4}{\theta^3+\theta^2+6} \left(1 + x + x^3\right) e^{-\theta x}\, dx = \frac{1}{2}$$

$$1 - \left(1 + \frac{m\theta^2 + \theta^3 m^3 + 3\theta^2 m^2 + 6\theta m}{\theta^3 + \theta^2 + 6}\right) e^{-\theta m} = \frac{1}{2}$$

$$\frac{(\theta^3 + \theta^2 + 6)}{2} - [(\theta^3 + \theta^2 + 6) + m\theta^2 + \theta^3 m^3 + 3\theta^2 m^2 + 6\theta m] e^{-\theta m} = 0 \tag{19}$$

3.4 Mean

The Mean of Juchez distribution is obtained as

$$E(X) = \mu = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

$$E(X) = \mu = \int_{0}^{\infty} x \frac{\theta^4}{\theta^3+\theta^2+6} \left(1 + x + x^3\right) e^{-\theta x}\, dx \tag{20}$$

$$= \frac{\theta^4}{\theta^3+\theta^2+6} \left[\int_{0}^{\infty} x e^{-\theta x} dx + \int_{0}^{\infty} x^2 e^{-\theta x} dx + \int_{0}^{\infty} x^4 e^{-\theta x} dx \right]$$

$$= \frac{\theta^4}{\theta^3+\theta^2+6} \left[\frac{1}{\theta^2} + \frac{2}{\theta^3} + \frac{24}{\theta^5}\right]$$

$$E(X) = \mu = \frac{\theta^3 + 2\theta^2 + 24}{\theta\, (\theta^3 + \theta^2 + 6)} \tag{21}$$

3.5 Moment Generating Function

The moment generating function of Juchez Distribution is derived as:

$$M_x(\mathrm{t}) = E\left(e^{tx}\right) = \int_{0}^{\infty} e^{tx} f(x)\, dx \tag{22}$$

$$= \int_{0}^{\infty} \mathrm{e}^{\mathrm{tx}} \frac{\theta^4}{\theta^3+\theta^2+6} \left(1 + x + x^3\right) e^{-\theta x}\, \mathrm{dx}$$

$$= \int_{0}^{\infty} \frac{\theta^4}{\theta^3+\theta^2+6} \left(1 + x + x^3\right) e^{-x(\theta - t)}\, \mathrm{dx}$$

Given that $\int_{0}^{\infty} x^m e^{-\theta x}\, \mathrm{dx} = \frac{\Gamma(m+1)}{\theta^{m+1}}$ and $(u + v)^{-n} = \sum_{r=0}^{\infty} \binom{r+a}{r} u^{-n-r} v^r$

$$= \frac{\theta^4}{\theta^3+\theta^2+6} \left[\frac{1}{\theta - t} + \frac{1}{(\theta - t)^2} + \frac{6}{(\theta - t)^4}\right]$$

$$= \frac{\theta^4}{\theta^3+\theta^2+6} \left[\frac{1}{\theta} \sum_{r=0}^{\infty} \frac{t^r}{\theta^r} + \frac{1}{\theta^2} \sum_{r=0}^{\infty} \binom{r+1}{r} \frac{t^r}{\theta^r} + \frac{6}{\theta^4} \sum_{r=0}^{\infty} \binom{r+6}{r} \frac{t^r}{\theta^r}\right.$$

$$M_x(\mathrm{t}) = \sum_{r=0}^{\infty} \left(\frac{\theta^3 + \theta^2 (r + 1)! + (r + 3)!}{\theta^r\, (\theta^3 + \theta^2 + 6)}\right) \frac{t^r}{r!} \tag{23}$$

3.6 Moment

The r^{th} moment of the Juchez distribution is obtained

$$E(x^r) = \frac{r!\, [\theta^3 + \theta^2 (r+1) + (r+1)(r+2)(r+3)]}{\theta^r\, (\theta^3 + \theta^2 + 6)} \tag{24}$$

Therefore, the first-four moments about origin of Juchez Distribution are given as:

$$\mu_1' = \frac{\theta^3 + 2\theta^2 + 24}{\theta(\theta^3 + \theta^2 + 6)} = \mu \qquad \mu_2' = \frac{2(\theta^3 + 3\theta^2 + 60)}{\theta^2(\theta^3 + \theta^2 + 6)} \tag{25}$$

$$\mu_3' = \frac{6(\theta^3 + 4\theta^2 + 120)}{\theta^3(\theta^3 + \theta^2 + 6)} \qquad \mu_4' = \frac{24(\theta^3 + 5\theta^2 + 210)}{\theta^4(\theta^3 + \theta^2 + 6)}$$

The central moment about the mean of Juchez distribution is:

$$\mu_n = E[(X - E[X])^n] = \sum_{j=0}^{n} \binom{n}{j} (-1)^{n-j} \mu_j' \mu^{n-j} \tag{26}$$

$$\mu_2 = \mu_2' - \mu^2$$

$$= \frac{\theta^6 + 4\theta^5 + 2\theta^4 + 84\theta^3 \; 60\theta^2 + 144}{\theta^2(\theta^3 + \theta^2 + 6)^2} = \sigma^2 \tag{27}$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu + 2\mu^3$$

$$= \frac{2(\theta^9 + 6\theta^8 + 6\theta^7 + 200\theta^6 + 270\theta^5 + 108\theta^4 + 324\theta^3 + 432\theta^2 + 864)}{\theta^3(\theta^3 + \theta^2 + 6)^3} \tag{28}$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu + 6\mu_2'\mu^2 - 3\mu^4$$

$$= \frac{3(3\theta^{12} + 24\theta^{11} + 44\theta^{10} + 968\theta^9 + 2336\theta^8 + 2016\theta^7 + 7488\theta^6 + 13248\theta^5 + 5760\theta^4 + 31104\theta^3 + 24192\theta^2 + 31104)}{\theta^4(\theta^3 + \theta^2 + 6)^4} \tag{29}$$

3.7 Coefficient of Variation, Skewness, Kurtosis, and Index of Dispersion

The coefficient of variation (CV), The coefficient of skewness ($\sqrt{\beta_1}$), The coefficient of kurtosis (β_2) and the index of dispersion (γ) of Juchez Distribution are thus obtained as:

$$\text{CV} = \frac{\sigma}{\mu_1'} = \frac{\sqrt{(\theta^6 + 4\theta^5 + 2\theta^4 + 84\theta^3 \; 60\theta^2 + 144)}}{(\theta^3 + 2\theta^2 + 24)} \tag{30}$$

$$\sqrt{\beta_1} = \frac{\mu_3}{\mu_2^{3/2}} = \frac{2(\theta^9 + 6\theta^8 + 6\theta^7 + 200\theta^6 + 270\theta^5 + 108\theta^4 + 324\theta^3 + 432\theta^2 + 864)}{(\theta^6 + 4\theta^5 + 2\theta^4 + 84\theta^3 \; 60\theta^2 + 144)^{3/2}} \tag{31}$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{3(3\theta^{12} + 24\theta^{11} + 44\theta^{10} + 968\theta^9 + 2336\theta^8 + 2016\theta^7 + 7488\theta^6 + 13248\theta^5 + 5760\theta^4 + 31104\theta^3 + 24192\theta^2 + 31104)}{(\theta^6 + 4\theta^5 + 2\theta^4 + 84\theta^3 \; 60\theta^2 + 144)^2} \tag{32}$$

$$\gamma = \frac{\sigma^2}{\mu_1'} = \frac{(\theta^6 + 4\theta^5 + 2\theta^4 + 84\theta^3 \; 60\theta^2 + 144)}{\theta(\theta^3 + 2\theta^2 + 24)(\theta^3 + \theta^2 + 6)} \tag{33}$$

4. FEATURES OF THE JUCHEZ DISTRIBUTION

4.1 Mean Residual Life Function (MRL)

In reliability studies, Mean Residual Life Function (MRL) is the expected additional lifetime, given that a component has survived until time t. This is defined as:

$$m(x) = E[X - x \mid X > x] = \frac{1}{1 - \mathrm{F(x)}} \int_x^{\infty} [1 - \mathrm{F(t)}]\, dt \tag{34}$$

Where we consider $\mathrm{A} = \frac{1}{1 - \mathrm{F(x)}}$

The mean residual life function of Juchez distribution is given as

$$m(x) = \mathrm{A} \int_x^{\infty} \left[1 - 1 - \left(1 + \frac{\theta t\,[\,\theta^2 + \theta^2 t^2 + 3\theta t + 6\,]}{\theta^3 + \theta^2 + 6} \right) e^{-\theta t} \right] dt \tag{35}$$

With $A = \dfrac{1}{1 - \left\{ 1 - \left(1 + \frac{\theta x\,[\,\theta^2 + \theta^2 x^2 + 3\theta x + 6\,]}{\theta^3 + \theta^2 + 6} \right) e^{-\theta x} \right\}}$

$$m(x) = \frac{\theta^3 + 2\theta^2 + \theta^3 x + \theta^3 x^2 + 6\theta^2 x^2 + 18\theta x + 24}{\theta(\,\theta^3 + \theta^2 + 6\,) + \theta x(\,\theta^2 + \theta^2 x^2 + 3x\theta + 6\,)} \tag{36}$$

Thus at $x = 0$, $\quad m(0) = \dfrac{(\,\theta^3 + 2\,\theta^2 + 24\,)}{\theta(\,\theta^3 + \theta^2 + 6\,)} = \mu$

4.2 Hazard Function

According to Gross and Clark [1975], hazard function accounts for the risk of failure of a system at varying times x. At the other hand Survival Function is the probability that a system survives beyond a given time $x, x \geq 0$. The Hazard Function of Juchez distribution is given as

$$H(x, \theta) = \frac{f(\,x,\ \theta)}{1 - F(\,x,\ \theta)} = \frac{f(\,x,\ \theta\,)}{S(\,x,\ \theta\,)} \tag{37}$$

where $S(\,x,\ \theta\,)$ is the survival function

$$= \frac{\frac{\theta^4}{\theta^3 + \theta^2 + 6}\,(\,1 + x + x^3\,)\,e^{-\theta x}}{\left[\,1 + \frac{\theta x\,[\,\theta^2 + \theta^2 x^2 + 3\theta x + 6\,]}{\theta^3 + \theta^2 + 6}\,\right] e^{-\theta x}}$$

$$H(x, \theta) = \frac{\theta^4(\,1 + x + x^3\,)}{(\,\theta^3 + \theta^2 + 6\,) + \theta x(\,\theta^2 + \theta^2 x^2 + 3x\theta + 6\,)} \tag{38}$$

4.3 Bonferroni and Lorenz Curve

Bonferroni [1930] gave a curve that measures for the conditional mean of a distribution; whereas, Dagum [1985] referred Lorenz curve as the measure of inequality of the variability of X. Let X be a non-negative continuous random variable, with positive and finite expected value μ, and distribution F; then Bonferroni curve is obtained as

$$B(p) = \frac{1}{p\mu} \int_0^q x\, f(x)dx \tag{36}$$

$$B(p) = \frac{1}{p\mu} \left[\int_0^{\infty} x\, f(x)dx - \int_q^{\infty} x f(x)dx \right] = \frac{1}{p\mu} \left[\mu - \int_q^{\infty} x\, f(x)dx \right] \tag{37}$$

While the Lorenz curve is obtained as

$$L(p) = \frac{1}{\mu}\int_0^q x\, f(x) dx \tag{38}$$

$$L(p) = \frac{1}{\mu}\left[\int_0^\infty x\, f(x) dx - \int_q^\infty x f(x) dx\right] = \frac{1}{\mu}\left[\mu - \int_q^\infty x\, f(x) dx\right] \tag{39}$$

The relationship between the Boneferroni curve and Lorenz curve is given as

$$B(p) = \frac{1}{\mu}\int_0^p F^{-1}(x) dx = \frac{L(p)}{p} \tag{40}$$

Where $\mu = E(X)$, $q = F^{-1}(p)$ and $p \in [0,1]$

Thus, when $X \sim Juchez(\theta)$, the $B(p)$ and $L(p)$ of Juchez distribution are defined as

$$B(p) = \frac{1}{p}\left(1 - \left[\frac{(\theta^3 + 2\theta^2 + 24) - q(\theta^4 + 2\theta^3 + 24) - q^2(\theta^4 + 12\theta^2) - 4\theta^3 q^3 - \theta^4 q^4)}{(\theta^3 + 2\theta^2 + 24)}\right]\right) \tag{41}$$

$$L(p) = \left(1 - \left[\frac{(\theta^3 + 2\theta^2 + 24) - q(\theta^4 + 2\theta^3 + 24) - q^2(\theta^4 + 12\theta^2) - 4\theta^3 q^3 - \theta^4 q^4)}{(\theta^3 + 2\theta^2 + 24)}\right]\right) \tag{42}$$

4.4 Stochastic Ordering

Given that $X \sim Juchez(\theta_1)$ and $Y \sim Juchez(\theta_2)$, and if $\theta_1 > \theta_2$ then $X \leq_{lr} Y$ and hence $X \leq_{hr} Y$, $X \leq_{mrl} Y$ and $X \leq_{st} Y$. Where lr, hr, mrl and st represent the likelihood ratio order, hazard rate order, mean residual life order and stochastic order respectively. Thus,

$$\frac{f_X(x)}{f_Y(x)} = \frac{\theta_1^4(\theta_2^3 + \theta_2^2 + 6)}{\theta_2^4(\theta_1^3 + \theta_1^2 + 6)} e^{x(\theta_2 - \theta_1)}, \quad x > 0. \tag{43}$$

If, for $\theta_2 > \theta_1$,

$$\frac{d}{dx}\frac{f_X(x)}{f_Y(y)} = (\theta_2 - \theta_1)\frac{f_X(x)}{f_Y(x)} < 0, \tag{44}$$

From equations (41) and (42), $\frac{f_X(x)}{f_Y(x)}$ is decreasing in x.That implies $X \leq_{lr} Y$.

Remark:

- $X \leq_{st} Y$ if $F_X(x) \geq F_Y(x)\ \forall x$;
- $X \leq_{hr} Y$ if $h_X(x) \geq h_Y(x)\ \forall x$;
- $X \leq_{mrl} Y$ if $m_X(x) \geq m_Y(x)\ \forall x$

These conditions hold if a random variable X is said to be lesser than a random variable Y. These implications are well known [Shaked and Shanthikumar, 1994]:

$$X \leq_{lr} Y \Rightarrow X \leq_{hr} Y \Rightarrow X \leq_{mrl} Y \text{ and } X \leq_{hr} Y \Rightarrow X \leq_{st} Y$$

4.5 Entropy Measure

Entropy measures the uncertainty, or randomness of a system, say probability distribution. The Rényi [1961] entropy of a random variable X, following the Juchez distribution is given by:

$$T_R(s) = \frac{1}{1-s}\log\left(\int f^s(x)\, dx\right) \quad \text{where } s > 0 \text{ and } s \neq 1 \tag{45}$$

$$= \frac{1}{1-s}\log\left(\int_0^\infty\left(\frac{\theta^4}{\theta^3+\theta^2+6}\right)^s (1+x+x^3)^s e^{-\theta s x}dx\right) \tag{46}$$

But $(1+a)^m = \sum_{i=0}^{\infty}\binom{m}{i} a^i$

$$= \frac{1}{1-s}\log\left(\int_0^\infty\left(\frac{\theta^4}{\theta^3+\theta^2+6}\right)^s \sum_{i=0}^{\infty}\binom{s}{i} (x+x^3)^i e^{-\theta s x}dx\right) \tag{47}$$

$$= \frac{1}{1-s}\left(\log \sum_{i=0}^{\infty}\binom{s}{i}\left(\frac{\theta^4}{\theta^3+\theta^2+6}\right)^s \int_0^\infty (1+x^2)^i x^i e^{-\theta s x}dx\right)$$

$$= \frac{1}{1-s}\log\left(\sum_{i=0}^{\infty}\binom{s}{i}\sum_{j=0}^{\infty}\binom{i}{j}\left(\frac{\theta^4}{\theta^3+\theta^2+6}\right)^s \int_0^\infty x^{2j+i} e^{-\theta s x}dx\right)$$

We have that $\int_0^\infty x^m e^{-\theta x}\,dx = \frac{\Gamma(m+1)}{\theta^{m+1}} = \frac{m!}{\theta^{m+1}}$

$$T_R(s) = \frac{1}{1-s}\log\left(\sum_{i=0}^{\infty}\binom{s}{i}\sum_{j=0}^{\infty}\binom{i}{j}\left(\frac{\theta^4}{\theta^3+\theta^2+6}\right)^s \frac{\Gamma(2j+i+1)}{(\theta s)^{2j+i+1}}\right) \tag{48}$$

$$T_R(s) = \frac{1}{1-s}\log\left(\sum_{i=0}^{\infty}\binom{s}{i}\sum_{j=0}^{\infty}\binom{i}{j}\frac{\theta^{4s-2j-i-1}}{(\theta^3+\theta^2+6)^s}\frac{(2j+i)!}{s^{2j+i+1}}\right) \tag{49}$$

4.6 Order Statistics

Let $X_1, X_2, \dots, X_n$ be a random sample of size n from Juchez Distribution. Let $X_1 < X_2 <, \dots, < X_n$ denote the corresponding order statistics. The pdf and the cdf of the kth order statistics say $Y = X_k$ is given by:

$$f_Y(y) = \frac{n!}{(k-1)!(n-k)!} F^{k-1}(y)\{1-F(y)\}^{n-k} f(y) \tag{50}$$

$$f_Y(y) = \frac{n!}{(k-1)!(n-k)!}\sum_{l=0}^{n-k}\binom{n-k}{l}(-1)^l F^{k+l-1}(y)f(y) \tag{51}$$

$$F_Y(y) = \sum_{j=k}^{n}\binom{n}{j} F^j(y)\{1-F(y)\}^{n-j} \tag{52}$$

$$F_Y(y) = \sum_{j=k}^{n}\sum_{l=0}^{n-j}\binom{n}{j}\binom{n-j}{l}(-1)^l F^{j+l}(y) \tag{53}$$

Thus the pdf and the cdf of kth order statistics of Juchez distribution are given by

$$f_Y(y) = \frac{n!\,\theta^3(1+x+x^3)e^{-\theta x}}{(\theta^3+\theta^2+6)(j-1)!(n-j)!}\sum_{l=0}^{n-j}\binom{n-j}{l}(-1)^l\left[1+\frac{\theta x[\theta^2+\theta^2x^2+3\theta x+6]}{\theta^3+\theta^2+6}\right]e^{-\theta x(i+j+1)} \tag{54}$$

$$F_Y(y) = \sum_{j=k}^{n}\binom{n}{j}\sum_{l=0}^{n-j}\binom{n-j}{l}\sum_{k}^{j}\binom{j}{k}\sum_{m=0}^{k}\binom{k}{m}(-1)^{j+l}\left[1+\frac{\theta x[\theta^2+\theta^2x^2+3\theta x+6]}{\theta^3+\theta^2+6}\right]e^{-\theta x(j+l)} \tag{55}$$

That implies that the pdf of minimum order statistics is obtained by substituting $j = k = 1$ in equation (52) to have:

$$f_{1:n} = \frac{n[\theta^3(1+x+x^3)e^{-\theta x}]}{(\theta^3+\theta^2+6)}\sum_{l=0}^{n-1}\binom{n-1}{l}(-1)^l\left[1+\frac{\theta x[\theta^2+\theta^2x^2+3\theta x+6]}{\theta^3+\theta^2+6}\right]e^{-\theta x(i+2)} \tag{56}$$

While the corresponding pdf of maximum order statistics is obtained by making the substitution of $j = k = n$ in equation (52)

$$f_{n:n} = \frac{n[\theta^3(1+x+x^3)e^{-\theta x}]}{(\theta^3+\theta^2+6)} \left[1 + \frac{\theta x[\theta^2+\theta^2 x^2+3\theta x+6]}{\theta^3+\theta^2+6}\right] e^{-\theta x(i+n+1)} \quad (57)$$

4.7 Limiting Distribution

If $X_1,..,X_n$ is a random sample, and if $\bar{X} = \frac{X_1 + \ldots + X_n}{n}$ denotes the sample mean then by the usual central limit theorem, $\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}}$ approaches the standard normal distribution $N(0,1)$ as $n \to \infty$.

There could be an interest in deriving the asymptotic of the extreme values $X_{n;n} = \max(X_1,..,X_n)$ and $X_{1;n} = \min(X_1,..,X_n)$. Bensid (2017) gave many examples on the Lindley family distribution.

The limiting distribution of sample minima and maxima of Juchez distribution is

$$\lim_{t\to 0}\frac{F(tx)}{F(t)} = x \lim_{t\to 0}\frac{f(tx)}{f(t)} \quad (58)$$

$$= x \log_{t\to 0}\frac{\theta^4(1+tx+t^3x^3)e^{-\theta tx}}{\theta^4(1+t+t^3)e^{-\theta t}}$$

$$\lim_{t\to 0}\frac{F(tx)}{F(t)} = x \text{, for } X_{1;n} \text{ minima} \quad (59)$$

$$\lim_{t\to\infty}\frac{1-F(tx)}{1-F(t)} = \lim_{t\to\infty}\left[\frac{\left(1+\frac{\theta^3(t+x)+\theta^3(t+x)^3+3\theta^2(t+x)^2+6\theta(t+x)}{\theta^3+\theta^2+6}\right)e^{-\theta(t+x)}}{\left(1+\frac{\theta^3 t+\theta^3 t^3+3\theta^2 t^2+6\theta t}{\theta^3+\theta^2+6}\right)e^{-\theta t}}\right] \quad (60)$$

$$= e^{-\theta x} \text{, for } X_{n;n} \text{ maxima} \quad (61)$$

4.8 Maximum Likelihood Estimator

Let $X_i, i = 1,2,3,\ldots,n$, be a random variable from Juchez Distribution, the maximum likelihood estimator (MLE) is obtained thus:

$$\mathrm{L}f(x,\theta) = \left(\frac{\theta^4}{(\theta^3+\theta^2+6)}\right)^n \prod_{i=1}^{n}(1+x+x^3)\, e^{-\theta\sum_{i=1}^{n}x_i} \quad (62)$$

$$ln\mathrm{L}f(x,\theta) = 4n\,ln\theta - n\,ln(\theta^3+\theta^2+6) + \sum_{i=1}^{n}\ln(1+x+x^3) - \theta\sum_{i=1}^{n}x_i \quad (63)$$

In estimation of MLE, the estimator is maximized at $\frac{\partial\, ln\mathrm{L}}{\partial\theta} = 0$, then

$$\frac{\partial\, ln\mathrm{L}f(x,\theta)}{\partial\theta} = \frac{4n}{\theta} - \frac{n(3\theta^2+2\theta)}{\theta^3+\theta^2+6} + 0 - \sum_{i=1}^{n}x_i = 0 \quad (64)$$

$$\frac{4n(\theta^3+\theta^2+6) - n\theta(3\theta^2+2\theta)}{\theta(\theta^3+\theta^2+6)} = \sum_{i=1}^{n}x_i \quad (65)$$

$$\frac{4(\theta^3+\theta^2+6) - \theta(3\theta^2+2\theta)}{\theta(\theta^3+\theta^2+6)} = \frac{\sum_{i=1}^{n}x_i}{n} = 0$$

$$(\theta^3+2\theta^2+24) - (\theta^4+\theta^3+6\theta)\bar{x} = 0 \quad (66)$$

MLE has the following properties:

- The estimator $\hat{\theta}_n$ of θ is consistent if $\hat{\theta}_n \xrightarrow{p} \theta \ as\ n \to \infty$. This also implies that

$$\lim_{n\to\infty} P\left(\left|\hat{\theta}_n - \theta\right| > \epsilon\right) = 0 \quad (67)$$

- The estimator $\hat{\theta}_n$ of θ is asymptotically normal:

$$\sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow{D} N\left(0, \frac{1}{I(\theta)}\right) \tag{68}$$

4.9 Simulation Study

Using numerical approach, the quantile function for the Juchez distribution can be obtained from this expression $x = F^{-1}(u)$, which is derived from $F(x) = u$; where $F(x)$ is the distribution function given by equation (14); where 0 < u < 1. And it provides for the generating of "n" random Juchez samples.

$$u = 1 - \left(1 + \frac{\theta x\,[\,\theta^2 + \theta^2 x^2 + 3\theta x + 6\,]}{\theta^3 + \theta^2 + 6}\right) e^{-\theta x} \tag{69}$$

$$1 - u = \left(1 + \frac{\theta x\,[\,\theta^2 + \theta^2 x^2 + 3\theta x + 6\,]}{\theta^3 + \theta^2 + 6}\right) e^{-\theta x} \tag{70}$$

$$ln\left(1 + \frac{\theta x\,[\,\theta^2 + \theta^2 x^2 + 3\theta x + 6\,]}{\theta^3 + \theta^2 + 6}\right) - \ln(1 - u) - \theta x = 0 \tag{71}$$

$$(\theta^3 + \theta^2 + 6)(1 - u) - [\,(\theta^3 + \theta^2 + 6\,) + \theta x(\theta^2 + \theta^2 x^2 + 3\theta x + 6)\,]\, e^{-\theta x} = 0 \qquad (70)$$

5. EMPIRICAL ANALYSIS

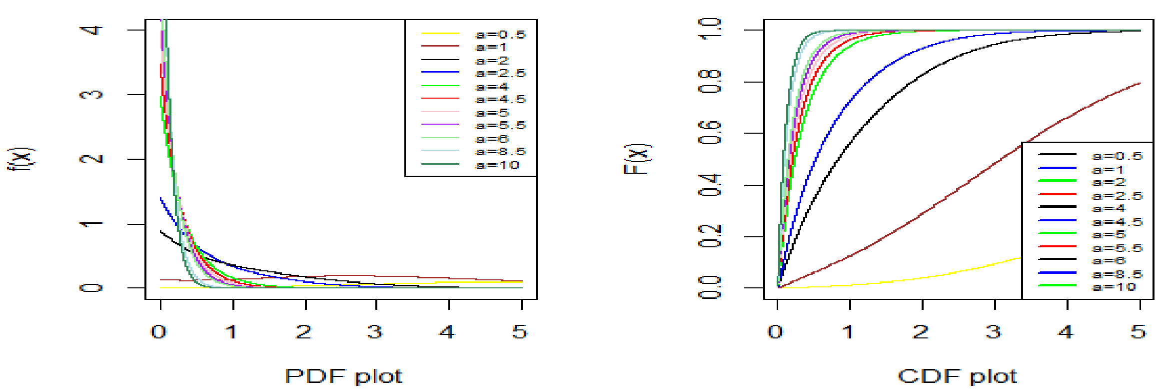

Figure 1. Pdf and CDF plots for Juchez distribution for different levels of parameters

In Figure 1, the plot for pdf of the Juchez distribution for selected values of θ shows that the distribution is positively skewed and unimodal; whereas the cdf is an increasing function at various parameter values and converges at $F(x) = 1$ as supposed.

Table 1. Mean, Median and Mode comparison for random simulated sample data using Juchez distribution

Statistics	Data1 (n = 10)	Data2 (n = 50)	Data3 (n = 100)	Data4 (n = 500)
Mode	2.000	1.000	4.000	2.000
Median	2.000	2.000	3.000	3.000
Mean	3.500	2.720	3.440	2.870

In Table 1, it is observed in data1 for $n = 10$, that *mode (M)* = *median (m)* < *mean* (μ). In data2 for $n = 50$, *mode (M)* < *median (m)* < *mean* (μ); whereas, in data3 for $n = 100$ *mode (M)* > *median (m)* < *mean* (μ). Finally, *mode (M)* < *median (m)* > *mean* (μ) as seen in data4 for $n = 500$. We could deduce clearly that as n increases the inequality tends not to show any patterned consistency.

As given by Lindley distribution**,** *mode (M)* < *median (m)* < *mean* (μ) under certain conditions; data 2 is seen to adhere to this**.** Abadir (2005), however, stated that "for a unimodal and positively skewed distributions whose first three moments exist, the inequality mode (M) < median (m) < mean (μ) does not necessarily hold". Consequently, data 1, 3 and 4 adhere to Abadir's proposition; and it is seen in the Juchez moment derivations in equation (25) that the first three moments exist.

Table 2. Coefficient of Variation (CV) comparison of different one-parameter distributions, valued at $\boldsymbol{\theta = 1}$.

Distributions	**CV = σ/μ**
Exponential	1
Lindley	0.8819
Akash	0.7693
Shanker	0.8819
Sujatha	0.7617
Ishita	0.7693
Aradhana	0.7551
Akshaya	0.6425
Juchez	0.6361

In Table 2, the coefficient of variation also known as the relative standard deviation (RSD) is compared across other one parameter probability distributions. The CV for Exponential Distribution equals 1, which implies that the standard deviation and mean are equal; this is different for other listed distributions. According to Everitt (1998), "higher CV of a model indicates greater dispersion around the mean of the model". By implication, lower values of CV, indicate greater precision of its model. Following the result obtained in Table 2, Juchez Distribution has the lowest variance when valued at $\theta = 1$. It is worthy of note that this trend is consistent for other parameter θ values. Therefore, Juchez Distribution, could be comparatively considered a more efficient model.

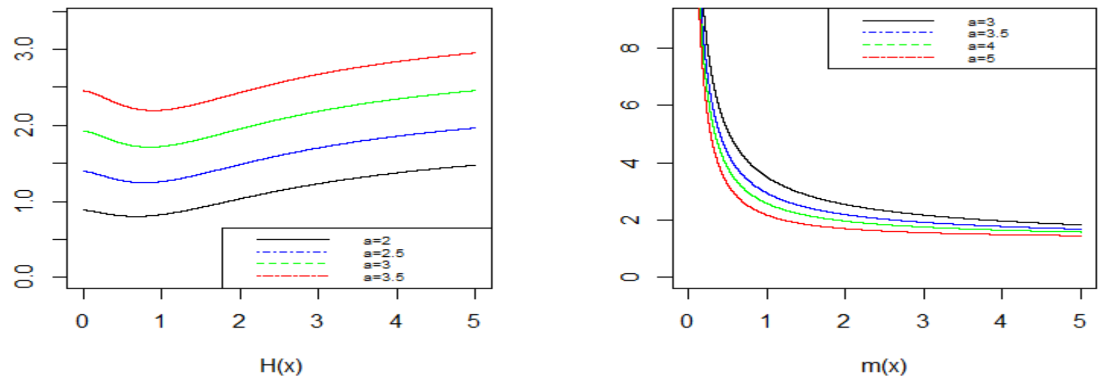

Figure 2. Hazard plots and Mean Residual Life plot for Juchez distribution for different levels of parameters

In figure 2, the two plots show both increasing and decreasing trend respectively. As a result, Juchez distribution is an increasing failure rate model. It is well-known that the MRL and the Hazard function have strong relationship with each other and also to the reliability function. Hence, both the MRL and the harzard functions are able to uniquely determine the distribution of the lifetime of items. In addition, these two functions usually have opposite monotonic trends and represent the ageing behavior of a component from different points of view. From the graphs in figures 2, it is confirmed that an increasing failure rate function implies a decreasing MRL function.

Remark: At $x = 0$, $m(0) = \frac{(\theta^3 + 2\theta^2 + 24)}{\theta(\theta^3 + \theta^2 + 6)} = \mu$; and $H(0) = f(0) = \frac{\theta^4}{\theta^3 + \theta^2 + 6}$. This quantity refers to the component failure of the distribution.

Table 3: Biasedness and Consistency of the MLE

N	$\hat{\theta}_{mle}$ at $\theta = 10$	$\hat{\theta}_{mle}$ at $\theta = 15$
20	10.4503	15.6409
50	10.1047	15.2298
80	10.0831	15.2037
100	10.0825	15.1972
200	10.0639	15.0498
400	10.0210	15.0272
600	10.0154	15.0253
800	10.0088	15.0179
1000	10.0049	15.0040

Table 3 shows that MLE is positively bias as $E(\hat{\theta}_{mle}) - \theta > 0$. In addition, as n increases, the MLE's tend to converge to the true parameter values with high probability; which gives a confirmation note to the consistency of the estimator. More so, this convergence will never meet up to equal the true parameter as n keeps increasing, hence we ascribe the MLE to be asymptotically normal.

Computation of the Average Bias and Mean Square Error for $M = 1000$ Monte Carlo Simulations; over the selected values of (n, θ).

$$Average\ Bias = \left[\frac{1}{M}\sum_{i=1}^{M}(\hat{\boldsymbol{\theta}}_i - \boldsymbol{\theta}) \right] \ and\ MSE = \left[\frac{1}{M}\sum_{i=1}^{M}(\hat{\boldsymbol{\theta}}_i - \boldsymbol{\theta})^2 \right] \quad (71)$$

Table 4: Average Bias of the Estimator $\hat{\boldsymbol{\theta}}$

N	$\boldsymbol{\theta = 8}$	$\boldsymbol{\theta = 10}$	$\boldsymbol{\theta = 15}$
20	0.3586	0.4503	0.6409
50	0.1556	0.1047	0.2298
80	0.0994	0.0831	0.2037
100	0.0475	0.0825	0.1972
200	0.0428	0.0639	0.0498
400	0.0167	0.0210	0.0272

Table 5: MSE of the estimator $\hat{\boldsymbol{\theta}}$

$\boldsymbol{N}$	$\boldsymbol{\theta = 8}$	$\boldsymbol{\theta = 10}$	$\boldsymbol{\theta = 15}$
20	3.0782	5.1541	12.1505
50	1.1013	1.7315	4.4184
80	0.5980	1.0819	2.7699
100	0.5085	0.7982	2.0735
200	0.2535	0.4146	1.0323
400	0.1258	0.1956	0.5009

From Tables 4 and 5, we deduce that the estimates of the average bias and the mean square error decrease as the sample size n increases. In addition, MSE estimates increases as θ increases, for each of the sample sizes.

Table 6: Statistical Table for the PDF of Juchez Distribution ($\boldsymbol{\theta = 0.1}$ to $\boldsymbol{\theta = 0.5}$)

X	$\boldsymbol{\theta = 0.1}$	$\boldsymbol{\theta = 0.2}$	$\boldsymbol{\theta = 0.25}$	$\boldsymbol{\theta = 0.3}$	$\boldsymbol{\theta = 0.35}$	$\boldsymbol{\theta = 0.4}$	$\boldsymbol{\theta = 0.45}$	$\boldsymbol{\theta = 0.5}$
1	0.000045	0.00065	0.0015	0.0029	0.0051	0.0083	0.0124	0.0178
2	0.00015	0.00195	0.0042	0.0080	0.0133	0.0203	0.0291	0.0397
3	0.00038	0.00450	0.0094	0.0167	0.0264	0.0384	0.0524	0.0678
4	0.00077	0.00820	0.0163	0.0275	0.0414	0.0573	0.0743	0.0916
5	0.00132	0.01275	0.0241	0.0387	0.0554	0.0729	0.0899	0.1054

6	0.00204	0.01777	0.0320	0.0488	0.0665	0.0832	0.0976	0.1088
7	0.00300	0.02290	0.0392	0.0569	0.0737	0.0878	0.0980	0.1039
8	0.00389	0.02783	0.0453	0.0626	0.0771	0.0874	0.0928	0.0936
9	0.00500	0.03232	0.0500	0.0658	0.0771	0.0831	0.0839	0.0805
10	0.00619	0.03619	0.0533	0.0667	0.0743	0.0762	0.0732	0.0668
11	0.00744	0.03936	0.0522	0.0656	0.0696	0.0678	0.0620	0.0538
12	0.00872	0.04178	0.0557	0.0630	0.0635	0.0589	0.0512	0.0423
13	0.0100	0.04438	0.0551	0.0593	0.0569	0.0502	0.0415	0.0326
14	0.0113	0.04466	0.0535	0.0548	0.0500	0.0419	0.0330	0.0247
15	0.0125	0.04435	0.0513	0.0499	0.0433	0.0346	0.0259	0.0184
16	0.0138	0.04476	0.0484	0.0448	0.0370	0.0281	0.0200	0.0135
17	0.0149	0.04353	0.0452	0.0398	0.0313	0.0226	0.0153	0.0098
18	0.0161	0.04229	0.0418	0.0350	0.0262	0.0180	0.0116	0.0071
19	0.0171	0.04071	0.0382	0.0305	0.0217	0.0142	0.0087	0.0050
20	0.0181	0.03886	0.0347	0.0263	0.0178	0.0111	0.0064	0.0036
21	0.0189	0.03682	0.0313	0.0226	0.0145	0.0086	0.0048	0.0025
22	0.0197	0.03406	0.0280	0.0192	0.0118	0.0066	0.0035	0.0017
23	0.0203	0.03241	0.0249	0.0163	0.0095	0.0051	0.0025	0.0012
24	0.0209	0.03015	0.0221	0.0137	0.0076	0.0039	0.0018	0.0008
25	0.0214	0.02790	0.0194	0.0115	0.0060	0.0029	0.0013	0.0006

In Table 6, the trend reveals that the pdf of Juchez Distribution is unimodal; and that the axiom $0 < P(x) < 1$ holds across the variable and the parameter values.

A fourteen-week (daily) observation was carried out over a facebook live-streaming program. The data represents a time or cycle-to-event data, which is the weekly average number of viewers before it went below 10,000, which was the target for outreach success.
11.2, 10.9, 13.2, 12.0, 11.5, 11.1, 10.8, 10.3, 13.8, 12.5, 12.3, 12.0, 11.1, 13.7, 14.3, 15.3, 13.1, 12.0, 11.8, 10.9, 13.5, 12.6, 13.4, 14.2, 11.6,13.7, 12.6, 11.6, 14.0, 11.0, 13.6, 12.0, 11.5, 11.9, 10.7, 12.6, 12.5, 13.7, 13.5, 12.4, 13.0, 13.2, 12.0, 14.3, 14.3, 12.5, 11.0, 12.0, 13.2, 12.0, 13.5, 13.2, 12.5, 11.6, 14.0, 12.9, 10.5, 13.4, 14.0, 10.5, 12.6, 13.4, 14, 12.6, 13, 12.8, 13.7, 12.7, 13.6, 14.5, 13.4, 12.9, 11.0, 15.1, 13.6, 12.4, 12.9, 11.2, 10.7, 12.3, 13.5, 12.6, 13.5, 12.3, 13.5, 12.4, 12.3, 11.2, 13.5, 10.3, 11.3 (in thousands).

Finally, we test for the flexibility of the Juchez distribution, in comparison with some renowned one parameter distributions. Literature has it that two or more parameter distributions usually show superiority over one parameter distributions due to its robustness; hence the comparative choice of similar one parameter distributions.

Among many tools, we employ: lnL (Log-Likelihood), AIC (Akaike Information Criterion) and BIC (Bayesian Information Criterion), for performance comparison. The models are given by:

$$AIC = -2lnL + 2k, \quad BIC = -2lnL + k\, ln * n \qquad (72)$$

where n is the number of observations, k is the number of estimated parameters and L is the value of the likelihood function evaluated at the parameter estimates

Table 7: Performance Comparison for Juchez Probability Distribution:

Model	**Parameter Estimate**	lnL	**AIC**	**BIC**	Rank
Juchez	0.3142	-255.36	512.73	515.24	1
Exponential	0.0795	-321.42	644.85	647.36	9
Akash	0.2342	-269.27	540.55	543.06	4
Lindley	0.1487	-292.66	587.31	589.82	8
Shanker	0.1561	-287.75	577.50	580.01	7
Sujatha	0.2270	-271.99	545.97	548.48	5
Aradhana	0.2212	-274.07	550.13	552.64	6
Amarendra	0.3074	-257.19	516.38	518.89	2
Akshaya	0.2946	-260.80	523.60	526.11	3

From Table 7, the best distribution corresponds to the smallest value in AIC, BIC statistics, and or the highest value in lnL. It can be easily seen from Table 7 that the Juchez distribution outperforms other distributions in terms of the inferential measures.

CONCLUSION

The paper aimed at proposing a new probability distribution suitable for modeling live-streaming data. Different distributions have their niche in modeling data from various fields of life. The empirical analyses carried out in this study are sufficient to project Juchez distribution as novel one parameter distribution with respect to live-streaming data modeling, which to the best of my knowledge is unprecedented in the field of distribution. Since all the models compared have one parameter, it follows that the Juchez distribution provides the better fit. A further back-up for this finding, can be obtained by observing the result in Table 2, where it is shown that the coefficient of variation for Juchez distribution is least among other listed distributions. These imply that the newly proposed distribution is more efficient in modeling live-streaming data.

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