Chris-Jerry Distribution and its Applications

Abstract

In this paper, a new one-parameter distribution named Chris-Jerry is suggested from two component mixture of Exponential (θ) *distribution and Gamma*$(3, \theta)$ *distribution with mixing proportion* $p = \frac{\theta}{\theta+2}$ *having a flexibility advantage in modeling lifetime data. The statistical properties are discussed and the maximum likelihood estimation procedure is used to obtain the parameter estimate. The Convolution of the product of Pareto random variable with the proposed Chris-Jerry distributed random variable is explored with its marginal density derived. To illustrate the usefulness, three sets of lifetime data are employed and LL, AIC, BIC and K-S statistics are obtained for Exponential, Ishita, Akash, Rama, Pranav, Rani, Lindley, Sujatha, Aradhana, Shanker and XGamma and the Chris-Jerry distributions.*

Keywords Chris-Jerry distribution, Exponential distribution, Gamma distribution, Component Mixture, Heavy-tailed distribution, New distributions

1 Introduction

Modeling lifetime data with heavy tail has been a problem among many researchers. Lindley, Exponential and Pareto are the oldest popular heavy-tailed distributions before a number of advances in the literature in the recent decades. One commonality among the standard heavy-tailed distributions is the parsimony with the models credit to the number of parameters.

Essentially, [1] proposed the extended Lomax distribution named McDonald distribution having five parameters hence exhibiting some complexities in mathematical manipulations. [2] proposed three heavy-tailed models based on the Student's t distribution with its scale parameter randomized that model financial data. [3] introduced and study a new family of continuous distributions called Kumaraswamy Weibull-generated family of distributions which is an extension of the Weibull-G family of probability distribution proposed by [4]. [5] was the first to explore two-components distribution to obtain a one-parameter distribution called Lindley distribution using Exponential distribution with scale parameter θ and a Gamma distribution having shape parameter 2 and scale parameter θ with mixing proportion $p = \frac{\theta}{\theta+1}$. [6] proposed the alpha power transformed power Lindley distribution, a generalization of the power Lindley distribution that provides a better fit. An extension of the Lindley distribution which offers a more flexible model for lifetime data was introduced by [7]. [8] derived a one-parameter distribution called Pranav distribution from two-distributions namely Exponential distribution with scale parameter θ and Gamma distribution having shape parameter 4 and scale parameter θ. [9] introduced a two-parameter lifetime distribution named, 'Shukla distribution' which includes several one parameter lifetime distributions. A new one-parameter lifetime distribution named Sujatha Distribution with an increasing hazard rate for modelling lifetime data was suggested by [10]. [11] studied a one-parameter lifetime distribution named Ishita distribution based on a two-component mixture of an Exponential distribution having a shape parameter θ and a Gamma distribution having a shape parameter 3 and scale parameter θ with mixing proportion $\frac{\theta^3}{\theta^3+2}$. [12] studied a one-parameter lifetime distribution named Akash distribution based on a two-component mixture of an Exponential distribution having a shape parameter θ and a Gamma distribution having a shape parameter 2 and scale parameter θ with mixing proportion $\frac{\theta}{\theta+1}$. [13] studied a one-parameter lifetime distribution named

Rani distribution based on a two-component mixture of an Exponential distribution having a shape parameter θ and a Gamma distribution having a shape parameter 5 and scale parameter θ with mixing proportion $\frac{\theta^5}{\theta^5+24}$. [14] studied a one-parameter lifetime distribution named Rama distribution based on a two-component mixture of an Exponential distribution having a shape parameter θ and a Gamma distribution having a shape parameter 4 and scale parameter θ with mixing proportion $\frac{\theta^3}{\theta^3+6}$. [15] studied a one-parameter lifetime distribution named XGamma distribution based on a two-component mixture of an Exponential distribution having a shape parameter θ and a Gamma distribution having a shape parameter 3 and scale parameter θ with mixing proportion $\frac{\theta}{\theta+1}$. [16] studied a one-parameter lifetime distribution named Aradhana distribution based on a two-component mixture of an Exponential distribution having a shape parameter θ and a Gamma distribution having a shape parameter 2 and scale parameter θ with mixing proportion $\frac{1}{\theta+1}$. [17] studied a one-parameter lifetime distribution named Shanker based on a two-component mixture of an Exponential distribution having a shape parameter θ and a Gamma distribution having a shape parameter 2 and scale parameter θ with mixing proportion $\frac{\theta^2}{\theta^2+1}$.

In this paper, a new one-parameter lifetime distribution having its probability density function(pdf) as

$$f_{CJ}(x,\theta) = \frac{\theta^2}{\theta+2}(1+\theta x^2)\mathrm{e}^{-\theta x}, x>0, \theta>0 \quad (1)$$

We call this distribution Chris-Jerry(CJ) distribution. The pdf (1) is a mixture of two distributions, Exponential distribution with scale parameter θ and Gamma distribution with shape and scale parameters 3 and θ respectively. The mixture is of the form $f_{CJ}(x,\theta) = pg_1(x,\theta) + (1-p)g_2(x,3,\theta)$ where $p = \frac{\theta}{\theta+2}$ is the mixing proportion. The cumulative density function(cdf) is given in equation (2).

$$F_{CJ}(x,\theta) = 1 - \left[1 + \frac{\theta x(\theta x+2)}{\theta+2}\right]\mathrm{e}^{-\theta x}, x>0, \theta>0 \quad (2)$$

For various parameter values, the pdf and cdf plots can be shown below

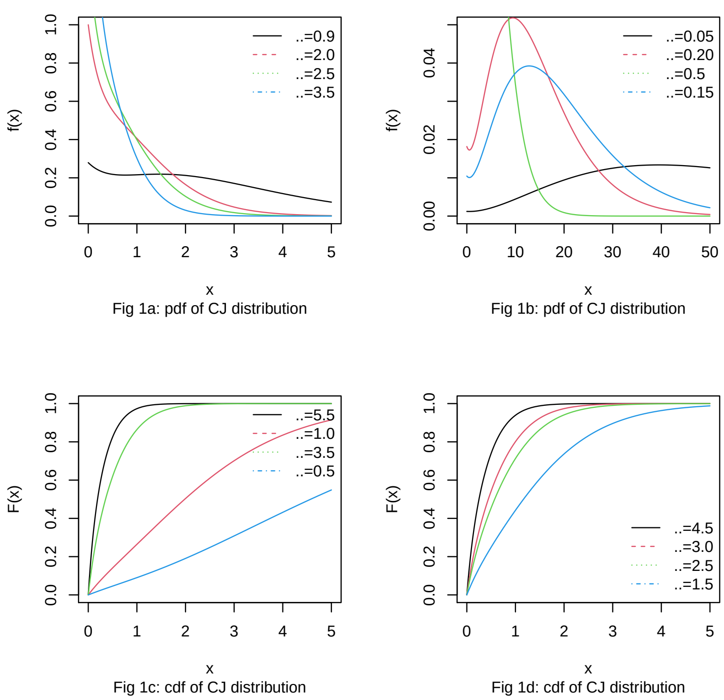

Fig 1a and 1b represent the pdf while Fig 1c and 1d represent the cdf of CJ distribution for various values of the parameter θ

2 Statistical Properties of Chris-Jerry Distribution

2.1 Moment

The rth non-central moment of a Chris-Jerry random variable X is given as

$$\mu_r^{'} = \frac{r!\left[\theta + (r+1)(r+2)\right]}{\theta^r(\theta+2)} \tag{3}$$

2.2 Mean

The arithmetic mean is obtained from equation (3) above by substituting $r = 1$

$$\mu = \frac{\theta + 6}{\theta(\theta+2)} \tag{4}$$

2.3 Other Useful Non-central Moments

The 2nd, 3rd and 4th non-central moment are respectively

$$\mu_2^{'} = \frac{2(\theta+12)}{\theta^2(\theta+2)} \tag{5}$$

$$\mu_3^{'} = \frac{6(\theta+20)}{\theta^3(\theta+2)} \tag{6}$$

and

$$\mu_4^{'} = \frac{24(\theta+30)}{\theta^4(\theta+2)} \tag{7}$$

2.4 Useful Central Moments

The 2nd, 3rd and 4th central moments are respectively

$$\sigma^2 = \frac{\theta^2 + 16\theta + 12}{\theta^2(\theta+2)^2} \tag{8}$$

$$\mu_3 = \frac{4\theta^3 + 72\theta^2 - 384}{\theta^3(\theta+2)^3} \tag{9}$$

and

$$\mu_4 = \frac{9\theta^4 + 864\theta^3 + 6024\theta^2 + 23304\theta - 720}{\theta^4(\theta+2)^4} \tag{10}$$

2.5 Coefficient of Skewness

The coefficient of skewness of Chris-Jerry distribution is given as

$$\gamma = \frac{4\theta^3 + 72\theta^2 - 384}{(\theta^2 + 16\theta + 12)^{\frac{3}{2}}} \tag{11}$$

2.6 Coefficient of Kurtosis

The coefficient of kurtosis of Chris-Jerry distribution is given as

$$\beta = \frac{9\theta^4 + 864\theta^3 + 6024\theta^2 + 2304\theta + 720}{(\theta^2 + 16\theta + 12)^2} \tag{12}$$

To examine the nature of the kurtosis and skewness, the graph of each has been provided in Fig. 2, for different values of θ

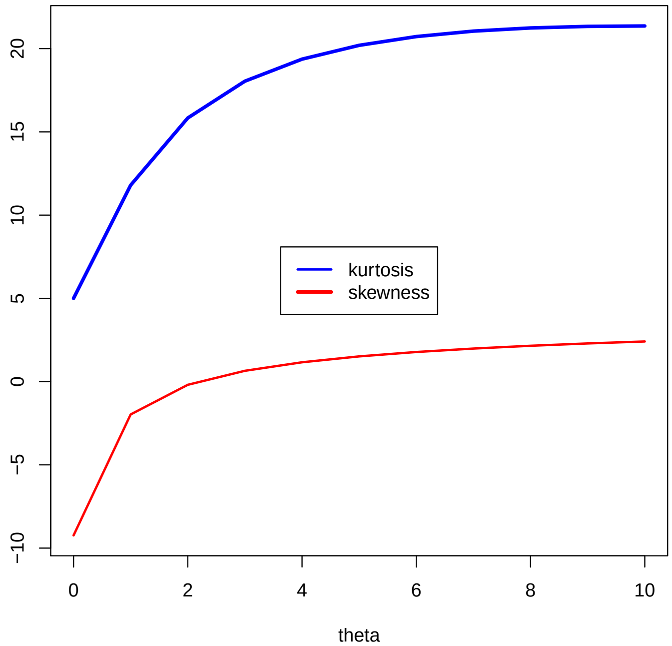

fig.2: Coefficient of kurtosis and skewness of CJ distribution

2.7 Coefficient of variation

$$\zeta = \frac{\sqrt{(\theta^2 + 16\theta + 12)}}{\theta + 6} \times \frac{100}{1} \tag{13}$$

2.8 Index of Dispersion

$$\eta = \frac{\sigma^2}{\mu_1^{'}} = \frac{\theta^2 + 16\theta + 12}{\theta(\theta + 2)(\theta + 6)} \tag{14}$$

2.9 The shape of the Chris-Jerry distribution: Mode

The mode x_0 of Chris-Jerry distribution is obtained by first taking the derivative of the pdf in equation (1)

$$\begin{aligned} \frac{d}{dx} f(x) &= \frac{\theta^2}{\theta + 2} \frac{d}{dx} (1 + \theta x^2) \mathrm{e}^{-\theta x} \\ &= \frac{\theta^2}{\theta + 2} \left[-\theta \mathrm{e}^{-\theta x} - \theta^2 x^2 \mathrm{e}^{-\theta x} + 2\theta x \mathrm{e}^{-\theta x} \right] \end{aligned} \tag{15}$$

It follows that for $\theta \leq 1$ then $\frac{d}{dx}f(x) = 0$

$$\begin{aligned} \frac{\theta^2}{\theta+2}\left[-\theta e^{-\theta x} - \theta^2 x^2 e^{-\theta x} + 2\theta x e^{-\theta x}\right] &= 0 \\ \theta x^2 - 2x + 1 &= 0 \end{aligned} \tag{16}$$

Resolving (16), the positive solution gives the mode, x_0 of the distribution.

$$x_0 = \frac{1+\sqrt{1-\theta}}{\theta} \tag{17}$$

2.10 Quantile function

The q-quantile of Chris-Jerry distribution is obtained using $F(x_q) = P(X \leq x_q) = q$ for $0 < q < 1$. Replace x with x_q in the cdf of Chris-Jerry distribution and equate to q

$$\begin{aligned} q &= 1 - \left[1 + \frac{\theta x_q(\theta x_q + 2)}{\theta + 2}\right] e^{-\theta x_q} \\ (\theta+2)(1-q) &= \left(2 + \theta + 2\theta x_q + \theta^2 x_q^2\right) e^{-\theta x_q} \end{aligned} \tag{18}$$

Solving the equation will give the quantile function x_q.

Theorem 1 (Relationship between mean, median and mode of Chris-Jerry distribution). Let $X \sim$ Chris-Jerry(θ). Then Mode(X) $< Median(X) < E(X)$

Proof. Let $x_0 = Mode(X)$; $x_{0.5} = Median(X)$ and $\mu = E(X)$, $\mu = \frac{\theta+6}{\theta(\theta+2)}$, $x_0 = \frac{1+\sqrt{1-\theta}}{\theta}$, $F(x_{0.5}) = 0.5$ It is easy to see that the theorem holds by the following substitution in the cdf in equation (2) for $|\theta| \leq 1$

$$\begin{aligned} F(\mu) &= 1 - \left[\frac{\theta^3 + 8\theta^2 + 32\theta + 48}{(\theta+2)^3}\right] e^{-\frac{\theta+6}{\theta+2}} \\ F(x_0) &= 1 - \left[\frac{5 + 3\sqrt{1-\theta}}{\theta+2}\right] e^{-1-\sqrt{1-\theta}} \end{aligned}$$

□

2.11 Stochastic Ordering of Chris-Jerry Distribution

The stochastic ordering of a non-negative continuous random variable is a vital tool for comparing the behaviour of system components. A random variable X is said to be smaller than another random variable Y in the

(i) Stochastic order $(X \leq_{st} Y)$ if $F_X(x) \geq F_Y(x)\ \forall x$

(ii) Hazard rate order $(X \leq_{hr} Y)$ if $h_X(x) \geq h_Y(x)\ \forall x$

(iii) Mean residual life order $(X \leq_{mrl} Y)$ if $m_X(x) \geq m_Y(x)\ \forall x$

(iv) Likelihood ratio order $(X \leq_{lr} Y)$ if $\frac{f_X(x)}{F_Y(x)}$ decreases in x

This implies that

$$X \leq_{lr} Y \Rightarrow X \leq_{hr} Y \Rightarrow X \leq_{st} Y \Rightarrow X \leq_{mrl} Y$$

Here, we prove that Chris-Jerry distribution is ordered with respect to the strongest "likelihood ratio" as shown in theorem below

Theorem 2. Let $X \sim CJ(\theta_1)$ and $Y \sim CJ(\theta_2)$. If $\theta_1 > \theta_2$ then $X \leq_{lr} Y$ hence $X \leq_{hr} Y$, $X \leq_{mrl} Y$ and $X \leq_{st} Y$

Proof.

$$\frac{f_X(x)}{f_Y(x)} = \frac{\frac{\theta_1^2}{\theta_1+2}(1+\theta_1 x^2)\mathrm{e}^{-\theta_1 x}}{\frac{\theta_2^2}{\theta_2+2}(1+\theta_2 x^2)\mathrm{e}^{-\theta_2 x}}$$
$$= \frac{\theta_1^2(\theta_2+2)(1+\theta_1 x^2)}{\theta_2^2(\theta_1+2)(1+\theta_2 x^2)}\mathrm{e}^{(\theta_2-\theta_1)x}$$

Taking natural log of the ratio will yield

$$\ln\frac{f_X(x)}{f_Y(x)} = \ln\frac{\theta_1^2(\theta_2+2)}{\theta_2^2(\theta_1+2)} + \ln\frac{1+\theta_1 x^2}{1+\theta_2 x^2} + (\theta_2-\theta_1)x$$

Differentiating the natural log of the ratio wrt x will yield

$$\frac{d}{dx}\ln\frac{f_X(x)}{f_Y(x)} = \frac{2x(\theta_1-\theta_2)}{(1+\theta_1 x^2)(1+\theta_2 x^2)} + (\theta_2-\theta_1)$$

If $\theta_2 > \theta_1$, $\frac{d}{dx}\ln\frac{f_X(x)}{f_Y(x)} < 0$, and $\frac{f_X(x;\theta_1)}{f_Y(x;\theta_2)}$is decreasing in x.

That is, $X \leq_{lr} Y$and hence, $X \leq_{hr} Y, X \leq_{mrl} Y$and $X \leq_{st} Y$

□

2.12 Maximum Likelihood Estimation of the Chris-Jerry Distribution Parameter

Let $(x_1, x_2, ..., x_n)$ be n random samples drawn from Chris-Jerry distribution, then the likelihood function is given as

$$\ell\big(f_{CJ}(x;\theta)\big) = \prod_{i=1}^{n}\frac{\theta^2}{\theta+2}(1+\theta x_i^2)\mathrm{e}^{-\theta x_i}$$
$$= \frac{\theta^{2n}}{(\theta+2)^n}\mathrm{e}^{-\theta\sum x_i}\prod_{i=1}^{n}(1+\theta x_i^2) \tag{19}$$

Taking the natural log of ℓ and differentiating wrt θ yields the following results

$$\psi = \ell(x;\theta) = 2n\ln\theta - n\ln(\theta+2) - \theta\sum_{i=1}^{n}x_i + \prod_{i=1}^{n}\ln(1+\theta x_i^2) \tag{20}$$

$$\frac{\partial\psi}{\partial\hat{\theta}} = \frac{2n}{\hat{\theta}} - \frac{n}{\hat{\theta}+2} - \sum x_i + \sum_{i=1}^{n}\left(\frac{x_i^2}{1+\hat{\theta}x_i^2}\right) \tag{21}$$

Set $\frac{\partial\psi}{\partial\hat{\theta}} = 0$, yields the following quadratic result

$$\frac{2n}{\hat{\theta}} - \frac{n}{\hat{\theta}+2} - \sum x_i + \sum_{i=1}^{n}\left(\frac{x_i^2}{1+\hat{\theta}x_i^2}\right) = 0 \tag{22}$$

The MLE is implemented using Newton-Raphson's numerical iterative method since it has no closed-form solution.

2.13 Moment Generating Function of Chris-Jerry Distribution

The moment generating function of a $X \sim$ Chris-Jerry(θ) is given by

$$\begin{aligned} M_X(t) = E(\mathrm{e}^{tx}) &= \int_0^\infty \mathrm{e}^{tx} f(x)dx \\ &= \frac{\theta^2}{\theta+2}\int_0^\infty \mathrm{e}^{tx}(1+\theta x^2)\mathrm{e}^{-\theta x}dx \\ &= \frac{\theta^2}{\theta+2}\left[\int_0^\infty \mathrm{e}^{-(\theta-t)x}dx + \theta\int_0^\infty x^2\mathrm{e}^{-(\theta-t)x}\right] \\ &= \frac{\theta^2}{\theta+2}\left[\frac{\Gamma(1)}{(\theta-t)} + \frac{\theta\Gamma(3)}{(\theta-t)^3}\right] \\ &= \frac{\theta^2\left[(\theta-t)^{-1} + 2\theta(\theta-t)^{-3}\right]}{\theta+2} \end{aligned} \tag{23}$$

2.14 Characteristic Function of Chris-Jerry Distribution

The moment generating function of a $X \sim$ Chris-Jerry(θ) is given by

$$\phi_X(it) = \frac{\theta^2\left[(\theta-it)^{-1} + 2\theta(\theta-it)^{-3}\right]}{\theta+2} \tag{24}$$

2.15 Distribution of the Order Statistics

Suppose that $X_1, X_1, .., X_n$ is is a random sample of $X_{(r)}$; $(r = 1, 2, ..., n$ are the r^{th} order statistics obtained by arranging X_r in ascending order of magnitude, $\ni X_1 \leq X_2 \leq ... \leq X_r$ and $X_1 = min(X_1, X_2, ..., X_r)$, $X_r = max(X_1, X_2, ..., X_r)$ then the probability density function of the r^{th} order statistics is given by

$$f_{r:n}(x;\theta) = \frac{n!}{(r-1)!(n-r)!} f_{CJ}(x;\theta)\left[F_{CJ}(x;\theta)\right]^{r-1}\left[1 - F_{CJ}(x;\theta)\right]^{n-r} \tag{25}$$

where $f(.)$ and $F(.)$ are respectively the pdf and cdf of Chris-Jerry distribution. Hence, we have

$$f_{r:n}(x;\theta) = \frac{n!}{(r-1)!(n-r)!}\frac{\theta^2}{\theta+2}\left(1+\theta x^2\right)\mathrm{e}^{-\theta x}\left\{1 - \left[1 + \frac{\theta x(\theta x+1)}{\theta+2}\right]\mathrm{e}^{-\theta x}\right\}^{r-1}\left\{\left[1 + \frac{\theta x(\theta x+1)}{\theta+2}\right]\mathrm{e}^{-\theta x}\right\}^{n-r} \tag{26}$$

The pdf of the largest order statistics is obtained by setting $r = n$

$$f_{n:n}(x;\theta) = \frac{n\theta^2}{\theta+2}\left(1+\theta x^2\right)\mathrm{e}^{-\theta x}\left\{1 - \left[1 + \frac{\theta x(\theta x+1)}{\theta+2}\right]\mathrm{e}^{-\theta x}\right\}^{n-1} \tag{27}$$

The pdf of the smallest order statistics is obtained by setting $r = 1$

$$f_{1:n}(x;\theta) = \frac{n\theta^2}{\theta+2}\left(1+\theta x^2\right)\left[1 + \frac{\theta x(\theta x+1)}{\theta+2}\right]^{n-1}\mathrm{e}^{-\theta nx} \tag{28}$$

2.16 Information Measure and Asymptotic Behaviour of Chris-Jerry distribution

Entropy is the quantity of uncertainty or randomness in a system. It is an information measure for non-negative $\omega \neq 1$. The Rény Entropy for Chris-Jerry distributed random variable X is

$$\begin{aligned} R_\omega(x) &= \lim_{n\to\infty}\left(I_\omega(f_n) - \log n\right) \\ &= \frac{1}{1-\omega}\log\int_0^\infty f^\infty(x)dx \end{aligned} \tag{29}$$

For $\omega \to 1$, we have the special case of Shannon Entropy $R_s(x)$

$$\begin{aligned} R_\omega(x) &= \frac{1}{1-\omega}\int_0^\infty \left\{\frac{\theta^2}{\theta+2}\left(1+\theta x^2\right)\mathrm{e}^{-\theta x}\right\}^\omega dx \\ &= \frac{\theta^{2\omega}}{(1-\omega)(\theta+2)^\omega}\sum_{j=0}^{\infty}\binom{\omega}{k}\theta^k\int_0^\infty x^{2\omega}\mathrm{e}^{-\theta\omega x}dx \\ &= \frac{\theta^{2\omega-2k-1}}{1-\omega}\left(\frac{1+\theta}{2+\theta}\right)^\omega\frac{\Gamma(2k+1)}{\theta^{2k+1}} \end{aligned} \tag{30}$$

The asymptotic behaviour of the Chris-Jerry distributed random variable is investigated by taking the limit of the pdf as $x \to 0$ and as $x \to \infty$.

$$\lim_{x\to 0}\frac{\theta^2}{\theta+2}\left(1+\theta x^2\right)\mathrm{e}^{-\theta x} = \frac{\theta^2}{\theta+2} \tag{31}$$

and

$$\lim_{x\to\infty}\frac{\theta^2}{\theta+2}\left(1+\theta x^2\right)\mathrm{e}^{-\theta x} = \frac{\theta^2}{\theta+2}\lim_{x\to\infty}\left(1+\theta x^2\right)\mathrm{e}^{-\theta x} = 0 \tag{32}$$

2.17 Survival Function and Failure Rate

Given a continuous distribution with pdf and cdf in equations (1) and (2), the survival function is given by

$$S_{CJ}(x;\theta) = 1 - F_{CJ}(x;\theta) = \left\{1 + \frac{\theta x(\theta x+2)}{\theta+2}\right\}\mathrm{e}^{-\theta x}; x,\theta > 0 \tag{33}$$

Notice that for Chris-Jerry distribution the survival function $S_{CJ}(x;\theta) = 1$ as $x \to 0$ and $S_{CJ}(x;\theta) = 0$ as $x \to \infty$. Also, the failure rate $h_{CJ}(x;\theta)$, an important tool in reliability measure and engineering is given by

$$h_{CJ}(x;\theta) = \frac{f_{CJ}(x;\theta)}{S_{CJ}(x;\theta)} = \frac{\theta^2(1+\theta x^2)}{\theta+2+\theta x(\theta x+2)}; x,\theta > 0 \tag{34}$$

For Chris-Jerry distribution, the failure rate exhibits the following behaviour;

(i) $h_{CJ}(0) = f_{CJ}(0) = \frac{\theta^2}{\theta+2}$, which is similar to the Lindley distribution

(ii) The function $h_{CJ}(x;\theta)$ is an increasing function in x and θ

(iii) $h_{CJ}(\infty) = 0$

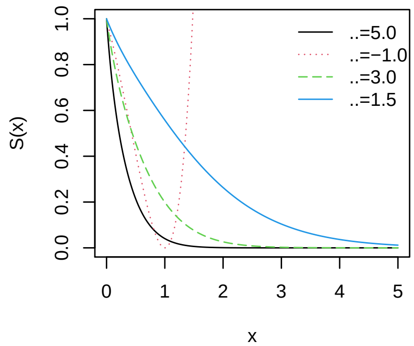

Fig3a: Survival function of CJ distribution

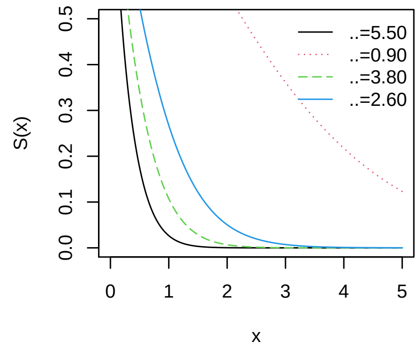

Fig3b: Survival function of CJ distribution

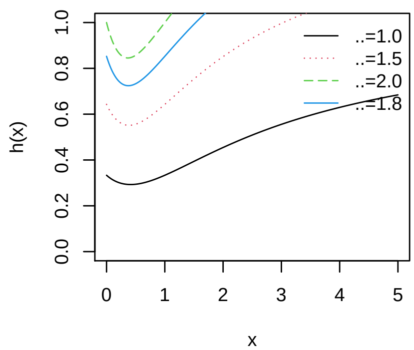

Fig3c: hazard function of CJ distribution

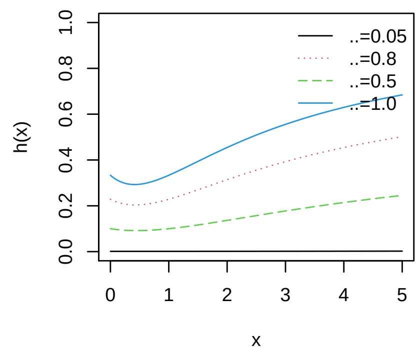

Fig3d: hazard function of CJ distribution

The figures 3a and 3b show the plots of survival function

and figures 3c and 3d are the plots of the hazard function for various parameter values

2.18 Stress-Strength Reliability

We examine the Stress-Strength Reliability of Chris-Jerry distribution. The stress-strength reliability measures the life of a component that possesses random strength X and subjected to random stress Y. When the applied stress Y is higher than the strength x of the system, that is $X < Y$, the component fails. For the component to function efficiently, the strength of the system must be greater than the stress applied to it. Hence, $R = P(Y < X)$ is the measure of the reliability of a component and find application in aging of concrete pressure vessels deteriorating of rocket motors, ceramic components and so on.

Theorem 3. Suppose X and Y are independent random variables denoting strength and stress of a component. We assume further that X and Y follow Chris-Jerry distribution with pdf given in equation (1), with parameter θ_1 and θ_2 respectively. Then, the stress-strength reliability is obtained as follows $R = P(Y < X) = \int_0^\infty P(Y < x|X = x) f_X(x)dx = \int_0^\infty f(x, \theta_1)F(x, \theta_2)dx$

Proof.

$$R = \frac{\theta_1^2}{\theta_1 + 2}\int_0^\infty (1 + \theta_1 x^2)\mathrm{e}^{-\theta x}\left\{1 - \left[1 + \frac{\theta_2 x(\theta_2 x + 2)}{\theta_2 + 2}\right]\mathrm{e}^{-\theta x}\right\}dx$$

$$= 1 - \frac{\theta_1^2}{\theta_1 + 2}\left\{\frac{2\theta_2^2(1 + \theta_1 + \theta_2)(\theta_2 + \theta_1)^2 + 2\theta_1\theta_2^2(13\theta_1 + \theta_2) + 2\theta_1(\theta_2 + 2)(\theta_2 + \theta_1)^2}{(\theta_2 + 2)(\theta_2 + \theta_1)^5}\right\}$$

□

Theorem 4 (**Convolution of the product of two independently and identically distributed Chris-Jerry (iidCJ) random variables**)**.** Let X_1 and X_2 be two iidCJ random variables, suppose $Y_1 = X_1X_2$ then the pdf of Y_1 is given as

$$f(y_1,\theta_1,\theta_2) = \frac{(\theta_1\theta_2)^2}{(\theta_1)(\theta_2+2)}\left\{\frac{\gamma(2,y_1)}{(\theta_2+\theta_1 y_1)^2} + (\theta_2+\theta_1 y_1^2)\frac{\gamma(3,y_1)}{(\theta_2+\theta_1 y_1)^3} + (\theta_1\theta_2 y_1^2)\frac{\gamma(5,y_1)}{(\theta_2+\theta_1 y_1)^5}\right\} \tag{35}$$

Proof. To determine the pdf of Y_1 we introduce a new random variable $Y_2 = X_2$ to make transformation from X_1 and X_2 to Y_1 and Y_2 a one-to-one linear transformation. The Jacobian of transformation $J = |\frac{1}{y_2}|$. The marginal pdf of Y_1 is

$$f(y_1) = \int_0^{y_1} |\frac{1}{y_2}| f\left(\frac{y_1}{y_2}, y_2\right) dy_2$$

. Since X_1 and X_2 are independent then

$$f\left(\frac{y_1}{y_2}\right) f(y_2) = f\left(\frac{y_1}{y_2}, y_2\right)$$

$$\text{where } f\left(\frac{y_1}{y_2}\right) = \frac{\theta_1^2}{\theta_1+2}\left(1+\theta_1\left(\frac{y_1}{y_2}\right)^2\right)\mathrm{e}^{-\theta_1\frac{y_1}{y_2}}$$

and

$$f(y_2) = \frac{\theta_2^2}{\theta_2+2}\left(1+\theta_2 y_2^2\right)\mathrm{e}^{-\theta_2 y_2}$$

$$f\left(\frac{y_1}{y_2}, y_1\right) = \frac{(\theta_1\theta_2)^2}{(\theta_1)(\theta_2+2)}\left(1+\frac{\theta_1 y_1^2}{y_2^2}\right)\left(1+\theta_2 y_2^2\right)\mathrm{e}^{-\theta_1 y_1 y_2^{-1} - \theta_2 y_2}$$

$$\therefore f(y_1) = \frac{(\theta_1\theta_2)^2}{(\theta_1)(\theta_2+2)}\left\{\frac{\gamma(2,y_1)}{(\theta_2+\theta_1 y_1)^2} + (\theta_2+\theta_1 y_1^2)\frac{\gamma(3,y_1)}{(\theta_2+\theta_1 y_1)^3} + (\theta_1\theta_2 y_1^2)\frac{\gamma(5,y_1)}{(\theta_2+\theta_1 y_1)^5}\right\}$$

□

Theorem 5 (**Convolution of the sum of two independently and identically distributed Chris–Jerry (iidCJ) random variables**)**.** Let X_1 and X_2 be two iidCJ random variables, suppose $Y_1 = X_1 + X_2$ then the pdf of Y_1 is given as

$$\begin{aligned} f(y_1) = \frac{(\theta_1\theta_2)^2\mathrm{e}^{-\theta_1 y_1}}{(\theta_1+2)(\theta_2+2)} & \left\{\frac{\mathrm{e}^{-y_1(\theta_2-\theta_1)}-1}{\theta_1-\theta 2} + (1+\theta_1+\theta_2+\theta_1\theta_2 y_1^2)\frac{\gamma(3,y_1)}{(\theta_2-\theta_1)^3}\right. \\ & \left. - 2\theta_1 y_1\frac{\gamma(2,y_1)}{(\theta_2-\theta_1)^2} - 2\theta_1\theta_2 y_1\frac{\gamma(4,y_1)}{(\theta_2-\theta_1)^4} + \theta_2\frac{\gamma(5,y_1)}{(\theta_2-\theta_1)^5}\right\} \end{aligned} \tag{36}$$

Proof. To determine the pdf of Y_1 we introduce a new random variable $Y_2 = X_2$ to make transformation from X_1 and X_2 to Y_1 and Y_2 a one-to-one linear transformation. The Jacobian of transformation $J = 1$. The marginal pdf of Y_1 is

$$f(y_1) = \int_0^{y_1} f(y_1-y_2, y_2) dy_2$$

. Since X_1 and X_2 are independent then

$$f(y_1-y_2) f(y_2) = f(y_1-y_2, y_2)$$

where $f(y_1-y_2)=\dfrac{\theta_1^2}{\theta_1+2}\Big(1+\theta_1(y_1-y_2)^2\Big)\mathrm{e}^{-\theta_1(y_1-y_2)}$

and

$$f(y_2)=\frac{\theta_2^2}{\theta_2+2}(1+\theta_2y_2^2)\mathrm{e}^{-\theta_2y_2}$$

$$\therefore f(y_1)=\frac{(\theta_1\theta_2)^2\mathrm{e}^{-\theta_1y_1}}{(\theta_1+2)(\theta_2+2)}\Bigg\{\frac{\mathrm{e}^{-y_1(\theta_2-\theta_1)}-1}{\theta_1-\theta 2}+(1+\theta_1+\theta_2+\theta_1\theta_2y_1^2)\frac{\gamma(3,y_1)}{(\theta_2-\theta_1)^3}$$
$$-2\theta_1y_1\frac{\gamma(2,y_1)}{(\theta_2-\theta_1)^2}-2\theta_1\theta_2y_1\frac{\gamma(4,y_1)}{(\theta_2-\theta_1)^4}+\theta_2\frac{\gamma(5,y_1)}{(\theta_2-\theta_1)^5}\Bigg\}$$

□

Theorem 6 (Convolution of the product of Pareto and Chris-Jerry distributed random variables). Let $X\sim$ Pareto(α,k) and $Y\sim$ Chris-Jerry(θ) be two independent random variables. Suppose $Z=XY$, then the pdf of Z is given as

$$f(Z)=\frac{\alpha k^\alpha\mathrm{e}^{-z}}{\theta+2}\Bigg\{\frac{\theta^{1-\alpha}}{(\alpha+1)}\sum_{r=1}^{\infty}\frac{Z^r}{\Gamma(\alpha+r+2)}+\theta^{-\alpha}\sum_{r=1}^{\infty}\frac{Z^{r+2}}{\Gamma(\alpha+r+3)}\Bigg\}\tag{37}$$

Proof. Since $X\sim$ Pareto(α,k) then the pdf of X is given by

$$f(x)=\frac{\alpha k^\alpha}{x^{\alpha+1}},x>k,\alpha>0,k\in\mathbb{R}^+\tag{38}$$

while the pdf of Chris-Jerry distribution is in equation (1). Refer to the theorem (4) on the convolution of the sum of products of independent and identically distributed random variables, the pdf of Z will be the marginal function given by

$$f(z)=\int_0^z\frac{1}{y}\frac{\alpha k^\alpha}{\left(\frac{z}{y}\right)^{\alpha+1}}\frac{\theta^2}{\theta+2}(1+\theta y^2)\mathrm{e}^{-\theta y}dy=\frac{\theta^2\alpha k^\alpha}{(\theta+2)z^{\alpha+1}}\int_0^z(1+\theta y^2)y^\alpha\mathrm{e}^{-\theta y}dy\tag{39}$$

This yields the following result

$$f(z)=\frac{\alpha k^\alpha}{(\theta+2)z^{\alpha+1}}\Bigg\{\frac{\gamma(\alpha+1,z)}{\theta^{\alpha-1}}+\frac{\gamma(\alpha+3,z)}{\theta^\alpha}\Bigg\}\tag{40}$$

Using a special result from incomplete Gamma function given by

$$\gamma(s,z)=z^s\Gamma(s)\mathrm{e}^{-z}\sum_{r=0}^{\infty}\frac{z^r}{\Gamma(s+r+1)}$$

We can further simplify the marginal pdf of Z to yield the following compact function

$$f(z)=\frac{\alpha k^\alpha\mathrm{e}^{-z}}{\theta+2}\Bigg\{\frac{\theta^{1-\alpha}}{(\alpha+1)}\sum_{r=1}^{\infty}\frac{z^r}{\Gamma(\alpha+r+2)}+\theta^{-\alpha}\sum_{r=1}^{\infty}\frac{z^{r+2}}{\Gamma(\alpha+r+3)}\Bigg\}\tag{41}$$

□

3 Applications

Chris-Jerry distribution has been fitted to some real lifetime data sets and it gives better fit than Exponential, Ishita, Akash, Rama, Pranav, Rani, Lindley, Sujatha, Aradhana, Shanker and XGamma distributions.

Table 1: [Data 1] shows the life of fatigue fracture of Kevlar 373/epoxy subjected to constant pressure at 90 % stress level until all had failed. Source: [18]

0.0251	0.0886	0.0891	0.2501	0.3113	0.3451	0.4763	0.5650	0.5671	0.6566	0.6748
0.6751	0.6753	0.7696	0.8375	0.8391	0.8425	0.8645	0.8851	0.9113	0.9120	0.9836
1.0483	1.0596	1.0773	1.1733	1.2570	1.2766	1.2985	1.3211	1.3503	1.3551	1.4595
1.4880	1.5728	1.5733	1.7083	1.7263	1.7460	1.7630	1.7746	1.8475	1.8375	1.8503
1.8808	1.8878	1.8881	1.9316	1.9558	2.0048	2.0408	2.0903	2.1093	2.1330	2.2100
2.2460	2.2878	2.3203	2.3470	2.3513	2.4951	2.5260	2.9911	3.0256	3.2678	3.4045
3.4846	3.7433	3.7455	3.9143	4.8073	5.4005	5.4435	5.5295	6.5541	9.0960	

Table 2: [Data 2] shows the Monthly actual taxes revenue (in 1000 million Egyptian pounds) in Egypt between January 2006 and November 2010

5.9	20.4	14.9	16.2	17.2	7.8	6.1	9.2	10.2	9.6	13.3	8.5	21.6	18.5
5.1	6.7	17	8.6	9.7	39.2	35.7	15.7	9.7	10	4.1	36	8.5	8
9.2	26.2	21.9	16.7	21.3	35.4	14.3	8.5	10.6	19.1	20.5	7.1	7.7	18.1
16.5	11.9	7	8.6	12.5	10.3	11.2	6.1	8.4	11	11.6	11.9	5.2	6.8
8.9	7.1	10.8											

Table 3: [Data 3] shows Values of Gross Written Life Insurance Premiums in Nigeria from year 2000 to 2020 (in million euros). Source: https://www.statista.com/statistics/885295/value-life-insurance-premiums-nigeria

13.4	24	25.4	28.8	35	66	79.1	108.9	119.7
138.2	116.6	184.6	195.3	267.5	327.6	388.3	448.9	484.9

To compare Exponential, Ishita, Akash, Rama, Pranav, Rani, Lindley, Sujatha, Aradhana, Shanker and XGamma distributions with the proposed Chris-Jerry distribution, LogLikelihood (LL), Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC) and the Kolmogorov-Smirnov statistics (K-S) are computed for the three real lifetime data sets in tables (1), (2) and (3) above.

The best distribution among others which gives much closer fit to lifetime data corresponds to the largest negative LL and lowest AIC, BIC and K-S statistics.

Table 4: MLE's, LL, AIC, BIC, KS and P-value of the fitted Distribution of datasets 1, 2 and 3

Data	Model	Parameters	S.E	LL	AIC	BIC	K-S	p
1	**Chris-Jerry**	**1.1726**	**0.0872**	**-124.2011**	**250.4023**	**252.7330**	**0.1154**	**0.2444**
	Ishita	1.1050	0.0621	-124.8408	251.6815	254.0123	0.1293	0.1440
	Akash	1.1324	0.0729	-124.5755	251.1510	253.4817	0.1231	0.1836
	Exponential	0.5104	0.0585	-127.1143	256.2287	258.5594	0.1663	0.0263
	Rama	1.4944	0.0767	-127.4547	256.9093	259.2400	0.1431	0.0805
	Pranav	1.4134	0.0637	-128.3959	258.7918	261.1225	0.1493	0.0606
	Rani	1.7315	0.0653	-133.1242	268.2485	270.5792	1.1521	0.0000
	Xgamma	1.0332	0.0818	-126.3260	254.6521	256.9828	0.1474	0.0662
2	**Chris-Jerry**	**0.2117**	**0.0162**	**-196.8246**	**395.6493**	**397.7268**	**0.1361**	**0.2047**
	Lindley	0.1392	0.0129	-200.6293	403.2586	405.3361	0.1922	0.0220
	Shanker	0.1462	0.0133	-198.5302	399.0605	401.1380	0.1706	0.0569
	Exponential	0.0741	0.0097	-212.5068	427.0136	429.0912	0.3034	0.0000
	XGamma	0.2029	0.0158	-199.2838	400.5676	402.6451	0.1666	0.0671
3	**Chris-Jerry**	**0.0175**	**0.0024**	**-116.3489**	**234.6978**	**235.5882**	**0.2498**	**0.1779**
	Ishita	0.0177	0.0024	-116.8004	235.6008	236.4912	0.2527	0.1686
	Akash	0.0177	0.0024	-116.7917	235.5835	236.4739	0.2526	0.1688
	Rama	0.0236	0.0028	-122.7121	247.4242	248.3145	0.2692	0.1220
	Sujatha	0.0177	0.0024	-116.6519	235.3038	236.1942	0.2519	0.1710
	Pranav	0.0236	0.0028	-122.7127	247.4253	248.3157	0.2692	0.1220
	Rani	0.0295	0.0031	-129.2550	260.5100	261.4003	1.2461	0.0000
	Aradhana	0.0176	0.0024	-116.5207	235.0414	235.9317	2.8027	0.0000

From table (4) it is easy to see that Chris-Jerry distribution gives the best fit than the Exponential, Ishita, Akash, Rama, Pranav, Rani, Lindley, Sujatha, Aradhana, Shanker and XGamma distributions since it has the least LL, AIC, BIC and K-S statistics.

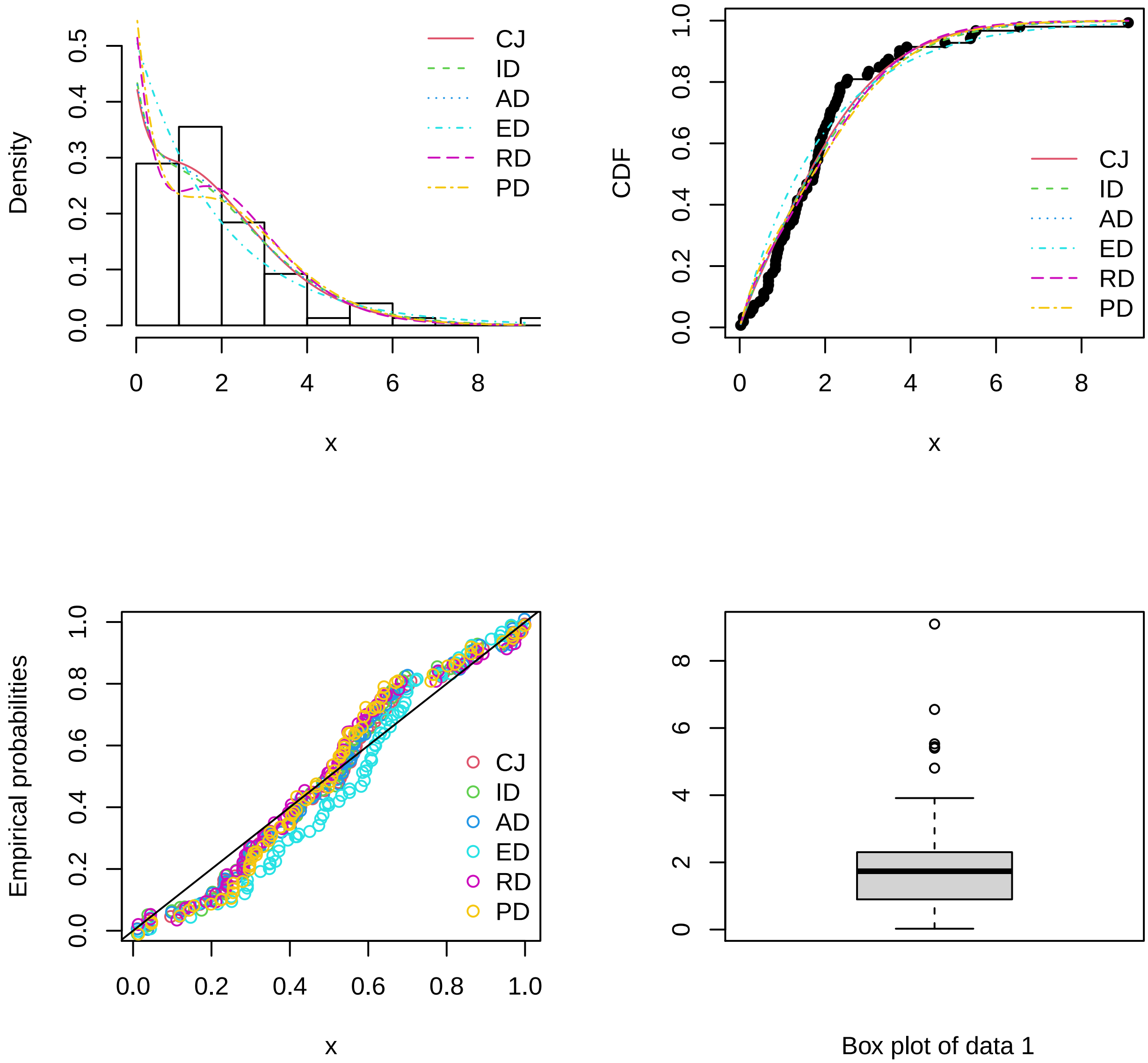

figure 4 :

The density, CDF, Emperical probability-plot and Box plot of Monthly actual taxes revenue (in 1000 million Egyptian pounds)in Egypt between January 2006 and November 2010

4 Conclusion

In this paper, a one-parameter lifetime distribution named, "Chris-Jerry distribution" has been introduced. Its various mathematical properties including shape, moments and useful measures, survival and hazard rate function , stochastic ordering, distribution of order statistics have been discussed. Furthermore, Renyi entropy measure and asymptotic behaviour of the proposed distribution have been derived. . The method of maximum likelihood estimation have also been discussed for estimating its parameter. Finally, the goodness of fit test using LL, AIC, BIC and K-S statistics Statistics for three real lifetime data- sets have been presented to demonstrate its applicability and superiority over Exponential, Ishita, Akash, Rama, Pranav, Rani, Lindley, Sujatha, Aradhana, Shanker and XGamma distributions.

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