GENERALIZED MOMENT GENERATING FUNCTIONS FOR SOME CONTINUOUS MULTIVARIATE PROBABILITY DISTRIBUTIONS

ABSTRACT

The traditional moment generating functions of random variables and their probability distributions are known to not exist for all distributions and/or at all points and, where they exist, serious difficult and tedious manipulations are needed for the evaluation of higher central and non-central moments. This paper developed the generalized multivariate moment generating function for some random vectors/matrices and their probability distribution functions with the intention to replace the traditional/conventional moment generating functions due to their simplicity and versatility. The new functions were developed for the multivariate gamma family of distributions, the multivariate normal and the dirrichlet distributions as a binomial expansion of the expected value of an exponent of a random vector/matrix about an arbitrarily chosen constant. The functions were used to generate moments of random vectors/matrices and their probability distribution functions and the results obtained were compared with those from existing traditional/conventional methods. It was observed that the functions generated same results as the traditional/conventional methods; in addition, they generated both central and non-central moments in the same simple way without requiring further tedious manipulations; they gave more information about the distributions, for instance while the traditional method gives skewness and kurtosis values of 0 and 3 respectively for p-variate multivariate normal distribution, the new methods gives $((0))_{p\times 1}$ and 3^p respectively and; they could generate moments of integral and real powers of random vectors/matrices.

KEYWORDS: Generalized Moments Generating Function, Multivariate Probability Distributions, Multivariate Normal Distribution, Multivariate Gamma Distribution, Dirrichlet Distribution.

1. INTRODUCTION

The significance of moments in explaining the characteristics of random variables and their probability distributions cannot be over emphasized in statistics. Basically, there are two types of moments namely moments about zero (crude moments) and moments about arbitrary points (central moment if the arbitrary point is mean of the distribution) (Pearson, 1900; Kenney & Keeping, 1962; Weisstein, 2002). The central moments are fundamental to the determination of such characteristics of probability distributions as the variance, skewness and kurtosis (Arua, et al., 1997).

The moment generating function is a very important mathematical device for generating moments of random variables and their probability distributions (Chukwu & Amuji, 2012). However, very serious setbacks of this method of generating moments are that it does not always exist for all probability distributions and that it can only generate crude moments by differentiating the moment generating function a certain number of times and evaluating at a zero value of some real coefficient of the variable in the transformation that determines the function (Chukwu & Amuji, 2012; William & Richard, 1973). To obtain the central moments from crude moments, some mathematical combinations of the crude moments of required order are applied. The process of differentiating to obtain crude moments through moment generating functions and eventually having to obtain central moments from crude moments by the mathematical combination of the crude moments is tedious and cumbersome (Oyeka, et al., 2010).

The univariate and bivariate alternative methods for finding the moments of random variables and their probability distributions have proved to be more versatile tools and are easier and quicker to apply in generating moments (Oyeka, et al., 2010; Oyeka, et al., 2008; Oyeka, et al., 2012).

The Generalized Moment Generating Functions of univariate random variables (Matthew, et al., 2017) were developed to eliminate the limitations inherent in the traditional methods and they were actually easier to apply. The method was developed as the expected value of $e^{(X^c+\lambda)}$ where c is an arbitrarily chosen value that is not necessarily an integer and may not be positive, λ is an arbitrarily chosen constant and X is a random variable that follows a defined probability distribution. Application of the method was illustrated with the Beta and Gamma family of Distributions and the Normal Distribution. The method proved useful in generating moments of both positive and negative integer, as well as, real powers of random variables and their probability distributions. The method was able to generate central and non-central moments in the same simple steps for continuous probability distributions. However, the multivariate extension of the method has not been developed. Hence, the Generalized Moment Generating Functions for some Continuous Multivariate Probability Distributions shall be developed and presented here.

2. GENERALIZED MOMENT GENERATING FUNCTIONS FOR SOME CONTINUOUS MULTIVARIATE PROBABILITY DISTRIBUTIONS

The generalized moment generating function for a continuous multivariate random variable $Y = X^c$ about a constant vector or matrix, λ, shall be denoted as $\boldsymbol{G}_n(c; \boldsymbol{\lambda})$.

Suppose $t \in \mathbb{R}$ is a $(p \times p)$ square matrix or a $(p \times 1)$ column vector, $Y = X^c$ and $c \in \mathbb{R}$.

Therefore,

$$M_{(Y;\lambda)}(t) = M_{(X^c;\lambda)}(t) = E\left(e^{t'(X^c+\lambda)}\right) \quad --------- \quad (1)$$

Equation 1 may be read as the moment generating function of X^c about λ and may be evaluated with the Maclaurin's series expansion as

$$E\left(e^{t'(X^c+\lambda)}\right) = E\left(\sum_{n=0}^{\infty} \frac{[t'(X^c+\lambda)]^n}{n!}\right) = \sum_{n=0}^{\infty} \frac{(t')^n}{n!} E(X^c+\lambda)^n$$

$$\therefore M_{(X^c;\lambda)}(t) = E(X^c + \lambda)^n \sum_{n=0}^{\infty} \frac{(t')^n}{n!} \text{ ---------(2)}$$

The coefficient of $\sum_{n=0}^{\infty} \frac{(t')^n}{n!}$ in Equation 2 yields the n^{th} moment of the random variable $Y = X^c$ and may be termed the Multivariate Generalized Moment Generating Function, $\boldsymbol{G}_n(c;\boldsymbol{\lambda})$. It can generate all conceivable moments of $\boldsymbol{X}^c$ about $\boldsymbol{\lambda}$. Obviously, if $c = 1$, $\lambda = 0$ and $n = 1$, Equation 2 yields the first moment of $\boldsymbol{X}$ about zero also called the mean of the distribution of $\boldsymbol{X}$.

If $c = 1$, $\lambda = -\mu$, and $n = 2$, we have from Equation 2 that $G_n(1;-\mu) = Var(X)$. That is;

$$Var(X) = E(X - \mu)^2$$

Higher moments of the distribution of X are similarly obtained by varying the value of n accordingly.

Equation 2 may be evaluated as

$$\boldsymbol{G}_n(c;\boldsymbol{\lambda}) = E(\boldsymbol{X}^c + \boldsymbol{\lambda})^n = E\left(\sum_{r=0}^{n} \binom{n}{r} \boldsymbol{\lambda}^{n-r} \boldsymbol{X}^{cr}\right) = \sum_{r=0}^{n} \binom{n}{r} \boldsymbol{\lambda}^{n-r} E(\boldsymbol{X}^{cr}) \text{ --(3)}$$

(Riordan, 1958).

The Generalized Multivariate Moment Generating Functions, $\boldsymbol{G}_n(c;\boldsymbol{\lambda})$ will be developed for the Multivariate Gamma, Normal and the Dirrichlet Distributions.

3. GENERALIZED MULTIVARIATE MOMENT GENERATING FUNCTION (GMMGF) FOR THE MULTIVARIATE GAMMA DISTRIBUTION

Let X be a positive-definite real $p \times p$ matrix distributed as Multivariate Gamma with shape parameter, α, scale parameter, β, and scale, Σ (a positive-definite real $p \times p$ matrix). Then, the probability density function, PDF, of X is given as,

$$f(X) = \frac{|\Sigma|^{-\alpha}}{\beta^{p\alpha}\Gamma_p(\alpha)} |X|^{\alpha-(p+1)/2} exp\left(tr\left(-\frac{1}{\beta}\Sigma^{-1}X\right)\right) \text{ -------- (4)}$$

where Γ_p is the multivariate gamma function (Gupta & Nagar, 1999; Royen, 2006).

If the shape parameter, $\alpha = \frac{\eta}{2}$, and the scale parameter, $\beta = 2$, the Multivariate Gamma Distribution reduces to the Wishart Distribution with sample size equal to η.

$$f(\boldsymbol{X}) = \frac{|v|^{-\frac{\eta}{2}}}{2^{\frac{\eta p}{2}}\Gamma_p\left(\frac{\eta}{2}\right)} |X|^{\frac{\eta-p-1}{2}} e^{-\frac{1}{2}tr(v^{-1}\boldsymbol{X})} \text{ ---------- } (5)$$

(Wishart, 1928).

Now, applying Equation 3, the generalized moment generating function for the gamma family of distributions is developed as

$$\boldsymbol{G}_n(c;\boldsymbol{\lambda}) = E(\boldsymbol{X}^c + \boldsymbol{\lambda})^n = E\left(\sum_{r=0}^{n} \boldsymbol{\lambda}^{n-r}.\boldsymbol{X}^{cr}.\binom{n}{r}\right) = \sum_{r=0}^{n}\binom{n}{r}\lambda^{n-r}E(\boldsymbol{X}^{cr})$$

where $E(X^{cr}) = \int_{-\infty}^{\infty} x^{cr} f(x)dx$, for p –variate gamma distribution from Equation 4 is

$$\sum_{r=0}^{n}\binom{n}{r}\lambda^{n-r}\int_{X>0}\frac{|\Sigma|^{-\alpha}}{\beta^{p\alpha}\Gamma_p(\alpha)}|X|^{cr+\alpha-(p+1)/2}exp\left(tr\left(-\frac{1}{\beta}\boldsymbol{\Sigma}^{-1}X\right)\right)d\boldsymbol{X}$$

$$= \sum_{r=0}^{n}\binom{n}{r}\lambda^{n-r}\frac{|\Sigma|^{-\alpha}}{\beta^{p\alpha}\Gamma_p(\alpha)}\int_{X>0}|X|^{cr+\alpha-(p+1)/2}exp\left(tr\left(-\frac{1}{\beta}\boldsymbol{\Sigma}^{-1}X\right)\right)d\boldsymbol{X}$$

$$= \sum_{r=0}^{n}\binom{n}{r}\boldsymbol{\lambda}^{n-r}\frac{|\boldsymbol{\Sigma}|^{\alpha+cr}\beta^{p(\alpha+cr)}\Gamma_p(\alpha+cr)}{\boldsymbol{\Sigma}^{\alpha}\beta^{p\alpha}\Gamma_p(\alpha)}$$

$$\therefore \boldsymbol{G}_n(c;\boldsymbol{\lambda}) = E(\boldsymbol{X}^c + \boldsymbol{\lambda})^n = \sum_{r=0}^{n}\binom{n}{r}\boldsymbol{\lambda}^{n-r}\frac{|\boldsymbol{\Sigma}|^{cr}\beta^{pcr}\Gamma_p(\alpha+cr)}{\Gamma_p(\alpha)} \text{ ------ } (6)$$

Substituting $\beta = 2$ and $\alpha = \frac{\eta}{2}$ in Equation 6 gives the multivariate generalized moment generating function of the wishart distribution as

$$\boldsymbol{G}_n(c;\boldsymbol{\lambda}) = \sum_{r=0}^{n}\binom{n}{r}\boldsymbol{\lambda}^{n-r}\frac{|\boldsymbol{\Sigma}|^{cr}2^{pcr}\Gamma_p\left(\frac{\eta}{2}+cr\right)}{\Gamma_p\left(\frac{\eta}{2}\right)} \text{ ---------- } (7)$$

Equation 7 depends on p, the number of variables, and η, the number of observations (sample size), which shows that it is the Multivariate Generalized Moment Generating Function, $\boldsymbol{G}_n(c;\boldsymbol{\lambda})$, of a p –variate extension of the chi-square random variable.

The practical application of this function is better appreciated where $n = 1$. That is,

$$n = 1, r = 0, 1 \; and \; c = 1$$

$$\boldsymbol{G}_1(1;\boldsymbol{\lambda}) = \sum_{r=0}^{1}\binom{1}{0}\boldsymbol{\lambda}^{1-r}\frac{\boldsymbol{\Sigma}^r\beta^{pr}\Gamma_p(\alpha+r)}{\Gamma_p(\alpha)} = \binom{1}{0}\boldsymbol{\lambda} + \binom{1}{1}\frac{\boldsymbol{\Sigma}\beta^p\Gamma_p(\alpha+1)}{\Gamma_p(\alpha)}$$

$$= \boldsymbol{\lambda} + \alpha\beta^{p}\Sigma \quad ------(8)$$

Now substituting λ with $-\mu$ in order to get the first central moment, it implies that

$$-\boldsymbol{\mu} + \alpha\beta^{p}\boldsymbol{\Sigma} = \left((0)\right)_{\boldsymbol{p}\times\boldsymbol{p}}$$

$$\therefore \boldsymbol{\mu} = \beta^{p}\alpha\boldsymbol{\Sigma} = \beta^{p-1}(\alpha\beta\boldsymbol{\Sigma}) ----- \quad (9)$$

The coefficient of β^{p-1} in Equation 9 is the mean of the distribution while β^{p-1} indicates that the dimension (number of variables) of the distribution is p.

4. GENERALIZED MULTIVARIATE MOMENT GENERATING FUNCTION (GMMGF) OF THE NORMAL DISTRIBUTION

A random variable, X, is said to have a univariate normal density if its density function is of the form:

$$f(X) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^{2}} ---- (10)$$

The joint density of independent normal variates, $x_1, x_2, \dots, x_i, \dots, x_p$ is

$$f\left(x_1, x_2, \dots, x_i, \dots x_p\right) = \frac{1}{(2\pi)^{\frac{p}{2}}\sigma_1 \dots \sigma_p} e^{-\frac{1}{2}\Sigma_{i=1}^{p}\left(\frac{x_i-\mu_i}{\sigma_i}\right)^{2}} ---- (11)$$

Let

$$\boldsymbol{X}_{(p\times 1)} = \begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_p \end{pmatrix}, \boldsymbol{\mu}_{(p\times 1)} = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \cdot \\ \cdot \\ \cdot \\ \mu_p \end{pmatrix} \text{ and } \boldsymbol{\Sigma}_{(p\times p)} = \begin{pmatrix} \sigma_1^2 & 0 \cdots & 0 \\ 0 & \sigma_2^2 \cdots & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 \dots & \sigma_p^2 \end{pmatrix}$$

Then,

$$f(\boldsymbol{X}) = \frac{1}{(2\pi)^{\frac{p}{2}}|\boldsymbol{\Sigma}|^{\frac{1}{2}}} exp\left[-\frac{1}{2}(\boldsymbol{X}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{X}-\boldsymbol{\mu})\right] ---- (12)$$

$$-\infty \leq \boldsymbol{\mu} \leq \infty, |\boldsymbol{\Sigma}| > 0$$

The covariance matrix of the random vector, $\boldsymbol{X}$, with correlated random variables is given as

$$\boldsymbol{\Sigma} = \begin{pmatrix} \sigma_{11} & \sigma_{12} \dots & \sigma_{1p} \\ \sigma_{21} & \sigma_{22} \dots & \sigma_{2p} \\ \vdots & \vdots & \vdots \\ \sigma_{p1} & \sigma_{p2} \dots & \sigma_{pp} \end{pmatrix}$$

By substituting this in Equation 12, it becomes the multivariate density function of the random vector of p −correlated random variables (Johnson & Wichern, 1992; Ogum, 2002; Onyeagu, 2003).

Thus $\boldsymbol{X} \sim N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$.

From Equation 3,

$$\boldsymbol{G}_n(c; \boldsymbol{\lambda}) = E(\boldsymbol{X}^c + \boldsymbol{\lambda})^n = \sum_{r=0}^{n} \binom{n}{r} \boldsymbol{\lambda}^{n-r} E(\boldsymbol{X}^{cr})$$

where

$$E(\boldsymbol{X}^{cr}) = \int_{-\infty}^{\infty} \boldsymbol{X}^{cr} f(\boldsymbol{X}) d\boldsymbol{X}_p$$

$$\therefore E(\boldsymbol{X}^{cr}) = \int_{-\infty}^{\infty} \frac{\boldsymbol{X}^{cr}}{(2\pi)^{\frac{p}{2}} |\boldsymbol{\Sigma}|^{\frac{1}{2}}} exp\left[-\frac{1}{2}(\boldsymbol{X}-\boldsymbol{\mu})'\Sigma^{-1}(\boldsymbol{X}-\boldsymbol{\mu})\right] d\boldsymbol{X}_p$$

$$E(\boldsymbol{X}^{cr}) = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \frac{\boldsymbol{X}^{cr}}{(2\pi)^{\frac{p}{2}} |\boldsymbol{\Sigma}|^{\frac{1}{2}}} e^{-\left[\frac{1}{2}\sum_{i=1}^{p}\left(\frac{x_i-\mu_i}{\sigma_i}\right)^2\right]} dx_1 \dots dx_p$$

Let

$$\left(\frac{x_i - \mu_i}{\sigma_i\sqrt{2}}\right)^2 = v_i; \Rightarrow v_i^{\frac{1}{2}} = \frac{x_i - \mu_i}{\sigma_i\sqrt{2}};\ x_i = \mu_i + \sigma_i\sqrt{2}v_i^{\frac{1}{2}};\ \frac{dx_i}{dv_i} = \sqrt{2}\sigma_i \frac{1}{2} v_i^{-\frac{1}{2}}$$

$$\therefore dx_i = \frac{\sigma_i v_i^{-\frac{1}{2}}}{\sqrt{2}} dv_i$$

Hence,

$$E(\boldsymbol{X}^{cr}) = \frac{1}{(2\pi)^{\frac{p}{2}} |\Sigma|^{\frac{1}{2}}} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \left(\mu_i + \sigma_i\sqrt{2}v_i^{\frac{1}{2}}\right)^{cr} . e^{-\frac{1}{2}\sum_{i=1}^{p}\left(\frac{x_i-\mu_i}{\sigma_i}\right)^2} dx_1 \dots dx_p$$

$$= \frac{2^p}{(2\pi)^{\frac{p}{2}}|\Sigma|^{\frac{1}{2}}}\int_0^{\infty} \dots \int_0^{\infty} \left(\mu_i + \sigma_i\sqrt{2}v_i^{\frac{1}{2}}\right)^{cr} e^{-\frac{1}{2}\sum_{i=1}^{p}\left(\frac{x_i-\mu_i}{\sigma_i}\right)^2} dx_1 \dots . dx_p$$

But,

$$\left(\mu_i + \sigma_i\sqrt{2}v_i^{\frac{1}{2}}\right)^{cr} = \sum_{t=0}^{cr}\binom{n}{t}\mu_i^{cr-t}(2\sigma_i^2 v_i)^{\frac{t}{2}}$$

Now,

$$E(\boldsymbol{X}^{cr}) = \frac{\sum_{t=0}^{cr}\binom{cr}{t}\mu_i^{cr-t}2^p(2^p\sigma_i^2)^{\frac{t}{2}}}{(2\pi)^{\frac{p}{2}}|\boldsymbol{\Sigma}|^{\frac{1}{2}}}\int_0^{\infty} \dots \int_0^{\infty} v_i^{\frac{t}{2}} e^{-\sum_{i=1}^{p} v_i}.\frac{\sigma_i v_i^{-\frac{1}{2}}}{\sqrt{2}} dv_1 \dots dv_p$$

$$E(\boldsymbol{X}^{cr}) = \frac{\sum_{t=0}^{cr}\binom{cr}{t}\mu_i^{cr-t}2^p(2^p\sigma_i^2)^{\frac{t}{2}}}{(2\pi)^{\frac{p}{2}}|\boldsymbol{\Sigma}|^{\frac{1}{2}}}.\frac{\sigma_i}{2^{\frac{p}{2}}}\int_0^{\infty} \dots \int_0^{\infty} v_i^{\left(\frac{t}{2}+\frac{1}{2}\right)-1} e^{-\sum_{i=1}^{p} v_i} dv_1 \dots dv_p$$

$$= \frac{\sum_{t=0}^{cr}\binom{cr}{t}\boldsymbol{\mu}^{cr-t}2^p(2^p\boldsymbol{\Sigma})^{\frac{t}{2}}|\boldsymbol{\Sigma}|^{\frac{1}{2}}}{(2\pi)^{\frac{p}{2}}.2^{\frac{p}{2}}|\boldsymbol{\Sigma}|^{\frac{1}{2}}}\left[\Gamma\left(\frac{t}{2}+\frac{1}{2}\right)\right]^p$$

$$\therefore E(\boldsymbol{X}^{cr}) = \frac{\sum_{t=0}^{cr}\binom{cr}{t}\boldsymbol{\mu}^{cr-t}(2^p\boldsymbol{\Sigma})^{\frac{t}{2}}}{\pi^{\frac{p}{2}}}\left[\Gamma\left(\frac{t}{2}+\frac{1}{2}\right)\right]^p$$

Hence,

$$\therefore \boldsymbol{G}_n(c;\boldsymbol{\lambda}) = \frac{\sum_{r=0}^{n}\binom{n}{r}\boldsymbol{\lambda}^{n-r}\sum_{t=0}^{cr}\binom{cr}{t}\boldsymbol{\mu}^{cr-t}(2^p\boldsymbol{\Sigma})^{\frac{t}{2}}\left[\Gamma\left(\frac{t}{2}+\frac{1}{2}\right)\right]^p}{\pi^{\frac{p}{2}}} ----(13)$$

Equation 13 is evaluated at even number values of t. That is, where $t = 0, 2, 4, \dots$

Examples:

For $c = 1$, $n = 1$, $r = 0,1$ and $t = 0$

$$\boldsymbol{G}_1(1;\boldsymbol{\lambda}) = \sum_{r=0}^{1}\binom{1}{0}\boldsymbol{\lambda}^{1-0}\frac{\left[\sum_{t=0}^{r}\binom{r}{t}\boldsymbol{\mu}^{r-t}(2^p\boldsymbol{\Sigma})^{\frac{t}{2}}\left[\Gamma\left(\frac{t}{2}+\frac{1}{2}\right)\right]^p\right]}{\pi^{\frac{p}{2}}}$$

$$\therefore \boldsymbol{G}_1(1;\boldsymbol{\lambda}) = \boldsymbol{\lambda} + \boldsymbol{\mu} \quad ----(14)$$

Suppose $\boldsymbol{\lambda} = -\boldsymbol{\mu}$, the first moment of $\boldsymbol{X}^1 = \boldsymbol{X}$ about $\boldsymbol{\mu}$ is obtained as

$$\boldsymbol{G}_1(1;-\mu) = -\boldsymbol{\mu} + \boldsymbol{\mu} = 0$$

as expected.

For $c = 1, n = 2, r = 0,1,2$ and $t = 0\,,2$; then

$$\boldsymbol{G}_2(1;\boldsymbol{\lambda}) = \sum_{r=0}^{2}\binom{2}{r}\boldsymbol{\lambda}^{2-r}\left[\frac{\sum_{t=0}^{r}\binom{r}{t}\boldsymbol{\mu}^{r-t}(2^p\boldsymbol{\Sigma})^{\frac{t}{2}}\left[\Gamma\left(\frac{t}{2}+\frac{1}{2}\right)\right]^p}{\pi^{\frac{p}{2}}}\right]$$

If $r = 0, t = 0$

$$\binom{2}{0}\boldsymbol{\lambda}^{2-0} = \boldsymbol{\lambda}^2$$

$$r = 1, t = 0$$

$$\binom{2}{1}\boldsymbol{\lambda}^{2-1}\left[\binom{1}{0}\boldsymbol{\mu}^{1-0}(2^p\boldsymbol{\Sigma})^{\frac{0}{2}}\frac{\left(\Gamma\left(\frac{0}{2}+\frac{1}{2}\right)\right)^p}{\pi^{\frac{p}{2}}}\right] = 2\boldsymbol{\lambda}\boldsymbol{\mu}$$

For $r = 2;\ t = 0,2$

$$\binom{2}{2}\boldsymbol{\lambda}^{2-2}\sum_{t=0}^{2}\binom{2}{t}\boldsymbol{\mu}^{2-t}(2^p\boldsymbol{\Sigma})^{\frac{t}{2}}\frac{\left[\Gamma\left(\frac{t}{2}+\frac{1}{2}\right)\right]^p}{\pi^p}$$

$$= \binom{2}{0}\boldsymbol{\mu}^2(2^p\boldsymbol{\Sigma})^0\frac{\left[\Gamma\left(\frac{1}{2}\right)\right]^p}{\pi^{\frac{p}{2}}} + \binom{2}{2}\boldsymbol{\mu}^0(2^p\Sigma)^{\frac{2}{2}}\frac{\left[\Gamma\left(\frac{3}{2}\right)\right]^p}{\pi^{\frac{p}{2}}} = \boldsymbol{\mu}^2 + (2^p\boldsymbol{\Sigma})\left(\frac{1}{2}\right)^p$$

$$= \boldsymbol{\mu}'\boldsymbol{\mu} + \boldsymbol{\Sigma}$$

$$\therefore \boldsymbol{G}_2(1;\boldsymbol{\lambda}) = \boldsymbol{\lambda}^2 + 2\boldsymbol{\lambda}\boldsymbol{\mu} + \boldsymbol{\mu}^2 + \boldsymbol{\Sigma} = \boldsymbol{\lambda}'\boldsymbol{\lambda} + 2\boldsymbol{\lambda}'\mu + \boldsymbol{\lambda}'\boldsymbol{\lambda} + \boldsymbol{\Sigma}\ \ ---\ \ (14)$$

Now, let $\boldsymbol{\lambda} = -\boldsymbol{\mu}$; hence the second moment of $\boldsymbol{X}^1 = \boldsymbol{X}$ about the mean, μ, is

$$_2(1;-\boldsymbol{\mu}) = (-\boldsymbol{\mu})^2 + 2(-\boldsymbol{\mu})(\boldsymbol{\mu}) + \boldsymbol{\mu}^2 + \boldsymbol{\Sigma} = 2\boldsymbol{\mu}^2 - 2\boldsymbol{\mu}^2 + \boldsymbol{\Sigma}$$

$$\therefore \boldsymbol{G}_2(1;-\boldsymbol{\mu}) = \boldsymbol{\Sigma}\ \ -----\ \ (15)$$

That is the variance-covariance matrix as expected.

Also, $\boldsymbol{G}_3(1;\boldsymbol{\lambda})$ is obtained from Equation 13 as follows:

$$n = 3;\ r = 0,1,2,3;\ t = 0,2$$

Now, where $r = 0$

$$\binom{3}{0}\boldsymbol{\lambda}^{3-0}\left[\binom{0}{0}\boldsymbol{\mu}^{0-0}(2^p\boldsymbol{\Sigma})^{\frac{0}{2}}\frac{\left[\Gamma\left(\frac{0}{2}+\frac{1}{2}\right)\right]^p}{\pi^{\frac{p}{2}}}\right] = \boldsymbol{\lambda}^3$$

where $r = 1; t = 0$

$$\binom{3}{1}\boldsymbol{\lambda}^2\left[\binom{1}{0}\boldsymbol{\mu}^{1-0}(2^p\boldsymbol{\Sigma})^{\frac{0}{2}}\left(\frac{\pi^{\frac{p}{2}}}{\pi^{\frac{p}{2}}}\right)\right] = 3\boldsymbol{\lambda}^2\boldsymbol{\mu}$$

where $r = 2; t = 0, 2$

$$\binom{3}{2}\boldsymbol{\lambda}^{3-2}\left[\binom{2}{0}\boldsymbol{\mu}^{2-0}(2^p\boldsymbol{\Sigma})^{\frac{0}{2}}\frac{\pi^{\frac{p}{2}}}{\pi^{\frac{p}{2}}} + \binom{2}{2}\boldsymbol{\mu}^{2-2}(2^p\boldsymbol{\Sigma})^{\frac{2}{2}}\frac{\left[\Gamma\left(\frac{2}{2}+\frac{1}{2}\right)\right]^p}{\pi^{\frac{p}{2}}}\right] = 3\boldsymbol{\lambda}\boldsymbol{\mu}^2 + 3\boldsymbol{\lambda}\boldsymbol{\Sigma}$$

where $r = 3;\ t = 0, 2$

$$\binom{3}{3}\boldsymbol{\lambda}^0\left[\binom{3}{0}\boldsymbol{\mu}^{3-0}(2^p\boldsymbol{\Sigma})^{\frac{0}{2}}\frac{\left[\Gamma\left(\frac{0}{2}+\frac{1}{2}\right)\right]^p}{\pi^{\frac{p}{2}}} + \binom{3}{2}\boldsymbol{\mu}^{3-2}(2^p\boldsymbol{\Sigma})^{\frac{2}{2}}\frac{\left[\Gamma\left(\frac{2}{2}+\frac{1}{2}\right)\right]^p}{\pi^{\frac{p}{2}}}\right]$$

$$= \boldsymbol{\mu}^3 + 3\boldsymbol{\mu}(2^p\boldsymbol{\Sigma})\frac{\left[\Gamma\left(\frac{3}{2}\right)\right]^p}{\pi^{\frac{p}{2}}} = \boldsymbol{\mu}^3 + 3\boldsymbol{\mu}\boldsymbol{\Sigma}$$

$$\therefore \boldsymbol{G}_3(1;\boldsymbol{\lambda}) = \boldsymbol{\lambda}^3 + 3\boldsymbol{\lambda}^2\boldsymbol{\mu} + 3\boldsymbol{\lambda}\boldsymbol{\mu}^2 + 3\boldsymbol{\lambda}\boldsymbol{\Sigma} + \boldsymbol{\mu}^3 + 3\boldsymbol{\mu}\boldsymbol{\Sigma} ---- (16)$$

Now, let $\lambda = -\mu$. Thus,

$$\boldsymbol{G}_3(1;-\boldsymbol{\mu}) = (-\boldsymbol{\mu})^3 + 3(-\boldsymbol{\mu})^2\boldsymbol{\mu} + 3(-\boldsymbol{\mu})\boldsymbol{\mu}^2 + 3(-\boldsymbol{\mu})\boldsymbol{\Sigma} + \boldsymbol{\mu}^3 + 3\boldsymbol{\mu}\boldsymbol{\Sigma}$$

$$\therefore \boldsymbol{G}_3(1;-\boldsymbol{\mu}) = -(\boldsymbol{\mu}'\boldsymbol{\mu})\boldsymbol{\mu} + 3(\boldsymbol{\mu}'\boldsymbol{\mu})\boldsymbol{\mu} - 3(\boldsymbol{\mu}'\boldsymbol{\mu})\boldsymbol{\mu} - 3\boldsymbol{\Sigma}\boldsymbol{\mu} + (\boldsymbol{\mu}'\boldsymbol{\mu})\boldsymbol{\mu} + 3\boldsymbol{\Sigma}\boldsymbol{\mu} = \big((0)\big)_{p\times 1} -- (17)$$

In the same reasoning, $\boldsymbol{G}_4(1;\boldsymbol{\lambda})$ is obtained as follows:

$$n = 4; r = 0, 1, 2, 3, 4; t = 0, 2, 4$$

where $r = 0; t = 0$

$$\binom{4}{0}\boldsymbol{\lambda}^{4-0}\left[\binom{0}{0}\boldsymbol{\mu}^{0-0}(2^p\boldsymbol{\Sigma})^{\frac{0}{2}}\frac{\left[\Gamma\left(\frac{0}{2}+\frac{1}{2}\right)\right]^p}{\pi^{\frac{p}{2}}}\right] = \boldsymbol{\lambda}^4$$

where $r = 1; t = 0$

$$\binom{4}{1}\boldsymbol{\lambda}^{4-1}\left[\binom{1}{0}\boldsymbol{\mu}^{1-0}(2^p\boldsymbol{\Sigma})^{\frac{0}{2}}\frac{\left[\Gamma\left(\frac{0}{2}+\frac{1}{2}\right)\right]^p}{\pi^{\frac{p}{2}}}\right] = 4\boldsymbol{\lambda}^3\boldsymbol{\mu}$$

where $r = 2; t = 0, 2$

$$\binom{4}{3}\boldsymbol{\lambda}^{4-3}\left[\binom{3}{0}\boldsymbol{\mu}^{3-0}(2^p\boldsymbol{\Sigma})^{\frac{0}{2}} + \binom{3}{2}\boldsymbol{\mu}^{3-2}(2^p\boldsymbol{\Sigma})^{\frac{2}{2}}\left(\frac{1}{2}\right)^p\frac{\pi^{\frac{p}{2}}}{\pi^{\frac{p}{2}}}\right] = 4\boldsymbol{\lambda}[\boldsymbol{\mu}^3 + 3\boldsymbol{\mu}\boldsymbol{\Sigma}]$$

$$= 4\boldsymbol{\lambda}\boldsymbol{\mu}^3 + 12\boldsymbol{\lambda}\boldsymbol{\mu}\boldsymbol{\Sigma}$$

where $r = 4;\ t = 0, 2, 4$

$$\binom{4}{4}\boldsymbol{\lambda}^{4-4}\left[\binom{4}{0}\boldsymbol{\mu}^{4-0}(2^p\boldsymbol{\Sigma})^{\frac{0}{2}} + \binom{4}{2}\boldsymbol{\mu}^{4-2}(2^p\boldsymbol{\Sigma})^{\frac{2}{2}}\left(\frac{1}{2}\right)^p\frac{\pi^{\frac{p}{2}}}{\pi^{\frac{p}{2}}} + \binom{4}{4}\boldsymbol{\mu}^{4-4}(2^p\boldsymbol{\Sigma})^{\frac{4}{2}}\frac{\left[\Gamma\left(\frac{5}{2}\right)\right]^p}{\pi^{\frac{p}{2}}}\right]$$

$$= (\boldsymbol{\mu}\boldsymbol{\mu}')(\boldsymbol{\mu}\boldsymbol{\mu}') + 6\boldsymbol{\mu}\boldsymbol{\mu}'\boldsymbol{\Sigma} + (2^p\boldsymbol{\Sigma})^2\left(\frac{3}{2}.\frac{1}{2}\right)^p\frac{\pi^{\frac{p}{2}}}{\pi^{\frac{p}{2}}} = (\boldsymbol{\mu}\boldsymbol{\mu}')(\boldsymbol{\mu}\boldsymbol{\mu}') + 6\boldsymbol{\mu}\boldsymbol{\mu}'\boldsymbol{\Sigma} + \frac{2^{2p}}{2^{2p}}.3^p\boldsymbol{\Sigma}'\boldsymbol{\Sigma}$$

$$= (\boldsymbol{\mu}\boldsymbol{\mu}')(\boldsymbol{\mu}\boldsymbol{\mu}') + 6\boldsymbol{\mu}\boldsymbol{\mu}'\boldsymbol{\Sigma} + 3^p\boldsymbol{\Sigma}'\boldsymbol{\Sigma}$$

$$\boldsymbol{G}_4(1;\boldsymbol{\lambda}) = \boldsymbol{\lambda}^4 + 4\boldsymbol{\lambda}^3\boldsymbol{\mu} + 6\boldsymbol{\lambda}^2\boldsymbol{\mu}^2 + 6\boldsymbol{\lambda}^2\boldsymbol{\Sigma} + 4\boldsymbol{\lambda}\mu^3 + 12\boldsymbol{\lambda}\boldsymbol{\mu}\boldsymbol{\Sigma} + \boldsymbol{\mu}^4 + 6\boldsymbol{\mu}^2\boldsymbol{\Sigma} + \boldsymbol{3}^p\boldsymbol{\Sigma}^2 - - - (18)$$

Now, let $\lambda = -\mu$; therefore, the fourth moment of $X^1 = X$ about its mean, μ, is obtained as

$$G_4(1;-\mu) = (\boldsymbol{\mu}\boldsymbol{\mu}')(\boldsymbol{\mu}\boldsymbol{\mu}') - 4(\boldsymbol{\mu}\boldsymbol{\mu}')(\boldsymbol{\mu}\boldsymbol{\mu}') + 6(\boldsymbol{\mu}\boldsymbol{\mu}')(\boldsymbol{\mu}\boldsymbol{\mu}') + 6\boldsymbol{\mu}\boldsymbol{\mu}'\boldsymbol{\Sigma} - 4(\boldsymbol{\mu}\boldsymbol{\mu}')(\boldsymbol{\mu}\boldsymbol{\mu}') - 12\boldsymbol{\mu}\boldsymbol{\mu}'\boldsymbol{\Sigma}$$
$$+ (\boldsymbol{\mu}\boldsymbol{\mu}')(\boldsymbol{\mu}\boldsymbol{\mu}') + 6\boldsymbol{\mu}\boldsymbol{\mu}'\boldsymbol{\Sigma} + 3^p\boldsymbol{\Sigma}'\boldsymbol{\Sigma}$$

$$= 12\boldsymbol{\mu}\boldsymbol{\mu}'\Sigma - 12\boldsymbol{\mu}\boldsymbol{\mu}'\boldsymbol{\Sigma} + 3^p\boldsymbol{\Sigma}'\boldsymbol{\Sigma}$$

$$\therefore \boldsymbol{G}_4(1;-\boldsymbol{\mu}) = 3^p\boldsymbol{\Sigma}'\boldsymbol{\Sigma}\ - - - - (19)$$

Using Equation 15 and Equation 17, the skewness of the distribution of $X^1 = X$ is obtained as

$$\boldsymbol{SK}(1) = \frac{\boldsymbol{G}_3(1;-\boldsymbol{\mu})}{[\boldsymbol{G}_2(1;-\boldsymbol{\mu})]^{\frac{3}{2}}} = \big((0)\big)_{p\times 1} \quad - - - -\ (20)$$

Thus, it is observed that the ratio of the third moment of $\boldsymbol{X}^1 = \boldsymbol{X}$ about its mean, $\boldsymbol{\mu}$, to the $\left(\frac{3}{2}\right)^{th}$ power of its second moment, its skewness, equals zero, 0 in p – dimensions.

Using Equation 15 and Equation 19, the kurtosis of the distribution of $X^1 = X$ is obtained as

$$\boldsymbol{K_U}(1) = \frac{\boldsymbol{G}_4(1; -\boldsymbol{\mu})}{[\boldsymbol{G}_2(1; -\boldsymbol{\mu})]^2} = \frac{3^p \Sigma' \Sigma}{\Sigma' \Sigma} = 3^p \quad ---- \quad (21)$$

This may be interpreted as mesokurtic because the base is equal to 3 or because the p^{th} root of $\boldsymbol{K_u}(1)$ is equal to 3. It may equally be termed a p $-$variate mesokurtic distribution.

But the traditional or conventional Moment Generating function of the multivariate normal distribution, which involves very tedious mathematical manipulation, gives scalar values of zero and 3 for the skewness and kurtosis of the multivariate normal distribution respectively (Onyeagu, 2003; Matthew, 2019).

5. GENERALIZED MULTIVARIATE MOMENT GENERATING FUNCTION (GMMGF) FOR DIRRICHLET (MULTIVARIATE BETA) DISTRIBUTION

The Dirrichlet distribution, often denoted $Dir(\alpha)$, is a family of continuous multivariate probability distributions parameterized by a vector α of positive reals. It is a multivariate generalization of the beta distribution (Kotz, Balakrishman, & Johnson, 2000).

The density function of the Dirichlet distribution is given as

$$f(x_1, \ldots, x_k) = \frac{1}{\beta(\alpha)} \prod_{i=1}^{k} x_i^{\alpha_i - 1} \qquad (22)$$

where

$$\beta(\alpha) = \frac{\prod_{i=1}^{k} \Gamma(\alpha_i)}{\Gamma\left(\sum_{i=1}^{k} \alpha_i\right)} \; and \; \alpha = (\alpha_1, \ldots, \alpha_k)$$

For $k \geq 2$ number of categories (integers), $\alpha_1, \ldots, \alpha_k$ concentration parameters, where $\alpha_i > 0$ with support variables: $x_1, \ldots, x_k \in (0, 1)$ and $\sum_{i=1}^{k} x_i = 1$.

$$\therefore f(x_i, \ldots x_k) = \frac{\Gamma\left(\sum_{i=1}^{k} \alpha_i\right)}{\prod_{i=1}^{k} \Gamma(\alpha_i)} \prod_{i=1}^{k} x_i^{\alpha_i - 1} \qquad (23)$$

Using equation (23),

$$E(X_i^{cr}) = \frac{\Gamma\left(\sum_{i=1}^{k} \alpha_i\right)}{\prod_{i=1}^{k} \Gamma(\alpha_i)} \int_0^1 \prod_{i=1}^{k} x_i^{\alpha_i + cr - 1} \, dx_i$$

$$\therefore E(X_i^{cr}) = \frac{\Gamma\left(\sum_{i=1}^{k} \alpha_i\right)}{\prod_{i=1}^{k} \Gamma(\alpha_i)} \frac{\prod_{i=1}^{k} \Gamma(\alpha_i + cr)}{\Gamma\left(\sum_{i=1}^{k} \alpha_i + cr\right)} \tag{24}$$

Hence,

$$\boldsymbol{G}_n(c; \boldsymbol{\lambda}) = \sum_{r=0}^{n} \binom{n}{r} \lambda^{n-r} \frac{\Gamma\left(\sum_{i=1}^{k} \alpha_i\right)}{\prod_{i=1}^{k} \Gamma(\alpha_i)} \frac{\prod_{i=1}^{k} \Gamma(\alpha_i + cr)}{\Gamma\left(\sum_{i=1}^{k} \alpha_i + cr\right)} \tag{25}$$

Now, let $n = 1, c = 1$ and $r = 0, 1$

$$\boldsymbol{G}_1(1; \lambda) = \lambda^1 + \frac{\Gamma\left(\sum_{i=1}^{k} \alpha_i\right)}{\prod_{i=1}^{k} \Gamma(\alpha_i)} \frac{\prod_{i=1}^{k} \Gamma(\alpha_i + cr)}{\Gamma\left(\sum_{i=1}^{k} \alpha_i + cr\right)} = \lambda + \frac{\alpha_i}{\sum_{i=1}^{k} \alpha_i}$$

Suppose $\lambda = -\mu$

$$\boldsymbol{G}_{\boldsymbol{1}}(1; -\mu) = -\mu + \frac{\alpha_i}{\sum_{i=1}^{k} \alpha_i} = 0$$

(First moment about the mean)

Hence,

$$E(X_i) = \frac{\alpha_i}{\sum_{i=1}^{k} \alpha_i} \tag{26}$$

Suppose $n = 2; r = 0, 1, 2; c = 1$ and $\lambda = \frac{-\alpha_i}{\sum_{i=1}^{k} \alpha_i}$ from Equation (25)

$$\boldsymbol{G}_2(1; \lambda) = \sum_{i=1}^{2} \binom{2}{r} \lambda^{2-r} \frac{\Gamma\left(\sum_{i=1}^{k} \alpha_i\right)}{\prod_{i=1}^{k} \Gamma(\alpha_i)} \frac{\prod_{i=1}^{k} \Gamma(\alpha_i + r)}{\Gamma\left(\sum_{i=1}^{k} \alpha_i + r\right)}$$

$r = 0;$

$$\binom{2}{0} \lambda^{2-0} \frac{\Gamma\left(\sum_{i=1}^{k} \alpha_i\right)}{\prod_{i=1}^{k} \Gamma(\alpha_i)} \frac{\prod_{i=1}^{k} \Gamma(\alpha_i)}{\Gamma\left(\sum_{i=1}^{k} \alpha_i\right)} = \lambda^2$$

$r = 1;$

$$\binom{2}{1} \lambda^{2-1} \frac{\Gamma\left(\sum_{i=1}^{k} \alpha_i\right)}{\prod_{i=1}^{k} \Gamma(\alpha_i)} \frac{\prod_{i=1}^{k} \Gamma(\alpha_i + 1)}{\Gamma\left(\sum_{i=1}^{k} \alpha_i + 1\right)} = 2\lambda \frac{\Gamma\left(\sum_{i=1}^{k} \alpha_i\right)}{\prod_{i=1}^{k} \Gamma(\alpha_i)} \frac{\alpha_i \prod_{i=1}^{k} \Gamma(\alpha_i)}{\sum_{i=1}^{k} \alpha_i \, \Gamma\left(\sum_{i=1}^{k} \alpha_i\right)} = 2\lambda \frac{\alpha_i}{\sum_{i=1}^{k} \alpha_i}$$

$r = 2$

$$\binom{2}{2} \lambda^0 \frac{\Gamma\left(\sum_{i=1}^{k} \alpha_i\right)}{\prod_{i=1}^{k} \alpha_i} \frac{\prod_{i=1}^{k} \Gamma(\alpha_i + 2)}{\Gamma\left(\sum_{i=1}^{k} \alpha_i + 2\right)} = \frac{\Gamma\left(\sum_{i=1}^{k} \alpha_i\right)(\alpha_i + 1)(\alpha_i) \prod_{i=1}^{k} \Gamma(\alpha_i)}{\prod_{i=1}^{k} \Gamma(\alpha_i) \left(\sum_{i=1}^{k} \alpha_i + 1\right)\left(\sum_{i=1}^{k} \alpha_i\right)\Gamma\left(\sum_{i=1}^{k} \alpha_i\right)}$$

$$= \frac{(\alpha_i + 1)(\alpha_i)\prod_{i=1}^{k}\Gamma(\alpha_i)}{\prod_{i=1}^{k}\Gamma(\alpha_i)\left(\sum_{i=1}^{k}\alpha_i + 1\right)\left(\sum_{i=1}^{k}\alpha_i\right)} = \frac{(\alpha_i + 1)(\alpha_i)}{\left(\sum_{i=1}^{k}\alpha_i + 1\right)\left(\sum_{i=1}^{k}\alpha_i\right)}$$

$$\therefore G_2(1;\ \lambda) = \lambda^2 + 2\lambda\frac{\alpha_i}{\sum_{i=1}^{k}\alpha_i} + \frac{(\alpha_i + 1)(\alpha_i)}{\left(\sum_{i=1}^{k}\alpha_i + 1\right)\left(\sum_{i=1}^{k}\alpha_i\right)} \tag{27}$$

Thus the second central moment; that is, $\lambda = \frac{-\alpha_i}{\sum_{i=1}^{k}\alpha_i}$ becomes

$$\boldsymbol{G}_2\left(1;\frac{-\alpha_i}{\sum_{i=1}^{k}\alpha_i}\right) = \frac{\alpha_i^2}{\left(\sum_{i=1}^{k}\alpha_i\right)^2} - 2\frac{\alpha_i^2}{\left(\sum_{i=1}^{k}\alpha_i\right)^2} + \frac{\alpha_i(\alpha_i+1)}{\left(\sum_{i=1}^{k}\alpha_i\right)\left(\sum_{i=1}^{k}\alpha_i+1\right)}$$

$$= \frac{\alpha_i\left[\left(\sum_{i=1}^{k}\alpha_i\right)(\alpha_i+1) - \alpha_i\left(\sum_{i=1}^{k}\alpha_i+1\right)\right]}{\left(\sum_{i=1}^{k}\alpha_i\right)^2\left(\sum_{i=1}^{k}\alpha_i+1\right)}$$

$$\therefore \boldsymbol{G}_2\left(1;\frac{-\alpha_i}{\sum_{i=1}^{k}\alpha_i}\right) = Var(X_i) = \frac{\alpha_i}{\left(\sum_{i=1}^{k}\alpha_i\right)^2}\frac{\left(\sum_{i=1}^{k}\alpha_i - \alpha_i\right)}{\left(\sum_{i=1}^{k}\alpha_i+1\right)}$$

Let $\sum_{i=1}^{k}\alpha_i = \alpha_0$. Thus,

$$Var(X_i) = \frac{\alpha_i}{\alpha_0^2}\frac{(\alpha_0 - \alpha_i)}{(\alpha_0 + 1)} \tag{28}$$

Now, for $n = 3;\ c = 1; r = 0,1,2,3$; we have

$r = 0$

$$\binom{3}{0}\lambda^{3-0}\frac{\Gamma\left(\sum_{i=1}^{k}\alpha_i\right)}{\prod_{i=1}^{k}\Gamma(\alpha_i)}\frac{\prod_{i=1}^{k}\Gamma(\alpha_i)}{\Gamma\left(\sum_{i=1}^{k}\alpha_i\right)} = \lambda^3$$

$r = 1$

$$\binom{3}{1}\lambda^{3-1}\frac{\Gamma\left(\sum_{i=1}^{k}\alpha_i\right)}{\prod_{i=1}^{k}\Gamma(\alpha_i)}\frac{\prod_{i=1}^{k}\Gamma(\alpha_i + 1)}{\Gamma\left(\sum_{i=1}^{k}\alpha_i + 1\right)} = 3\lambda^2\frac{\Gamma\left(\sum_{i=1}^{k}\alpha_i\right)}{\prod_{i=1}^{k}\alpha_i}\frac{\alpha_i\prod_{i=1}^{k}\Gamma(\alpha_i)}{\sum_{i=1}^{k}\alpha_i\,\Gamma\left(\sum_{i=1}^{k}\alpha_i\right)}$$

$$= 3\lambda^2\frac{\alpha_i}{\sum_{i=1}^{k}\alpha_i}$$

$r = 2$

$$\binom{3}{2}\lambda^{3-2}\frac{\Gamma\left(\sum_{i=1}^{k}\alpha_i\right)}{\prod_{i=1}^{k}\Gamma(\alpha_i)}\frac{\prod_{i=1}^{k}\Gamma(\alpha_i + 2)}{\Gamma\left(\sum_{i=1}^{k}\alpha_i + 2\right)}$$

$$= 3\lambda\frac{\Gamma\left(\sum_{i=1}^{k}\alpha_i\right)}{\prod_{i=1}^{k}\Gamma(\alpha_i)}\frac{\alpha_i(\alpha_i + 1)\prod_{i=1}^{k}\Gamma(\alpha_i)}{\left(\sum_{i=1}^{k}\alpha_i\right)\left(\sum_{i=1}^{k}\alpha_i + 1\right)\Gamma\left(\sum_{i=1}^{k}\alpha_i\right)}$$

$r = 3$

$$\binom{3}{3}\lambda^{3-3}\frac{\Gamma\left(\sum_{i=1}^{k}\alpha_i\right)}{\prod_{i=1}^{k}\Gamma(\alpha_i)}\frac{\prod_{i=1}^{k}\Gamma(\alpha_i + 3)}{\Gamma\left(\sum_{i=1}^{k}\alpha_i + 3\right)}$$

$$= \frac{\Gamma\left(\sum_{i=1}^{k}\alpha_i\right)}{\prod_{i=1}^{k}\Gamma(\alpha_i)}\frac{\alpha_i(\alpha_i + 1)(\alpha_i + 2)\prod_{i=1}^{k}\Gamma(\alpha_i)}{\left(\sum_{i=1}^{k}\alpha_i\right)\left(\sum_{i=1}^{k}\alpha_i + 1\right)\left(\sum_{i=1}^{k}\alpha_i + 2\right)\Gamma\left(\sum_{i=1}^{k}\alpha_i\right)}$$

$$\therefore G_3(1;\lambda) = \lambda^3 + 3\lambda^2\frac{\alpha_i}{\sum_{i=1}^{k}\alpha_i} + 3\lambda\frac{\alpha_i(\alpha_i + 1)}{\left(\sum_{i=1}^{k}\alpha_i\right)\left(\sum_{i=1}^{k}\alpha_i + 1\right)}$$
$$+ \frac{\alpha_i(\alpha_i + 1)(\alpha_i + 2)}{\left(\sum_{i=1}^{k}\alpha_i\right)\left(\sum_{i=1}^{k}\alpha_i + 1\right)\left(\sum_{i=1}^{k}\alpha_i + 2\right)} \qquad (29)$$

Now, let $\lambda = \frac{-\alpha_i}{\sum_{i=1}^{k}\alpha_i}$. Hence to obtain the third central moment of the Dirrichlet distribution we have;

$$G_3\left(1;\frac{-\alpha_i}{\sum_{i=1}^{k}\alpha_i}\right) =$$
$$\frac{-\alpha_i^3}{\left(\sum_{i=1}^{k}\alpha_i\right)^3} + 3\frac{\alpha_i^2}{\left(\sum_{i=1}^{k}\alpha_i\right)^2}\frac{\alpha_i}{\sum_{i=1}^{k}\alpha_i} - 3\frac{\alpha_i}{\sum_{i=1}^{k}\alpha_i}\frac{\alpha_i}{\sum_{i=1}^{k}\alpha_i}\frac{(\alpha_i+1)}{\left(\sum_{i=1}^{k}\alpha_i+1\right)} + \frac{\alpha_i(\alpha_i+1)(\alpha_i+2)}{\left(\sum_{i=1}^{k}\alpha_i\right)\left(\sum_{i=1}^{k}\alpha_i+1\right)\left(\sum_{i=1}^{k}\alpha_i+2\right)}$$

$$= \frac{2\alpha_i^3\left(\sum_{i=1}^{k}\alpha_i + 1\right)\left(\sum_{i=1}^{k}\alpha_i + 2\right) - 3\alpha_i(\alpha_i^3 - \alpha_i^2) + \left(\sum_{i=1}^{k}\alpha_i\right)^2(\alpha_i^3 + 3\alpha_i^2 + 2\alpha_i)}{\left(\sum_{i==1}^{k}\alpha_i\right)^3\left(\sum_{i=1}^{k}\alpha_i + 1\right)\left(\sum_{i=1}^{k}\alpha_i + 2\right)}$$

$$= \frac{3\alpha_i^3\sum_{i=1}^{k}\alpha_i\left(\sum_{i=1}^{k}\alpha_i + 1\right) + \alpha_i^3\left[\left(\sum_{i=1}^{k}\alpha_i\right)^2 + 4\right] - 3\alpha_i^2\sum_{i=1}^{k}\alpha_i\left(1 - \sum_{i=1}^{k}\alpha_i\right) + 2\alpha_i\left(\sum_{i=1}^{k}\alpha_i\right)^2}{\left(\sum_{i=1}^{k}\alpha_i\right)^3\left(\sum_{i=1}^{k}\alpha_i + 1\right)\left(\sum_{i=1}^{k}\alpha_i + 2\right)}$$

$$\therefore G_3\left(1;\frac{-\alpha_i}{\sum_{i=1}^{k}\alpha_i}\right) = \frac{3\alpha_i^3\alpha_0(\alpha_0 + 1) + \alpha_i^3(\alpha_0 + 4) - 3\alpha_i^2\alpha_0(1 - \alpha_0) + 2\alpha_i\alpha_0^2}{\alpha_0^3(\alpha_0 + 1)(\alpha_0 + 2)} \qquad (30)$$

Hence skewness can be determined as

$$Sk(1) = \frac{3\alpha_i^3\alpha_0(\alpha_0 + 1) + \alpha_i^3(\alpha_0 + 4) - 3\alpha_i^2\alpha_0(1 - \alpha_0) + 2\alpha_i\alpha_0^2}{\alpha_0^3(\alpha_0 + 1)(\alpha_0 + 2)} * \frac{\alpha_0^3(\alpha_0 + 1)^{\frac{3}{2}}}{\alpha_i^3(\alpha_0 - \alpha_i)^{\frac{3}{2}}}$$

$$\therefore Sk(1) = \frac{3\alpha_i^3\alpha_0(\alpha_0 + 1) + \alpha_i^3(\alpha_0 + 4) - 3\alpha_i^2\alpha_0(1 - \alpha_i) + 2\alpha_i\alpha_0^2}{(\alpha_i^3\alpha_0^2 + 3\alpha_i^3\alpha_0 + 2\alpha_i^3)}\frac{(\alpha_0 + 1)^{\frac{3}{2}}}{(\alpha_0 - \alpha_i)^{\frac{3}{2}}} \qquad (31)$$

Hence the Dirrichlet distribution is positively skewed since $\alpha_i > 0;\ \forall\, i$.

Now, let $n = 4; c = 1; r = 0, 1, 2, 3, 4$

$r = 0$

$$\binom{4}{0}\lambda^{4-0}\frac{\Gamma\left(\sum_{i=1}^{k}\alpha_i\right)}{\prod_{i=1}^{k}\Gamma(\alpha_i)}\frac{\prod_{i=1}^{k}\Gamma(\alpha_i)}{\Gamma\left(\sum_{i=1}^{k}\alpha_i\right)} = \lambda^4$$

$r = 1$

$$\binom{4}{1}\lambda^{4-1}\frac{\Gamma\left(\sum_{i=1}^{k}\alpha_i\right)}{\prod_{i=1}^{k}\Gamma(\alpha_i)}\frac{\prod_{i=1}^{k}\Gamma(\alpha_i+1)}{\Gamma\left(\sum_{i=1}^{k}\alpha_i+1\right)} = 4\lambda^3\frac{\Gamma\left(\sum_{i=1}^{k}\alpha_i\right)}{\prod_{i=1}^{k}\Gamma(\alpha_i)}\frac{\alpha_i}{\left(\sum_{i=1}^{k}\alpha_i\right)}\frac{\prod_{i=1}^{k}\Gamma(\alpha_i)}{\Gamma\left(\sum_{i=1}^{k}\alpha_i\right)}$$

Hence, for $r = 1$ we have

$$\binom{4}{1}\lambda^{4-1}\frac{\Gamma\left(\sum_{i=1}^{k}\alpha_i\right)}{\prod_{i=1}^{k}\Gamma(\alpha_i)}.\frac{\prod_{i=1}^{k}\Gamma(\alpha_i+1)}{\Gamma\left(\sum_{i=1}^{k}\alpha_i+1\right)} = 4\lambda^3\frac{\alpha_i}{\left(\sum_{i=1}^{k}\alpha_i\right)}$$

$r = 2$

$$\binom{4}{2} = \lambda^{4-2}\frac{\Gamma\left(\sum_{i=1}^{k}\alpha_i\right)}{\prod_{i=1}^{k}\Gamma(\alpha_i)}.\frac{\prod_{i=1}^{k}\Gamma(\alpha_i+2)}{\Gamma\left(\sum_{i=1}^{k}\alpha_i+2\right)} = 6\lambda^2\frac{\Gamma\left(\sum_{i=1}^{k}\alpha_i\right)}{\prod_{i=1}^{k}\Gamma(\alpha_i)}\frac{\alpha_i(\alpha_i+1)\Gamma(\alpha_i)}{\left(\sum_{i=1}^{k}\alpha_i\right)\left(\sum_{i=1}^{k}\alpha_i+1\right)\Gamma\left(\sum_{i=1}^{k}\alpha_i\right)}$$

$$= 6\lambda^2\frac{\alpha_i}{\left(\sum_{i=1}^{k}\alpha_i\right)}\frac{(\alpha_i+1)}{\left(\sum_{i=1}^{k}\alpha_i+1\right)}$$

$r = 3$

$$\binom{4}{3}\lambda^{4-3}\frac{\Gamma\left(\sum_{i=1}^{k}\alpha_i\right)}{\prod_{i=1}^{k}\Gamma(\alpha_i)}\frac{\prod_{i=1}^{k}\Gamma(\alpha_i+3)}{\Gamma\left(\sum_{i=1}^{k}\alpha_i+3\right)}$$

$$= 4\lambda\frac{\Gamma\left(\sum_{i=1}^{k}\alpha_i\right)}{\prod_{i=1}^{k}\Gamma(\alpha_i)}\frac{\alpha_i(\alpha_i+1)(\alpha_i+2)\prod_{i=1}^{k}\Gamma(\alpha_i)}{\left(\sum_{i=1}^{k}\alpha_i\right)\left(\sum_{i=1}^{k}\alpha_i+1\right)\left(\sum_{i=1}^{k}\alpha_i+2\right)\Gamma\left(\sum_{i=1}^{k}\alpha_i\right)}$$

$$= 4\lambda\frac{\alpha_i}{\left(\sum_{i=1}^{k}\alpha_i\right)}\frac{(\alpha_i+1)(\alpha_i+2)}{\left(\sum_{i=1}^{k}\alpha_i+1\right)\left(\sum_{i=1}^{k}\alpha_i+2\right)}$$

$r = 4$

$$\binom{4}{4}\lambda^{4-4}\frac{\Gamma\left(\sum_{i=1}^{k}\alpha_i\right)}{\prod_{i=1}^{k}\Gamma(\alpha_i)}\frac{\prod_{i=1}^{k}\Gamma(\alpha_i+4)}{\Gamma\left(\sum_{i=1}^{k}\alpha_i+4\right)}$$

$$= \frac{\Gamma\left(\sum_{i=1}^{k}\alpha_i\right)}{\prod_{i=1}^{k}\Gamma(\alpha_i)}\frac{\alpha_i(\alpha_i+1)(\alpha_i+2)(\alpha_i+3)\prod_{i=1}^{k}\Gamma(\alpha_i)}{\left(\sum_{i=1}^{k}\alpha_i\right)\left(\sum_{i=1}^{k}\alpha_i+1\right)\left(\sum_{i=1}^{k}\alpha_i+2\right)\left(\sum_{i=1}^{k}\alpha_i+3\right)\Gamma\left(\sum_{i=1}^{k}\alpha_i\right)}$$

$$= \frac{\alpha_i(\alpha_i + 1)(\alpha_i + 2)(\alpha_i + 3)}{\left(\sum_{i=1}^{k} \alpha_i\right)\left(\sum_{i=1}^{k} \alpha_i + 1\right)\left(\sum_{i=1}^{k} \alpha_i + 2\right)\left(\sum_{i=1}^{k} \alpha_i + 3\right)}$$

Hence,

$$G_4(1;\lambda) = \lambda^4 + 4\lambda^3 \frac{\alpha_i}{\sum_{i=1}^{k} \alpha_i} + 6\lambda^2 \frac{\alpha_i(\alpha_i + 1)}{\left(\sum_{i=1}^{k} \alpha_i\right)\left(\sum_{i=1}^{k} \alpha_i + 1\right)}$$

$$+4\lambda \frac{\alpha_i(\alpha_i + 1)(\alpha_i + 2)}{\left(\sum_{i=1}^{k} \alpha_i\right)\left(\sum_{i=1}^{k} \alpha_i + 1\right)\left(\sum_{i=1}^{k} \alpha_i + 2\right)}$$
$$+ \frac{\alpha_i(\alpha_i + 1)(\alpha_i + 2)(\alpha_i + 3)}{\left(\sum_{i=1}^{k} \alpha_i\right)\left(\sum_{i=1}^{k} \alpha_i + 1\right)\left(\sum_{i=1}^{k} \alpha_i + 2\right)\left(\sum_{i=1}^{k} \alpha_i + 3\right)} \quad (32)$$

Now, let $\lambda = \frac{-\alpha_i}{\sum_{i=1}^{k} \alpha_i}$, hence the fourth central moment becomes

$$G_4\left(1;\frac{-\alpha_i}{\sum_{i=1}^{k} \alpha_i}\right) = \frac{\alpha_i^4}{\left(\sum_{i=1}^{k} \alpha_i\right)^4} - 4\frac{\alpha_i^3}{\left(\sum_{i=1}^{k} \alpha_i\right)^3}\frac{\alpha_i}{\left(\sum_{i=1}^{k} \alpha_i\right)} + 6\frac{\alpha_i^2}{\left(\sum_{i=1}^{k} \alpha_i\right)^2}\frac{\alpha_i(\alpha_i + 1)}{\left(\sum_{i=1}^{k} \alpha_i\right)\left(\sum_{i=1}^{k} \alpha_i + 1\right)}$$
$$- 4\frac{\alpha_i}{\sum_{i=1}^{k} \alpha_i}\frac{\alpha_i(\alpha_i + 1)(\alpha_i + 2)}{\left(\sum_{i=1}^{k} \alpha_i\right)\left(\sum_{i=1}^{k} \alpha_i + 1\right)\left(\sum_{i=1}^{k} \alpha_i + 2\right)}$$
$$+ \frac{\alpha_i(\alpha_i + 1)(\alpha_i + 2)(\alpha_i + 3)}{\left(\sum_{i=1}^{k} \alpha_i\right)\left(\sum_{i=1}^{k} \alpha_i + 1\right)\left(\sum_{i=1}^{k} \alpha_i + 2\right)\left(\sum_{i=1}^{k} \alpha_i + 3\right)}$$

$$= \frac{\alpha_i}{\sum_{i=1}^{k} \alpha_i}\left\{\frac{6\alpha_i^3 \sum_{i=1}^{k} \alpha_i + 6\alpha_i^2 \sum_{i=1}^{k} \alpha_i - 3\alpha_i^3 \sum_{i=1}^{k} \alpha_i - 3\alpha_i^3}{\left(\sum_{i=1}^{k} \alpha_i\right)^3\left(\sum_{i=1}^{k} \alpha_i + 1\right)}\right.$$
$$\left. + \frac{\left(\sum_{i=1}^{k} \alpha_i\right)(6 - 3\alpha_i^3 - 6\alpha_i^2 - 21\alpha_i) - 12\alpha_i^2(\alpha_i - 4)}{\left(\sum_{i=1}^{k} \alpha_i\right)\left(\sum_{i=1}^{k} \alpha_i + 1\right)\left(\sum_{i=1}^{k} \alpha_i + 2\right)\left(\sum_{i=1}^{k} \alpha_i + 3\right)}\right\}$$

$$\therefore G_4\left(1;\frac{-\alpha_i}{\sum_{i=1}^{k} \alpha_i}\right)$$
$$= \frac{\alpha_i}{\alpha_0}\left\{\frac{(\alpha_0 + 2)(\alpha_0 + 3)(3\alpha_i^3\alpha_0 + 6\alpha_i^2\alpha_0 - 3\alpha_i^3)}{\alpha_0^3(\alpha_0 + 1)(\alpha_0 + 2)(\alpha_0 + 3)}\right.$$
$$\left. + \frac{\alpha_0^2[\alpha_0(6 - 3\alpha_i^3 - 6\alpha_i^2 - 21\alpha_i)] - 12\alpha_i^2(\alpha_i + 4)}{\alpha_0^3(\alpha_0 + 1)(\alpha_0 + 2)(\alpha_0 + 3)}\right\} \quad (33)$$

Hence the kurtosis of the distribution may be obtained as

$$ku(1) = \left\{\frac{(\alpha_0 + 2)(\alpha_0 + 3)(3\alpha_i^3\alpha_0 + 6\alpha_i^2\alpha_0 - 3\alpha_i^3)}{\alpha_0^3(\alpha_0 + 1)(\alpha_0 + 2)(\alpha_0 + 3)} + \frac{\alpha_0^2[\alpha_0(6 - 3\alpha_i^3 - 6\alpha_i^2 - 21\alpha_i)] - 12\alpha_i^2(\alpha_i + 4)}{(\alpha_0 + 1)(\alpha_0 + 2)(\alpha_0 + 3)}\right\} \times \frac{(\alpha_0 + 1)^2}{\alpha_i(\alpha_0 - \alpha_0)^2}$$

$$\therefore ku(1) = \frac{6\alpha_i^2\alpha_0^3 + 3\alpha_i^3\alpha_0^2 + 18\alpha_i^3\alpha_0 + 39\alpha_i^2\alpha_0 + 24\alpha_0^3 + 15\alpha_0^4 - 66\alpha_i^2\alpha_0^2 + 3\alpha_0^5}{(\alpha_0^3 + 6\alpha_0^2 + 11\alpha_0 + 6)} \times \frac{(\alpha_0 + 1)^2}{\alpha_i(\alpha_0 - \alpha_i)^2} \qquad (34)$$

The value of $ku(1)$ is positive and may be less than, equal to or greater than 3 depending on the values of $\alpha_i \forall\ i = 1, 2, \dots, k$ where $\alpha_0 = \sum_{i=1}^{k} \alpha_i$.

6. DISCUSSION OF RESULTS/CONCLUSION

The Generalized Multivariate Moment Generating Functions for some Continuous Multivariate Probability Distributions were successfully developed and presented in this paper. The functions were found to generate the central and non-central moments in the same easy way; no further calculus is required to evaluate moments as is the case with the traditional/conventional methods; the method gives room for the manipulation of the powers of random variable to accommodate real and negative powers; the method gives similar results as its traditional counterparts and gives additional information in the area of the dimension of distributions. Most importantly, the method is easy to apply.

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