

Original Research Article

A Common Fixed-Point Theorem of Mappings on S -metric spaces.

ABSTRACT In this paper, we prove a common fixed-point theorem of two mappings in S -metric space. The theorem uses an implicit contractive condition, putting functions of class F_6 . With the change of the form of these functions of class F_6 we have different results for common fixed points of two functions.

Keywords: S -metric space, common fixed point, contractive condition, function of class F_6 , weakly compatible mappings

1. INTRODUCTION

In 2012, S. Sedghi et. Al [1] introduced the notions of S -metric space. In this paper [1] and in [2] [3] the authors showed some properties of S -metric spaces and proved some fixed points results in these spaces. [4]

Recently, Özgür et al [5] and Prudhvi [4] [6] obtained some results in S -metric spaces.

The results for the fixed points in different spaces are given considering functions that satisfy explicit contractive conditions. On the other hand, lately some results are verified taking in consideration more general contractive conditions given in implicit way from Popa and Berinde [8] [9] [10], Duraj and Hoxha [11] [12].

In 2021, Patriciu [7] proved one general fixed point theorem for mappings which satisfy a cyclical contractive condition using a new type of implicit relation.

In this paper, we prove some common fixed point results for two of weakly compatible mappings in S -metric space using a new implicit relation. The presented theorems extend various known fixed points results.[3-7]

2. PRELIMINARIES

Definition 2.1. ([1] [2]) Let X be a non-empty set. An S -metric on X is a function $S: X^3 \rightarrow \mathbb{R}^+$ such that satisfies the following conditions:

(S_1): $S(x, y, z) = 0$ if and only if $x = y = z$.

(S_1): $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$ for all $x, y, z, a \in X$.

The pair (X, S) is called an S -metric space.

Example 2.2. [7] Let $X = \mathbb{R}$ be the set of real numbers and $S(x, y, z) = |x - z| + |y - z|$. Then $S(x, y, z)$ is an S -metric on \mathbb{R} which is named the usual S -metric on X .

Example 2.3. Let $X = \mathbb{R}$ be the set of real numbers and $S(x, y, z) = \begin{cases} 0, & x = y = z \\ \max\{x, y, z\}, & \text{otherwise} \end{cases}$. Then (X, S) is an S -metric space.

Lemma 2.4 [1] If (X, S) is an S -metric space on a nonempty set X , then $S(x, x, y) = S(y, y, x)$ for all $x, y \in X$.

Definition 2.5

- A sequence $\{x_n\}$ in (X, S) is called convergent to x , (denoted $\lim_{n \rightarrow \infty} x_n = x$ or $x_n \rightarrow x$) if $S(x_n, x_n, x) \rightarrow 0$ as $n \rightarrow \infty$.
- A sequence $\{x_n\}$ in (X, S) is called a Cauchy sequence, if $S(x_n, x_n, x_m) \rightarrow 0$ as $n, m \rightarrow \infty$.
- (X, S) is called complete if every Cauchy sequence is convergent in it.

Lemma 2.6 ([1] [3]) Let (X, S) be a S -metric space. If $x_n \rightarrow x$ and $y_n \rightarrow y$ then $S(x_n, x_n, y_n) \rightarrow S(x, x, y)$.

Lemma 2.7 ([1] [2]) Let (X, S) be a S-metric space. If $x_n \rightarrow x$. Then $\lim_{n \rightarrow \infty} x_n$ is unique.

Definition 2.8 Let T and g be a self-mappings of a nonempty set X . A point $x \in X$ is said to be a common fixed point of T and g if $x = Tx = gx$.

Definition 2.9. [13] Let (X, S) and (X', S') be two S -metric spaces and let $f: (X, S) \rightarrow (X', S')$ be a function. Then f is said to be continuous at a point $a \in X$ if for every sequence $\{x_n\}$ in X , completes $S(x_n, x_n, f(a)) \rightarrow a$, implies $S'(f(x_n), f(x_n), f(a)) \rightarrow 0$. A function f is continuous at X and if and only if it is continuous at all $a \in X$.

3. Main results

Let us consider the class \mathcal{F}_6 of all the functions lower semi-continuous $F: \mathbb{R}_+^6 \rightarrow \mathbb{R}$. Satisfying the following conditions:

(F_1): F is non decreasing in t .

(F_2): There exists $k \in [0, 1/2)$ such that for all $u, v \geq 0$ $F(u, v, v, u, 0, 2u + v) \leq 0$ implies $u \leq kv$.

(F_3): $F(t, t, 0, 0, t, t) > 0, \forall t > 0$.

Example 3.1. [7]

- $F(t_1, t_2, \dots, t_6) = t_1 - at_2 - bt_3 - b \max\{t_3, t_4, t_5, t_6\}$ where $a, b \geq 0$ and $a + 3b < \frac{1}{2}$.
- $F(t_1, t_2, \dots, t_6) = t_1 - at_2 - bt_3 - ct_4 - d \max\{t_5, t_6\}$ where $a, b, c, d \geq 0$ and $a + b + c + 3d < \frac{1}{2}$.
- $F(t_1, t_2, \dots, t_6) = t_1 - at_2 - d \max\{t_3, t_4\} - bt_5 - ct_6$, where $a, b, c, d \geq 0$ and $a + 3c + d < \frac{1}{2}$.
- $F(t_1, t_2, \dots, t_6) = 1 - a \max\{t_4 + t_5, t_3 + t_6\} - bt_2$, where $a, b \geq 0$ and $4a + b < \frac{1}{2}$.
- $F(t_1, t_2, \dots, t_6) = t_1 - k \max\{t_2, t_3, t_4, t_5, t_6\}$ where $k \in [0, \frac{1}{3})$.
- $F(t_1, t_2, \dots, t_6) = t_1 - k \max\{t_2, t_3, t_4, \frac{t_5 + t_6}{3}\}$ where $k \in [0, \frac{1}{2})$.

Theorem 3.2. Let (X, S) be a complete S-metric space and let $T, g: X \rightarrow X$ when T is continuous in X . If the inequality:

$$F(S(Tx, gy, gy), S(x, y, y), S(x, Tx, Tx), S(y, gy, gy), S(y, Tx, Tx), S(x, gy, gy)) \leq 0 \quad (1)$$

holds for all $x, y \in X$ and $F \in \mathcal{F}_6$, then T and g have a common fixed point in X .

Proof:

Let x_0 be an arbitrary point in X . Then $x_1 = Tx_0$ and $x_2 = gx_1$.

By the condition (1) of the Theorem 3.2 for $x = x_0, y = x_1$, we have

$$F(S(Tx_0, gx_1, gx_1), S(x_0, x_1, x_1), S(x_0, Tx_0, Tx_0), S(x_1, gx_1, gx_1), S(x_1, Tx_0, Tx_0), S(x_0, gx_1, gx_1)) \leq 0$$

$$F(S(x_1, x_2, x_2), S(x_0, x_1, x_1), S(x_0, x_1, x_1), S(x_1, x_2, x_2), S(x_1, x_1, x_1), S(x_0, x_2, x_2)) \leq 0$$

Considering (S_2) and Lemma 2.4, we have

$$S(x_0, x_2, x_2) = S(x_2, x_2, x_0) \leq S(x_2, x_2, x_1) + S(x_2, x_2, x_1) + S(x_0, x_0, x_1) = 2S(x_2, x_2, x_1) + S(x_0, x_1, x_1).$$

Using (S_1), we have $S(x_1, x_1, x_1) = 0$.

Consequently,

$$F(S(x, x_2, x_2), S(x_0, x_1, x_1), S(x_0, x_1, x_1), S(x_1, x_2, x_2), 0, 2S(x_1, x_2, x_2) + S(x_0, x_1, x_1)) \leq 0.$$

By (F_2) exists $k \in [0, \frac{1}{2})$ such that $S(x_1, x_2, x_2) \leq kS(x_0, x_1, x_1)$.

Continuing this process, we can define inductively the sequence $\{x_n\}$ as follows:

$$x_{2n+1} = Tx_{2n} \text{ and } x_{2n} = gx_{2n-1}.$$

Replacing $x = x_{2n}$ and $y = x_{2n+1}$ in (1) we have:

$$F(S(Tx_{2n}, gx_{2n+1}, gx_{2n+1}), S(x_{2n}, x_{2n+1}, x_{2n+1}), S(x_{2n}, Tx_{2n}, Tx_{2n}), S(x_{2n+1}, gx_{2n+1}, gx_{2n+1}), S(x_{2n+1}, Tx_{2n}, Tx_{2n}), S(x_{2n}, gx_{2n+1}, gx_{2n+1})) \leq 0.$$

$$F(S(x_{2n+1}, x_{2n+2}, x_{2n+2}), S(x_{2n}, x_{2n+1}, x_{2n+1}), S(x_{2n}, x_{2n+1}, x_{2n+1}), S(x_{2n+1}, x_{2n+2}, x_{2n+2}), S(x_{2n+1}, x_{2n+1}, x_{2n+1}), S(x_{2n}, x_{2n+2}, x_{2n+2})) \leq 0.$$

Using (S_2) and Lemma 2.4, we have:

$$\begin{aligned}
S(x_{2n}, x_{2n+2}, x_{2n+2}) &= S(x_{2n+2}, x_{2n+2}, x_{2n}) \\
&\leq S(x_{2n+2}, x_{2n+2}, x_{2n+1}) + S(x_{2n+2}, x_{2n+2}, x_{2n+1}) + S(x_{2n}, x_{2n+1}, x_{2n+1}) \\
&= 2S(x_{2n+2}, x_{2n+2}, x_{2n+1}) + S(x_{2n}, x_{2n+1}, x_{2n+1})
\end{aligned}$$

$$S(x_{2n+1}, x_{2n+1}, x_{2n+1}) = 0.$$

So,

$$F(S(x_{2n+1}, x_{2n+2}, x_{2n+2}); S(x_{2n}, x_{2n+1}, x_{2n+1}); S(x_{2n}, x_{2n+1}, x_{2n+1}); S(x_{2n+1}, x_{2n+2}, x_{2n+2}), 0, 2S(x_{2n+2}, x_{2n+2}, x_{2n+1}) + S(x_{2n}, x_{2n+1}, x_{2n+1})) \leq 0.$$

Due to (F_2) , there exists $k \in [0, \frac{1}{2})$ such that $S(x_{2n+1}, x_{2n+2}, x_{2n+2}) \leq kS(x_{2n}, x_{2n+1}, x_{2n+1})$.

Similarly, we obtain $S(x_{2n}, x_{2n+1}, x_{2n+1}) \leq kS(x_{2n-1}, x_{2n}, x_{2n})$.

Hence, we have

$$S(x_{2n+1}, x_{2n+2}, x_{2n+2}) \leq kS(x_{2n}, x_{2n+1}, x_{2n+1}) \leq k^2S(x_{2n-1}, x_{2n}, x_{2n}) \leq \dots \leq k^{2n}S(x_0, x_1, x_1),$$

where $n \in \mathbb{N}$; (2)

or $S(x_n, x_{n+1}, x_{n+1}) \leq k^n S(x_0, x_1, x_1)$ where $n \in \mathbb{N}$. (3)

Let us show that the sequence $\{x_n\}$ is Cauchy. Using (S_2) we have:

$$\begin{aligned}
S(x_n, x_n, x_m) &= S(x_m, x_m, x_n) \leq S(x_m, x_m, x_{n+1}) + S(x_m, x_m, x_{n+1}) + S(x_n, x_{n+1}, x_{n+1}) \leq \\
&\leq 2S(x_m, x_m, x_{n+1}) + S(x_n, x_{n+1}, x_{n+1}) \\
&\leq 2S(x_m, x_m, x_{n+1}) + k^n S(x_0, x_1, x_1) \\
&\leq 2[S(x_m, x_m, x_{n+2}) + S(x_m, x_m, x_{n+2}) + S(x_{n+1}, x_{n+1}, x_{n+2})] + k^n S(x_0, x_1, x_1) \\
&\leq 2^2 S(x_m, x_m, x_{n+2}) + 2k^{n+1} S(x_0, x_1, x_1) + k^n S(x_0, x_1, x_1) \leq \dots \\
&\leq 2^{m-n-1} k^{m-1} S(x_0, x_1, x_1) + 2^{m-n-2} k^{m-2} S(x_0, x_1, x_1) + \dots + 2k^{n+1} S(x_0, x_1, x_1) + k^n S(x_0, x_1, x_1) \\
&= [(2k)^{m-n-1} + (2k)^{m-n-2} + 2k + 1] k^n S(x_0, x_1, x_1) \\
&= \frac{[(2k)^{m-n-1}]}{2k-1} k^n S(x_0, x_1, x_1).
\end{aligned}$$

Taking limit when $n, m \rightarrow \infty$ and since $k \in [0, \frac{1}{2})$, we have that $\lim_{m, n \rightarrow \infty} S(x_n, x_n, x_m) = 0$, hence the sequence $\{x_n\}$ is Cauchy.

Since (X, S) is complete, it follows that $\{x_n\}$ is convergent to a point $u \in X$. So the $\lim_{n \rightarrow \infty} S(x_n, x_n, u) = 0$ (4).

We have to prove that u is a common fixed point of T and g .

For $x = x_{2n}$ and $y = u$ by (1) we obtain:

$$\begin{aligned}
&F(S(Tx_{2n}, gu, gu), S(x_{2n}, u, u), S(x_{2n}, Tx_{2n}, Tx_{2n}), S(u, gu, gu), \\
&\quad S(u, Tx_{2n}, Tx_{2n}), S(x_{2n}, gu, gu)) \leq 0 \\
&F(S(x_{2n+1}, gu, gu), S(x_{2n}, u, u), S(x_{2n}, x_{2n+1}, x_{2n+1}), \\
&\quad S(u, gu, gu), S(u, x_{2n+1}, x_{2n+1}), S(x_{2n}, gu, gu)) \leq 0.
\end{aligned}$$

By taking the limit when $n \rightarrow \infty$ and using (3) and (4), we have:

$$F(S(u, gu, gu), S(u, u, u), S(u, u, u), S(u, gu, gu), S(u, u, u), S(u, gu, gu)) \leq 0$$

$$F(S(u, gu, gu); 0; 0; S(u, gu, gu), 0, S(u, gu, gu)) \leq 0$$

Using (F_1) , we get

$$F(S(u, gu, gu); 0; 0; S(u, gu, gu), 0, 2S(u, gu, gu)) \leq 0$$

Considering (F_2) , we obtain $S(u, gu, gu) \leq k \cdot 0 = 0$ and $S(u, gu, gu) = 0$ and by (S_1) it follows that $u = gu$. Hence u is a fixed point of g .

Since T is continuous form (4), we have that $\lim_{n \rightarrow \infty} S(x_{2n}, x_{2n}, u) = 0$ and we get $\lim_{n \rightarrow \infty} S(Tx_{2n}, Tx_{2n}, Tu) = 0$.

But $0 = \lim_{n \rightarrow \infty} S(Tx_{2n}, Tx_{2n}, Tu) = \lim_{n \rightarrow \infty} S(x_{2n+1}, x_{2n+1}, Tu) = S(u, u, Tu)$

From (S_1) we have $Tu = u$ and u is a fixed point of T . Hence $u = gu = Tu$ is common fixed point of T and g .

Remark 3.3. If we take, $T \equiv g$ then the above theorem reduces to Theorem 3 of [7].

Remark 3.4. For the different functions $F \in \mathcal{F}_6$ (see Example 3.1). We can take the corollaries of Theorem 3.2.

4. CONCLUSION

In this article are given results for the common fixed points of two points in S-metric space which satisfy general contraction given in implicit way. Taking different contractions as in Example 3.1, there are obtained different results about the fixed common points in S-metric space.

Furthermore, in Theorem 3.2 we have take only one of the functions continuous, concretely the function T while for the function g the continuity is not necessary.

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