On the Estimation of Stress Strength Reliability Parameter of Weibull-Rayleigh Distribution

Abstract

The reliability of a component depends on the stress conditions of the operating environment, which are uncertainty and should be modeled as random. This article deals with the estimation of the stress-strength reliability parameter of Weibull-Rayleigh distribution. Let X and Y be two independent random variables, where X and Y follow Weibull-Rayleigh distribution. The maximum likelihood estimator and the approximate maximum likelihood estimator of the stress-strength reliability are obtained. Other properties of the Weibull-Rayleigh distributions are derived. Two real data applications are given for showing the flexibility of the Weibull-Rayleigh stress-strength reliability.

Keywords: Maximum Likelihood Estimation, Reliability; Weibull-Rayleigh; Stress-Strength; Uncertainty

1.0 Introduction

Reliability is defined as the ability of a system or component to perform its required functions under stated conditions for a specified period (IEEE, 1990). With an increase in the demand placed on the reliability of systems, it becomes even more important to ensure a clear match between the results provided by methods and techniques used in reliability estimation and the actual nature of the failure. This, however, is not an easy task; on the other hand, there is a requirement to conduct the estimation with minimal cost and use of resources. On the other hand, over-simplification of the estimation process in terms of the underlying mathematical model or limited data results in a mismatch that makes the whole study meaningless (Gertsbakh and Kordonskiy; 1969). Most of the investigation in reliability estimation comes from a methodology known as lifetime data analysis. The data recorded is then used to construct a ‘time-to-failure’ distribution, using Maximum Likelihood Estimation (MLE) method, which makes it possible to estimate parameters that describe the distribution.

The stress-strength model is one that is used to compute reliability. It has been known in the mechanical as follows, the stress is the mechanical loads and forces, while the strength in the physical effort that can resist the loads to perform its required function when the stress exceeds the strength, the failures occur. If X represents the strength and Y represents the stress, the main theme of statisticians is to estimate the probability of failure or reliability of this model. Since the reliability concept is general, so the stress strength model can be applied in different fields outside of the scope of mechanics (Saracoglu *et al*., 2012). Although the model is very simple it's largely applicable in fields of reliability, engineering, manufacturing etc. Since its emergence, various researchers have produced some research articles on different statistical distributions based on this model. For recent references see, Gunasekera (2015), Wang et al.(2018), Sharma (2018), Cetinkaya and Genc (2019), and Bai et al. (2019).

The Stress-Strength reliability of a system defines the probability that the system will function properly until the strength exceeds stress. Due to the manufacturing variability and uncertain factors, the strength of the system varies and also when the system is put to use, it is subjected to the stress which is again random in nature. These manufacturing variables and uncertain factors can be used material, production style, humidity, temperature of the environment etc. The genesis of this problem can be seen in Birnbaum et al. (1956). Later, Birnbaum and Mc-Carty (1958) studied statistical properties of the model. Although the model is very simple in nature but it is largely applicable in fields of reliability, engineering, manufacturing etc. since its emergence, various researches have produced research article on different statistical distributions based on this model. For recent references see, Gumasekera (2015), Wang et al. (2018), Sharma (2018), Certinkaya and Genc (2019) and Bai et al. (2019). In probability and statistics, different convoluted distributions have been generated but only a few have been used for further study especially in the area of reliability modelling. More so, many known convoluted continuous probability distributions stress strength reliability parameter using MLE have not been estimated.

In this research, we will make use of Weibull-Rayleigh distribution (WRD) proposed by Akarawak, et al. (2013) and further derive some characterizations of WRD. Moreover, determine their reliability function of stress strength and also estimate their parameters using the MLE method. This research therefore aimed at analyzing the stress-strength of WRD and its application to health data. We derived some characterizations of Weibull-Rayleigh distribution, determined the reliability function of its stress-strength, estimate the parameters of the reliability function of stress-strength using the MLE method, apply the stress-strength analysis of WRD to real dataset, and compare the results of WRD stress-strength model to that of Weibull and Rayleigh distributions. The study will set the pace for applied statisticians, econometricians, and other users of statistics especially in the area of modelling to apply the WRD reliability function of stress strength and estimate the parameters of the reliability function of stress-strength using MLE. The WRD Stress-Strength model to fit data that are not normally distributed and where Weibull or Rayleigh distributions fail to fit the data appropriately. Many other derivations, characterizations, and applications of the WRD.

2.0 Materials and Method

2.1 Stress-Strength Analysis

Stress-Strength analysis has been used in mechanical component design. The probability of failure is based on the probability of the stress exceeding strength. The following equation is used to calculate the expected probability of failure, F:

$$F = P[\text{Stress} > \text{Strength}] = \int_{-\infty}^{+\infty} f_y(y)\left[\int_{y}^{+\infty} f_x(x)dx\right]dy$$

The expected probability of success or the expected reliability, *R*, is calculated as:

$$R = P[\text{Stress} < \text{Strength}] = \int_{-\infty}^{+\infty} f_y(y)\left[\int_{-\infty}^{y} f_x(x)dx\right]dy$$

The equations above assume that both stress and strength are in the positive domain. For general cases, the expected reliability can be calculated using the following equation:

$$R = P(X_1 < X_2) = \iint_{(-\infty)}^{x} f(x_1, x_2)dx_1dx_2$$

2.1.1 Stress-Strength Model

Let *X* and *Y* are the independent random variables. Then the stress-strength reliability R is calculated as

$$\begin{aligned} R &= P(X > Y) \\ &= \int_0^\infty \int_0^x f(x,y)dy\,dx \\ &= \int_0^\infty \left[\int_0^x f_Y(y)dy\right] f_X(x)\,dx \\ &= \int_0^\infty F_Y(x)f_X(x)\,dx \end{aligned}$$

The estimation of stress-strength parameter plays an important role in the reliability analysis. For example, if *X* is the strength of a system which is subjected to stress *Y*, then the parameter *R* measures the system performance which is frequently used in the context of mechanical reliability of a system.

2.1.2 Variation in Model Parameters

If both the stress and strength distributions are estimated from data sets, then there are uncertainties associated with the estimated distribution parameters. These uncertainties will cause some degree of variation of the probability calculated from the stress-strength analysis. Therefore, we can use these uncertainties to estimate the confidence intervals on the calculated probability. To get the confidence intervals, we first calculate the variance of the reliability based on Taylor expansion by ignoring the

2^{nd} order term. *R*, the approximation for the variance is Variance of *f(x)* and $R_2(x)$ can be estimated from the Fisher Information Matrix. Once the variance of the expected reliability is obtained, the two-sided confidence intervals can be calculated using

Where
CL is the confidence level
α is *1- CL*
$\omega = exp\ z_{1-}\ \alpha \sqrt{Var(R)}/R(1- R)\}$
$z_{1-}\ \alpha$ is the $1- {}^{\alpha}/_{2}$ percentile of a standard normal distribution.

If the upper bound *(U)* and lower bound *(L)* are not infinite and zero *(0)*, respectively, then the calculated variance of *R* is the adjusted by $(1 / F_1(U) – F_1(L))^2$

Assume the distributions for stress and strength are known. For the stress-strength equation

We know that the calculated reliability is the expected value of the probability that a strength value is larger than a stress value. Since both strength and stress are random variables from their distributions, the reliability is also a random variable.

2.2 Developing WRD Stress-Strength Reliability Model

In probability and statistics, the Weibull distribution is one of the most important continuous probability distributions, it was, first introduced by W. Weibull in 1939 when he was studying the issue of structural strength and life data analysis, and was formally named after him later in 1951. Due to the result of research conducted by Gnedenko (1943), no matter what the original distribution of the variable is, the asymptotic distribution of the minimum could only be three different forms. The Weibull distribution is one of them. Since Weibull distribution is established on the weakest link model, which could sufficiently reflect the defect of material and the effects of stress concentration, it has been considered as appropriate model to describe strength of fiber material in practical application. The Weibull distribution is a very flexible life distribution model with two parameters.

The pdf of Weibull distribution is given as:

$$f(x) = \begin{cases} \frac{\beta}{\alpha}\left(\frac{X}{\alpha}\right)^{\beta-1}\ e^{-\left(\frac{X}{\alpha}\right)^{\beta}}, & x \geq 0 \\ 0 & x < 0 \end{cases}$$

The parameters α, and β are positive constant, i.e. α, β > 0.
The Weibull distribution function, $f(x)$ is reduced to Exponential distribution when β = 1.
The function $f(x)$, increases when β > 1 and decreases when β < 1.
Also, the Rayleigh distribution named for William Strutt, Lord Rayleigh, is the distribution of the magnitude of two-dimensional random vector whose coordinates are independent, identically distributed, mean zero (0) normal variables.
In probability theory and statistic, the Rayleigh distribution is a continuous probability distribution for positive-valued random variable.

$$f(x) = \frac{2x}{a} exp\left(\frac{x^2}{a}\right), a > 0, 0 < x < \infty .$$

The combination of Weibull and Rayleigh distribution using the T-X transformation technique developed by Alzaatreh *et al.* (2013) produced a convoluted distribution called Weibull_rayleigh distributionproposed by Akarawak *et al.* (2013).

2.2.1 Properties of Weibull-Rayleigh Distribution (WRD).

Akarawak et al (2013) proposed WRD and derived the pdf *f(x)* and cdf *F(x)* of the distribution respectively as:

$$f(x) = \frac{2a}{x}\left\{\frac{x^2}{k}\right\}^a \exp\left\{-\left[\frac{x^2}{k}\right]^a\right\}, x > 0, a, k > 0; \quad (1)$$

$$F(x) = 1 - e^{-\left[\frac{x^2}{k}\right]^a} \quad (2)$$

where a is the shape parameter and k is the scale parameter. Equations (1) and (2) are the pdf and cdf of WRD.

The Survival and Hazard functions are respectively given in equations (3) and (4).

$$S(x) = e^{-\left(\frac{x^2}{K}\right)^a} \quad (3)$$

and

$$h(x) = \frac{2a}{x}\left(\frac{x^2}{K}\right)^a = \frac{2ax^{2a-1}}{K^a} \quad (4)$$

The moment of WRD is displayed in equation (5)

$$E(X_r') = K^{\frac{r}{2}}\Gamma\left(\frac{r}{2a} + 1\right) \quad (5)$$

While the moment generating function is given in equation (6)

$$M_X(t) = \sum_{r=0}^{\infty} K^{\frac{r}{2}}\Gamma\left(\frac{r}{2a} + 1\right) \quad (6)$$

and the Shannon Entropy (*SE*) of WRD is given in equation (7).

$$\text{SE} = \text{E}\log(x) - \log(2a) - a\log[\text{E}(x^2)] + a\log(\text{k}) - \text{E}\left(\frac{X^2}{K}\right)^2 \quad (7)$$

2.2.2 Further Properties of WRD Derived

In this section, further statistical properties of WRD that were not derived by Akarawak et al.(2013) are derived.

Cumulative Hazard Function of WRD

The cumulative hazard function of WRD would be derived.

By definition, the cumulative hazard function of a probability distribution is given by

$$H(x) = -\log[1 - F_x(x)] \quad (8)$$

Substitute the cdf of WRD in equation (2) into equation (8) to have

$$H(x) = -log\left\{1 - (1 - e^{-\left(\frac{x^2}{K}\right)^a})\right\} \quad (9)$$

Solve equation (9) further to have equation (10)

$$H(x) = -Log\left\{e^{-\left[\frac{x^2}{K}\right]^a}\right\} \quad (10)$$

$$H(x) = \left[\frac{x^2}{K}\right]^a \quad (11)$$

Thus, equation (11) provides the cumulative hazard function of WRD.

Quantile Function $Q_{(p)}$ of WRD

From equation (2) which is the cdf of WRD, we have

$$F_X(x) = 1 - e^{-\left[\frac{x^2}{K}\right]^a} \quad (12)$$

$$e^{-\left[\frac{x^2}{K}\right]^a} = 1 - F_X(x)$$

Take log of both sides

$$-\left[\frac{x^2}{K}\right]^a = \log\left[1 - F_X(x)\right]$$

$$x = \{-k^a \log\left[1 - F_X(x)\right]\}^{1/2a}$$

Letting $p = F_X(x)$, then

$$x = \{-k^a \log(1-p)\}^{1/2a}$$

$$Q_{(p)} = \{-k^a \log(1-p)\}^{1/2a} \quad (13)$$

Equation (13) is the quantile function of WRD, where p is uniformly generated in the closed interval [0, 1]. The first, second and third quartiles of WRD would be derived by letting p = 0.25, 0.5 and 0.75 respectively.

Consequently, the first quantile is given by:

$$Q_{(0.5)} = (0.3010k^a)^{1/2a} \quad (14)$$

The second quantile, which is the median, is given by:

$$Q_{(0.25)} = (0.1250k^a)^{1/2a} \quad (15)$$

and the third quantile is given by

$$Q_{(0.75)} = (0.6021k^a)^{1/2a} \quad (16)$$

Skewness and Kurtosis of WRD

To obtain the skewness and kurtosis of WRD, we employ the formular of Galton(1883) and Moors(1988).

Accordingly, the Galtons Skewness (S_K) and Moors' Kurtosis (K) respectively are given by:

$$S_K = \frac{Q_{0.75} - 2Q_{0.5} + Q_{0.25}}{Q_{0.75} - Q_{0.25}} \quad (17)$$

$$K = \frac{Q_{0.875} - Q_{0.625} - Q_{0.375} + Q_{0.125}}{Q_{0.375} - Q_{0.25}} \quad (18)$$

For p = 0.125, we have

$$Q_{(0.125)} = \left(-k^a \log(1-0.125)\right)^{1/2a}$$
$$Q_{(0.125)} = (0.0580k^a)^{1/2a} \tag{19}$$
For p = 0.375, we have
$$Q_{(0.375)} = \left(-k^a \log(1-0.375)\right)^{1/2a}$$
$$Q_{(0.375)} = (0.2041k^a)^{1/2a} \tag{20}$$
For p = 0.625, we have
$$Q_{(0.625)} = \left(-k^a \log(1-0.625)\right)^{1/2a}$$
$$Q_{(0.625)} = (0.4260k^a)^{1/2a} \tag{21}$$
And also for p = 0.875, we have
$$Q_{(0.875)} = \left(-k^a \log(1-0.875)\right)^{1/2a}$$
$$Q_{(0.875)} = (0.9031k^a)^{1/2a} \tag{22}$$
$$S_K = \frac{(0.6021k^a)^{1/2a} - 2(0.3010k^a)^{1/2a} + (0.1250k^a)^{1/2a}}{(0.6021\mathrm{K}^a)^{1/2a} - (0.1250k^a)^{1/2a}} \tag{23}$$
Equation (23) is the skewness of WRD
$$K = \frac{Q_{0.875} - Q_{0.625} - Q_{0.375} + Q_{0.125}}{Q_{0.375} - Q_{0.25}}$$
$$K = \frac{(0.9031k^a)^{1/2a} - (0.4260k^a)^{1/2a} - (0.2041k^a)^{1/2a} + (0.0580k^a)^{1/2a}}{(0.2041k^a)^{1/2a} - (0.1250k^a)^{1/2a}} \tag{24}$$
Equation (24) is the Kurtosis of WRD

The Characteristics Function of WRD

The characteristics function has many useful and important properties which give it a central role in statistical theory. Its approach is particularly useful in analysis of linear combination of independent random variables. A representation for the characteristics function is given by

$$\varphi_x(t) = E\left[e^{itx}\right] = \int_0^\infty e^{itx} f(x)dx \tag{25}$$

From (1) let $y = x^{2a}$

$$y^{1/a} = x^2$$

$$\frac{1}{a} y^{\frac{1}{a}-1} \frac{dy}{dx} = 2x$$

$$dx = \frac{\frac{1}{a} y^{\frac{1}{a}-1}}{2x} = \frac{y^{\frac{1}{a}-1}}{2ax}$$

Now, we have

$$\varphi_y(t) = E\left[e^{ity}\right] = \int_0^{\infty} e^{ity} \cdot \frac{2a}{x}\frac{y}{K^a} e^{-\frac{y}{k^a}} \cdot \frac{y^{\frac{1}{a}-1}}{2ax} dy \tag{26}$$

$$= \frac{1}{K^a}\int_0^{\infty} e^{-\frac{y}{k^a}+ ity}\ dy = \frac{1}{K^a}\int_0^{\infty} e^{-\left(\frac{1}{K^a}-it\right)y}\ dy$$

Let $\frac{\Gamma(\alpha)}{\beta^{\alpha}} = \int y^{\alpha-1}e^{-\beta y}\, dy$ (27)

$$\alpha - 1 = 0 \Rightarrow \alpha = 1 \text{ and } \beta = \frac{1}{K^a} - it$$

$$\varphi_y(t) = E\left(e^{ity}\right) = \frac{1}{k^a} \cdot \frac{\Gamma(1)}{\left(\frac{1}{k^a}- it\right)} = \frac{1}{k^a\left(\frac{1}{k^a}- it\right)} = \frac{1}{1-itk^a} \tag{28}$$

Equation (28) is the characteristics function of WRD

Mode of WRD

From equation (1) which is the *pdf* of WRD.

$$f(x) = \frac{2a}{x}\left\{\frac{x^2}{k}\right\}^a\ exp\left\{-\left[\frac{x^2}{k}\right]^2\right\}$$

Differentiating the pdf with respect to x gives

$$f'(x) = -\frac{2a}{x^2}\left\{\frac{X^2}{k}\right\}^a exp\left\{-\left(\frac{x^2}{k}\right)\right\}^a + 2ax\left\{\frac{x^2}{k}\right\}^{a-1} \exp\left\{-\left(\frac{x^2}{k}\right)\right\}^a + 4a^2\left\{\frac{x^2}{k}\right\}^{2a-1} \exp\left\{-\left(\frac{x^2}{k}\right)\right\}^a \tag{29}$$

Equate $f'(x) = 0$, gives:

$$0 = \frac{2a}{x^2}\left\{\frac{X^2}{k}\right\}^a exp\left\{-\left(\frac{x^2}{k}\right)\right\}^a + 2ax\left\{\frac{x^2}{k}\right\}^{a-1} \exp\left\{-\left(\frac{x^2}{k}\right)\right\}^a + 4a^2\left\{\frac{x^2}{k}\right\}^{2a-1} \exp\left\{-\left(\frac{x^2}{k}\right)\right\}^a \tag{30}$$

Multiply both side by x^2

$$2ax^3\left\{\frac{x^2}{k}\right\}^{a-1} \exp\left\{-\left(\frac{x^2}{k}\right)\right\}^a + 4ax^2\left\{\frac{x^2}{k}\right\}^{2a-1} \exp\left\{-\left(\frac{x^2}{k}\right)\right\}^a = 0$$

$$2aX^3\left\{\frac{X^2}{K}\right\}^{a-1} \exp\left\{-\left(\frac{X^2}{K}\right)\right\}^a = 2a\left\{\frac{x^2}{k}\right\}^a exp\left\{-\left(\frac{x^2}{k}\right)\right\}^a - 4ax^2\left\{\frac{x^2}{k}\right\}^{2a-1} \exp\left\{-\left(\frac{x^2}{k}\right)\right\}^a$$

Divide both sides by $2a\left\{\frac{x^2}{k}\right\}^{a-1} exp\left\{-\left(\frac{x^2}{k}\right)\right\}^a$, we have

$$x^3 = \frac{2a\left\{\frac{x^2}{k}\right\}^a exp\left\{-\left(\frac{x^2}{k}\right)\right\}^a - 4ax^2\left\{\frac{x^2}{k}\right\}^{2a-1} \exp\left\{-\left(\frac{x^2}{k}\right)\right\}^a}{2a\left\{\frac{x^2}{k}\right\}^{a-1} exp\left\{-\left(\frac{x^2}{k}\right)\right\}^a}$$

$$x = \sqrt[3]{\frac{2a\left\{\frac{x^2}{k}\right\}^a exp\left\{-\left(\frac{x^2}{k}\right)\right\}^2 - 4ax^2\left\{\frac{x^2}{k}\right\}^{2a-1} \exp\left\{-\left(\frac{x^2}{k}\right)\right\}^a}{2a\left\{\frac{x^2}{k}\right\}^{a-1} exp\left\{-\left(\frac{x^2}{k}\right)\right\}^a}} \tag{31}$$

Equation (31) is a mode of WRD.

2.3 Weibull Distribution Stress-Strength Model

Let X be the strength of the Component which is subjected to Stress Y with parameter b and k, respectively. The probability density function and cumulative distribution function of X and Y is given by:

$f_X(x) = b_2 k_2 x^{k-1} e^{-b_2 x^{k_2}}$ (32)

$F_Y(x) = 1 - e^{-b_1 x^{k_1}}$ (33)

The Stress-Strength reliability R can be expressed as

R = P(X >Y) (34)

$= \int_0^\infty \int_0^x f(x,y) dy\, dx$ (35)

$= \int_0^\infty \left[\int_0^x f_Y(y) dy\right] f_X(x)\, dx$

$= \int_0^\infty F_Y(x) f_X(x)\, dx$

$R = \int_0^\infty \left(1 - e^{-b_1 X^{K_1}}\right)\left(b_2 k_2 X^{K_2-1} e^{-b_2 X^{K_2}}\right) dx$

$R = b_2 k_2 \{\int_0^\infty X^{K_2-1} e^{-b_2 X^{K_2}} dx - \int_0^\infty x^{k-1}\, e^{-\left(b_1 X^{K_1} + b_2 X^{K_2}\right)} dx$

For $\int_0^\infty x^{k_2-1} e^{-b_2 X^{k_2}} dx$ (36)

Let U = $x^{k_2} \Rightarrow du/_{dx} = k_2 x^{k_2-1}$

$\Rightarrow dx = \frac{du}{k_2 x^{k_2-1}}$

We have $\int_0^\infty x^{k_2-1} e^{-b_2 U} \cdot \frac{du}{k_2 x^{k_2-1}}$ (37)

$\frac{1}{k_2} \int_0^\infty e^{-b_2 U}\, du$ (38)

From Gamma distribution $\frac{\Gamma(\alpha)}{\beta^\alpha} = \int_0^\infty U^{\alpha-1} e^{-\beta u}\, du$ (39)

Compare equation (38) and (39) together we have

$\alpha - 1 = 0 \Rightarrow \alpha = 1$ and $\beta = -b_2$

$\frac{\Gamma(\alpha)}{\beta^\alpha} = \frac{\Gamma(1)}{(-b_2)^1} = \frac{1}{-b_2}$

Implies that,

$\int_0^\infty x^{k_2-1} e^{-b_2 X^{k_2}}\, dx = \frac{1}{b_2 k_2}$ (40)

Substituting equation (40) into (35) gives

$R = b_2 K_2 \left\{\frac{1}{b_2 K_2} - \int_0^\infty X^{K_2-1}\, e^{-\left(b_1 X^{K_1} + b_2 X^{K_2}\right)} dx\right.$ (41)

$R = 1 - b_2 K_2 \int_0^\infty X^{K_2-1}\, e^{-\left(b_1 X^{K_1} + b_2 X^{K_2}\right)} dx$ (42)

From the above, we observed that parameter R is the function of parameters b_1, b_2, k_1 and k_2. Therefore, for maximum likelihood estimate (MLE) of R, we need to obtain the MLEs of b_1, b_2, k_1 and k_2.

Using invariance principle

If $k_1 = k_2$, we have

From $\int_0^\infty x^{k-1}\, e^{-\left(b_1 x^{k_1} + b_2 x^{k_2}\right)} dx$ (43)

$\int_0^\infty x^{k_2-1}\, e^{-(b_1x^{k_1}+b_2x^{k_2})} dx$

Let v = $x^{k_2} \Rightarrow {dv}/{dx} = k_2x^{k_2-1}$

$dx = \frac{dv}{k_2x^{k_2-1}}$

Which implies that,

$\int_0^\infty x^{k_2-1}\, e^{-(b_1+b_2)\,V} \cdot \frac{dv}{k_2x^{k_2-1}}$ (44)

$\frac{1}{K_2}\int_0^\infty e^{-(b_1+b_2)\,V}\, dv$ (45)

From Gamma function $\frac{\Gamma(\alpha)}{\beta^\alpha} = \int_0^\infty V^{\alpha-1}\, e^{-\beta V}\, dv$ (46)

$\alpha - 1 = 0 \Rightarrow \alpha = 1$

$\beta = b_1 + b_2$

$$\frac{\Gamma(\alpha)}{\beta^\alpha} = \frac{\Gamma(1)}{(b_1 + b_2)} = \frac{1}{b_1 + b_2}$$

$\frac{1}{K_2}\int_0^\infty e^{-(b_1+b_2)\,V}\, dv = \frac{1}{K_2(b_1+b_2)}$ (47)

Then

$\int_0^\infty X^{K_2-1}\, e^{-(b_1+b_2)\,X^{K_2}}\, dx = \frac{1}{K_2(b_1+b_2)}$ (48)

Substituting equation (48) into equation (45) gives

$R_1 = 1 - \frac{b_2}{b_1+b_2} = \frac{b_1}{b_1+b_2}$ (49)

Maximum Likelihood Estimation for R of Weibull Distribution

The main aim of this section is to derive the MLE of *R* and *R₁* in (70) and (78)

Now let $x_1, x_2, \dots, x_n$ and $y_1, y_2, \dots, y_m$ are two independent observations from WD (b_1, k_1) and WD (b_2, k_2), respectively. Then, the log – likelihood function of $b_1, b_2, k_1 and\ k_2$ is given by

l= L (b, k, xi, yj) *i* = 1,2, …,n, *j*=1,2,…,m

$= \prod_{i=1}^{n} f(x_i)\ \prod_{j=1}^{m} g(y_j)$ (50)

$= \prod_{i=1}^{n} b_i x^{k_i-1} e^{-b_i \sum_{i=1}^{n} x_i^{k_i}}\ \prod_{j=1}^{m} b_2 x^{k_2-1} e^{-b_2 \sum_{j=1}^{n} x_i^{k_i}}$ (51)

Taking the Logarithm gives

$Ln(L) = nIn(b_1) + nIn(k_1) + (k_1 - 1)\sum_{i=1}^{n} In(x_i) - b_1x^{k_1} + mIn\,(b_2) + mIn\,(k_2) + (k_2 - 1)\sum_{j=1}^{m} In(y_j) - b_2y^{k_2}$ (52)

Differentiating with respect *b₁*, *b₂*, *k₁* and *k₂* gives the following system of equations

$\frac{\delta \text{Ln(L)}}{\delta b_1} = \frac{n}{b_1} \sum_{i=1}^{n} x_1{}^{k_1} = 0$ (53)

$\frac{n}{b_1} = \sum_{i=1}^{n} x_1{}^{k_1} = 0$

$$\hat{b}_1 = \frac{\sum_{i=1}^{n} x_1{}^{k_1}}{n}$$

$$\frac{\delta \mathrm{Ln(L)}}{\delta \mathrm{b}_2} = \frac{m}{\mathrm{b}_2} - \sum_{j=1}^{m} y_j{}^{k_2} = 0 \quad (54)$$

$$\frac{m}{\mathrm{b}_2} = \sum_{j=1}^{m} y_j{}^{k_2} = 0$$

$$\hat{b}_2 = \frac{\sum_{j=1}^{m} y_j{}^{k_2}}{m}$$

$$\frac{\delta \mathrm{Ln(L)}}{\delta \mathrm{k}_1} = \frac{n}{\mathrm{k}_1} + \sum_{i=1}^{n} In(x_i) - b_i x^{k_1} In(x_i) = 0 \quad (55)$$

$$b_i x^{k_1} In(x_i) - \frac{n}{\mathrm{k}_1} = \sum_{i=1}^{n} In(x_i)$$

$$b_i \mathrm{k}_1 x^{k_1} In(x_i) - \mathrm{k}_1 \sum_{i=1}^{n} In(x_i) = \mathrm{n}$$

$$\frac{\delta \mathrm{Ln(L)}}{\delta \mathrm{k}_2} = \frac{m}{\mathrm{k}_2} + \sum_{j=1}^{m} In(y_j) - b_2 y^{k_2} In(y_j) = 0 \quad (56)$$

$$b_2 y^{k_2} In(y_j) - \frac{m}{\mathrm{k}_2} = \sum_{j=1}^{m} In(y_j)$$

$$b_2 y^{k_2} In(y_j) + \mathrm{k}_2 \sum_{j=1}^{m} In(y_j) = m$$

Since the results of the equation (55) and (56) are not in closed form R package will be used to estimate $\hat{k}_1$ and k_2

Hence, using the invariance properties of MLEs, then MLE of the parameters R and R_1 are given by

$$\hat{R} = 1 - \hat{\mathrm{b}}_2 \hat{\mathrm{k}}_2 \int_0^\infty \mathrm{X}^{\hat{\mathrm{K}}_2 - 1} \, \mathrm{e}^{-\left(\hat{\mathrm{b}}_1 \mathrm{X}^{\hat{\mathrm{K}}_1} + \hat{\mathrm{b}}_2 \mathrm{X}^{\hat{\mathrm{K}}_2}\right)} dx \quad (57)$$

$$\hat{R}_1 = \frac{\hat{\mathrm{b}}_1}{\hat{\mathrm{b}}_1 + \hat{\mathrm{b}}_2} \quad (58)$$

2.4 Rayleigh Distribution Stress-Strength Model

Let *X* be the strength of the compound which is subjected to stress *Y* with parameter σ. The probability density function and cumulative distribution function of *X* and *Y* is given by

$$f_X(x) = \frac{x}{\sigma_2{}^2} \, \mathrm{e}^{-\frac{x^2}{2\sigma_2{}^2}}$$

$$F_y(x) = 1 - \mathrm{e}^{-\frac{x^2}{2\sigma_1{}^2}}$$

The stress-strength reliability R can be calculated as:

$$R = \int_0^\infty \mathrm{F_y}(x) \, \mathrm{f_X}(x) \, dx$$

$$R = \int_0^\infty \left(1 - \mathrm{e}^{-\frac{x^2}{2\sigma_1{}^2}}\right) \left(\frac{x}{\sigma_2{}^2} \, \mathrm{e}^{-\frac{x^2}{2\sigma_2{}^2}}\right) dx$$

$$R = \frac{1}{\sigma_2{}^2}\int_0^\infty \left(x\mathrm{e}^{-\frac{x^2}{2\sigma_2{}^2}} - x\mathrm{e}^{-\left(\frac{1}{\sigma_1{}^2}+\frac{1}{\sigma_2{}^2}\right)\frac{x^2}{2}} \right) dx \quad (59)$$

For $\int_0^\infty x\mathrm{e}^{-\frac{x^2}{2\sigma_2{}^2}}\, dx$ (60)

Let w = $\frac{x^2}{2} \Rightarrow \frac{dw}{dx} = x$

$$dx = \frac{dw}{x}$$

Equation (60) becomes

$$\int_0^\infty x\mathrm{e}^{-\frac{\mathrm{w}}{\sigma_2{}^2}} . \frac{dw}{x}\, dx$$

$$\int_0^\infty \mathrm{e}^{-\frac{\mathrm{w}}{\sigma_2{}^2}}\, dw \quad (61)$$

From Gamma distribution

$$\frac{\Gamma(\alpha)}{\beta^\alpha} = \int_0^\infty \mathrm{w}^{\alpha-1}\,\mathrm{e}^{-\beta \mathrm{w}}\, dw \quad (62)$$

Compare equation (61) and (62) together, we have,

$$\alpha - 1 = 0 \Rightarrow \alpha = 1$$

$$\beta = \frac{1}{\sigma_2{}^2} \text{ and } \frac{\Gamma(\alpha)}{\beta^\alpha} = \frac{\Gamma(1)}{\left(\frac{1}{\sigma_2{}^2}\right)^1} = \sigma_2{}^2$$

Implies that

$$\int_0^\infty x\mathrm{e}^{-\frac{x^2}{2\sigma_2{}^2}}\, dx = \sigma_2{}^2 \quad (63)$$

Substitute equation (98) into (93), we have

$$R = 1 - \frac{1}{\sigma_2{}^2}\int_0^\infty x\mathrm{e}^{-\left(\frac{1}{\sigma_1{}^2}+\frac{1}{\sigma_2{}^2}\right)\frac{x^2}{2}}\, dx \quad (64)$$

From the above, we observed that parameter R is the function of parameters σ_1 and σ_2. Therefore, for maximum likelihood estimate (MLE) of R, we used to obtain the MLEs of σ_1 and σ_2 using invariance principle.

If $\sigma_1 = \sigma_2$, we have

From $\int_0^\infty x\mathrm{e}^{-\left(\frac{1}{\sigma_1{}^2}+\frac{1}{\sigma_2{}^2}\right)\frac{x^2}{2}}\, dx$ (65)

$$\int_0^\infty x\mathrm{e}^{-\left(\frac{1}{\sigma_2{}^2}+\frac{1}{\sigma_2{}^2}\right)\frac{x^2}{2}}\, dx$$

$$\int_0^\infty x\mathrm{e}^{-\left(\frac{2}{\sigma_2{}^2}\right)\frac{x^2}{2}}\, dx$$

Let $t = \frac{x^2}{2} \Rightarrow \frac{dt}{dx} = x$

$$dx = \frac{dt}{x}$$

Which implies that

$$\int_0^\infty x\mathrm{e}^{-\left(\frac{2}{\sigma_2{}^2}\right)t} . \frac{dt}{x} \quad (66)$$

$\int_0^\infty e^{-\left(\frac{2}{\sigma_2{}^2}\right)t}\ dt$

From Gamma distribution, $\frac{\Gamma(\alpha)}{\beta^\alpha} = \int_0^\infty t^{\alpha-1}\, e^{-\beta t}\ dt$ (67)

Comparing equations (66) and (67) gives

$$\alpha - 1 = 0 \Rightarrow \alpha = 1$$

$$\beta = \frac{2}{\sigma_2{}^2} \text{ and } \frac{\Gamma(\alpha)}{\beta^\alpha} = \frac{\Gamma(1)}{\left(\frac{2}{\sigma_2{}^2}\right)} = \frac{1}{\frac{2}{\sigma_2{}^2}} = \frac{\sigma_2{}^2}{2}$$

Implies that

$$\int_0^\infty x e^{-\left(\frac{2}{\sigma_2{}^2}\right)\frac{x^2}{2}}\ dx = \frac{\sigma_2{}^2}{2} \quad (68)$$

Substituting equation (68) into (64) gives

$$\boldsymbol{\hat{R}_1 = 1 - \frac{1}{\sigma_2^2}\left(\frac{\sigma_1^2\sigma_2^2}{\sigma_1^2+\sigma_2^2}\right) = \frac{\sigma_2^2}{\sigma_1^2+\sigma_2^2}}$$

Maximum Likelihood Estimation for R of Rayleigh Distribution

The main aim of this section is to derive the MLE of R. $x_i\, y_j$

Now let $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_m$ are two independent observations from RD (σ_1) and RD (σ_2), respectively. Then, the log – likelihood function of $\sigma_1 and\ \sigma_2$ is given by

$$l = L(\sigma_, xi\ yj) \qquad i = 1,2, \ldots, n,\ j = 1,2,\ldots,m$$

$$= \prod_{i=1}^{n} f(x_i)\ \prod_{j=1}^{m} g(y_j)$$

$$= \prod_{i=1}^{n} \frac{x}{\sigma^2{}_1}\, e^{-\sum_{i=1}^{n}\frac{x^2}{\sigma^2}{}_{1i}}\ \prod_{j=1}^{m} \frac{y}{\sigma^2{}_2}\, e^{-\sum_{i=1}^{m}\frac{y^2}{2\sigma^2}{}_{1i}}$$

$$= \left(\frac{1}{\sigma^2}\right)^n \prod_{i=1}^{n} \sum x_i\, e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n} x^2}\ \left(\frac{1}{\sigma^2}\right)^m \prod_{j=1}^{m} \sum y_i\, e^{-\frac{1}{2\sigma^2}\sum_{j=1}^{m} y_i^2} \quad (69)$$

Taking the Logarithm of the both sides of the equation, then implies

$$\text{Ln(L)} = -2n\ln(\sigma_1) + \sum_{i=1}^{n} In(x_i) - \frac{1}{2\sigma^2}\sum_{i=1}^{n} x_i^2 - 2mIn(\sigma_2)_2 + \sum_{i=1}^{m} In(y_j) - \frac{1}{2\sigma^2{}_2}\sum_{j=1}^{m} y_j^2 \quad (70)$$

Differentiating with respect σ_1 and σ_2 get the following normal equation.

$$\frac{\delta \text{Ln(L)}}{\delta\sigma_1} = \frac{-2n}{\sigma_1} + \frac{1}{\sigma_1{}^3} \quad (71)$$

$$-2n\sigma_1{}^2 + 1 = 0$$

$$2n\sigma_1{}^2 = 1$$

$$\hat{\sigma}_1 = \frac{1}{\sqrt{2n}}$$

$$\frac{\delta \text{Ln(L)}}{\delta\sigma_2} = \frac{-2m}{\sigma_2} + \frac{1}{\sigma_2{}^3} = 0 \quad (72)$$

$$-2m\sigma_2{}^2 + 1 = 0$$

$$\hat{\sigma}_2 = \frac{1}{\sqrt{2m}}$$

Hence, using the invariance properties of MLEs, the MLE of the parameters R is given by

$$\hat{R} = 1 - \frac{1}{\hat{\sigma}_2{}^2}\int_0^\infty x e^{-\left(\frac{1}{\hat{\sigma}_2{}^2}+\frac{1}{\hat{\sigma}_2{}^2}\right)\frac{x^2}{2}}\,dx \tag{73}$$

2.5 Weibull Rayleigh Distribution Stress-Strength Model

Let X be the strength of the component which is subjected to stress Y with parameters a and k, respectively. The probability density function and cumulative distribution function of X and Y is given by

$$f_X(x) = \frac{2a_2}{x}\left(\frac{x^2}{k_2}\right)^{a_2} e^{-\left(\frac{x^2}{k_2}\right)^{a_2}} \tag{74}$$

$$F_Y(x) = 1 - e^{-\left(\frac{x^2}{k_1}\right)^{a_1}} \tag{75}$$

The stress-strength reliability R can be expressed as:

$$R = \int_0^\infty F_y(x)\, f_X(x)\, dx$$

$$R = \int_0^\infty \left[1 - e^{-\left(\frac{x^2}{k_1}\right)^{a_1}}\right]\left[\frac{2a_2}{x}\left(\frac{x^2}{k_2}\right)^{a_2} e^{-\left(\frac{x^2}{k_2}\right)^{a_2}}\right] dx \tag{76}$$

$$R = \frac{2a_2}{k_2{}^{a_2}}\int_0^\infty \left(x^{2a_2-1}e^{-\left(\frac{x^2}{k_2}\right)^{a_2}} - x^{2a_2-1}e^{-\left\{\left(\frac{x^2}{k_1}\right)^{a_1}+\left(\frac{x^2}{k_2}\right)^{a_2}\right\}}\right) dx \tag{77}$$

$$\text{For } \int_0^\infty x^{2a_2-1}e^{-\left(\frac{x^2}{k_2}\right)^{a_2}}\, dx \tag{78}$$

Let t = $x^{2a_2} \Rightarrow \frac{dt}{dx} = 2a_2 x^{2a_2-1}$

$$dx = \frac{dt}{2a_2 x^{2a_2-1}}$$

Equation (78) becomes

$$\int_0^\infty x^{2a_2-1}e^{-\frac{t}{K_2{}^{a_2}}}\cdot\frac{dt}{2a_2 x^{2a_2-1}} \tag{79}$$

$$\frac{1}{2a_2}\int_0^\infty e^{-\frac{t}{K_2{}^{a_2}}}\, dt$$

From Gamma distribution, $\frac{\Gamma(\alpha)}{\beta^\alpha} = \int_0^\infty t^{\alpha-1}\, e^{-\beta t}\, dt$ (80)

$$\alpha - 1 = 0 \Rightarrow \alpha = 1$$

$\beta = \frac{1}{K_2{}^{a_2}}$ and $\frac{\Gamma(\alpha)}{\beta^\alpha} = \frac{\Gamma(1)}{\left(\frac{1}{K_2{}^{a_2}}\right)} = K_2{}^{a_2}$

Equation (79) becomes

$$\frac{1}{2a_2}\int_0^\infty e^{-\frac{t}{K_2{}^{a_2}}}\, dt = \frac{K_2{}^{a_2}}{2a_2} \tag{81}$$

$$R = 1 - \frac{2a_2}{{k_2}^{a_2}} \int_0^{\infty} \left(x^{2a_2 - 1} e^{-\left\{ \left(\frac{x^2}{k_1}\right)^{a_1} + \left(\frac{x^2}{k_2}\right)^{a_2} \right\}} \right) dx$$

$$R = 1 - \frac{2a_2}{{k_2}^{a_2}} \int_0^{\infty} \left(x^{2a_2 - 1} e^{-\left\{ \left(\frac{x^2}{k_1}\right)^{a_2} + \left(\frac{x^2}{k_2}\right)^{a_2} \right\}} \right) dx$$

$$R = 1 - \frac{2a_2}{k_2^{a_2}} \int_0^{\infty} \left[x^{2a_2 - 1} e^{-\left(\frac{x^{2a_2}}{k_1^{a_2}} + \frac{x^{2a_2}}{k_2^{a_2}} \right)} \right] dx$$

Maximum Likelihood of Weibull-Rayleigh Parameters

l= L (a, k, xi, yj) *i* = 1,2, …,n, *j*=1,2,…,m

$= \prod_{i=1}^{n} f(x_i) \ \prod_{j=1}^{m} g(y_j)$

$$= \prod_{i=1}^{n} \ \sum_{i=1}^{n} (\frac{2a_i}{x_i})(\frac{x^2}{k_2})^{a_i} e^{-\sum_{i=1}^{m}(\frac{x^2}{k_2})^{a_i}}{}_{1i} \prod_{j=1}^{m} \sum_{i=1}^{m} (\frac{2a_2}{yj})(\frac{y^2}{k_2})^{a_2} e^{-\sum_{i=1}^{m}(\frac{x^2}{k_2})^{a_2}}{}_{1i} \quad (82)$$

$$= \frac{2^n a^n}{k^{na}{}_1} e^{-\frac{1}{k_1^a}\sum_{i=1}^{n}(x_i)^{2a_i}}{}_{1i} \prod_{i=1}^{n} x_i^{2a-1} \ \ \frac{2^m {a_2}^m}{{k_2}^{na}{}_2} e^{-\frac{1}{k_2^a}\sum_{j=1}^{n}(y_i)^{2a_2}}{}_{1i} \quad (83)$$

Taking the Logarithm of the both sides of the equation, then implies

$$\text{In(L)} = \text{nIn}(2) + \text{nIn}\,(a_1) - \text{n}a_1\text{In}(k_1) - \frac{1}{k_1^a} \sum_{i=1}^{n} {x_i}^{2a_i} + (2a_1 - 1)\sum_{i=1}^{n} In\,(x_i) + m\text{In}(2) + \text{nIn}\,(a_2) - \text{m}a_2\text{In}(k_2) - \frac{1}{k_2^a} \sum_{j=1}^{n} {y_j}^{2a_2} + (2a_2 - 1)\sum_{i=1}^{m} In\,(y_j) \quad (84)$$

Differentiating with respect *a₁*, *a₂*, *k₁* and *k₂* get the following normal equation.

$$\frac{\delta \text{Ln(L)}}{\delta a_1} = \frac{n}{a_1} - \text{nIn}(k_1) - k_1^a \text{In}(k_1) \ \sum_{i=1}^{n} {x_i}^{2a_i} + 2\sum_{i=1}^{n} In\,(x_i) + \frac{2x^{a_1} In \sum_{i=1}^{n} x_i}{k_1^a} \quad (85)$$

$$\frac{\delta \text{Ln(L)}}{\delta a_2} = \frac{m}{a_2} - \text{mIn}(k_2) - k_2^a \text{In}(k_2) \ \sum_{j=1}^{m} y^{2a_2} + 2\sum_{i=1}^{m} In\,(y_j) + \frac{2y^{2a_2} In \sum_{i=1}^{m} y_j}{k_2^a} \quad (86)$$

$$\frac{\delta \text{Ln(L)}}{\delta k_1} = \frac{-na_1}{k_1} + \frac{a_1}{k_{1a-1}} = 0$$

$$\frac{-na_1}{k_1} = \frac{a_1}{k_{1a-1}}$$

$$k_{1^{2-a_1}} = n$$

$$\widehat{K}_1 = n^{\frac{1}{2} - a^1}$$

$$\frac{\delta \text{Ln(L)}}{\delta k_2} = \frac{-ma_1}{k_2} + \frac{a_2}{k_{2a-1}} = 0$$

$$\frac{-ma_1}{k_2} = \frac{a_2}{k_2 a-1}$$

$$k_{2^{2-a2}} = m$$

$$\widehat{K}_2 = m^{\frac{1}{2}-a_2}$$

Hence, using the Invariance properties of MLEs, the MLE of the parameters R is given

$$\hat{R} = 1 - \frac{2a_2}{k_2 a_2} \int_0^\infty x^{2a_2-1} e^{-(\frac{x^2}{k_1})^{a_1} + (\frac{x^2}{k_2})^{a_2}}$$

$$\hat{R}_1 = \frac{K_2^{a2}}{K_1 a^2 + K_2 a^2}$$

3 Results and Discussion

3.1 Data Presentation

In this article, we considered data sets, from two groups of patients suffering from head and neck cancer disease collected from Lagos University Teaching Hospital (LUTH). This data would be used as stress-strength reliability data to demonstrate whether the Weibull-Rayleigh distribution can be a better model than other ones. The first group comprises 34 cancer patients diagnosed of head cancer, while the second group comprises 75 patients diagnosed of neck cancer. The two groups then proceed for treatment. Out of the 109 patients, 61 were treated using radiotherapy (X), whereas 48 patients were treated using a combined radiotherapy and chemotherapy (Y). The datasets are described using different charts and tables.

Table 1. Descriptive Statistics of Head and Neck Cancer

Distribution	**Head Cancer**	**Neck Cancer**
Min	5.547	11.700
Max	87.576	80.020
Median	51.317	44.760
1st Quartile	38.550	36.670
3rd Quartile	66.129	53.790
Mean	49.777	45.320
Standard Deviation	21.011	14.050
Skewness	-0.455	0.099
Kurtosis	2.508	2.874

Table 1 shows that data on head cancer is negatively skewed, while that of neck cancer is slightly positively. The deviation in neck cancer is less than that of head cancer. Figure 1 depicts that there are more female cancer patients than male cancer patients. Figure 2 depicts that most of the cancer patients were alive at the end of the treatment, while only very few died of cancer, some outcomes were unknown may be due to lose of contact.

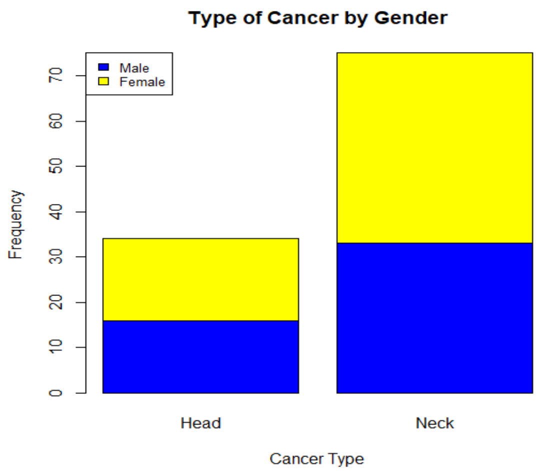


Figure 1: Type of Cancer by Gender of Patients

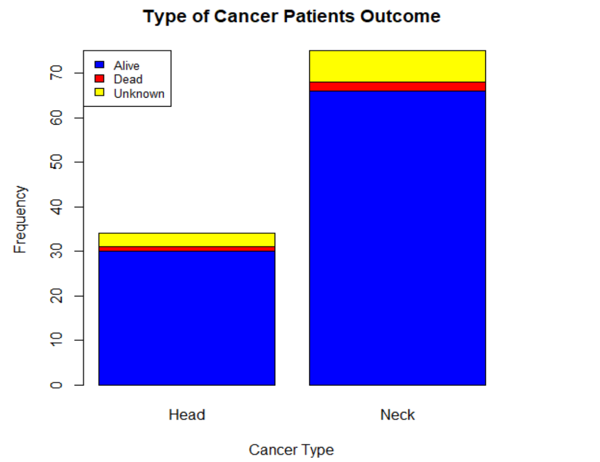


Figure 2: Type of Cancer by Outcome of Patients

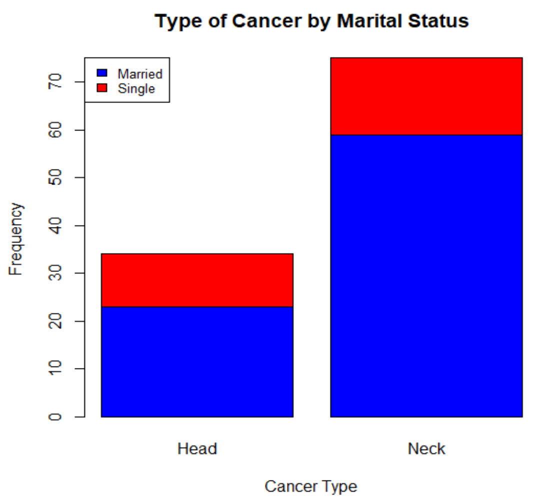


Figure 3: Type of Cancer by Marital Status of Patients

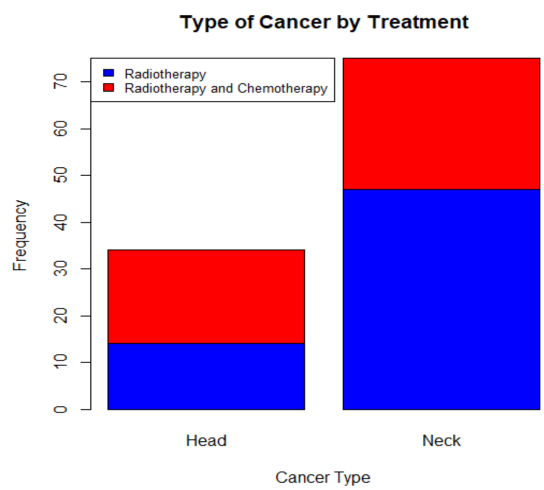


Figure 4: Type of Cancer by Treatment type

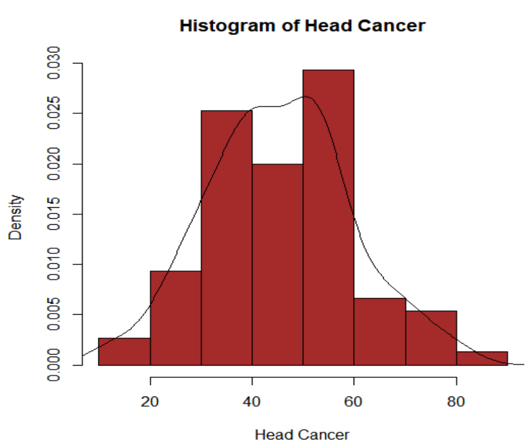


Figure 5: Histogram for Head Cancer

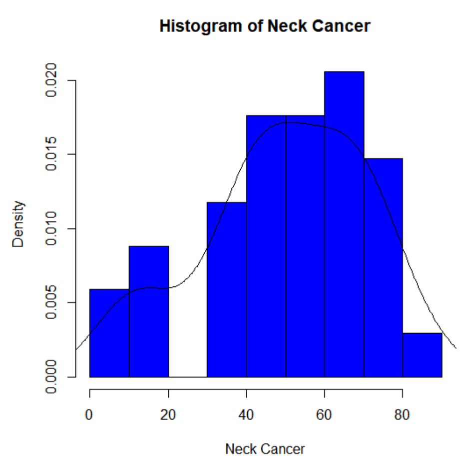
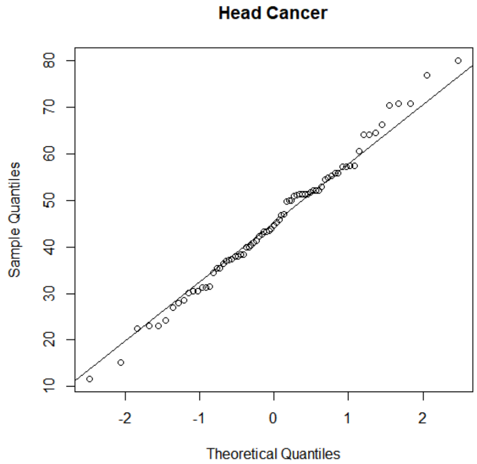


Figure 6: Histogram for Neck Cancer



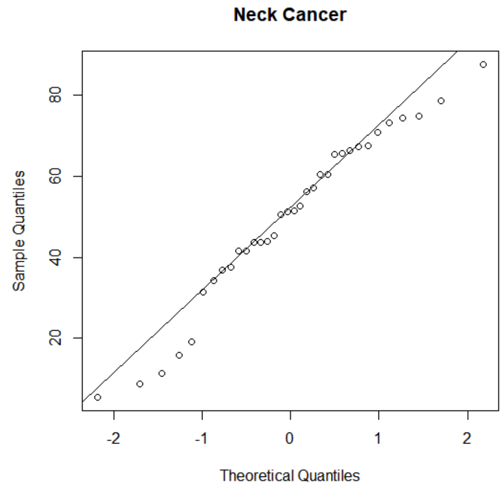


Figure 7: QQ Plot for Head Cancer   Figure 8: QQ Plot for Neck Cancer

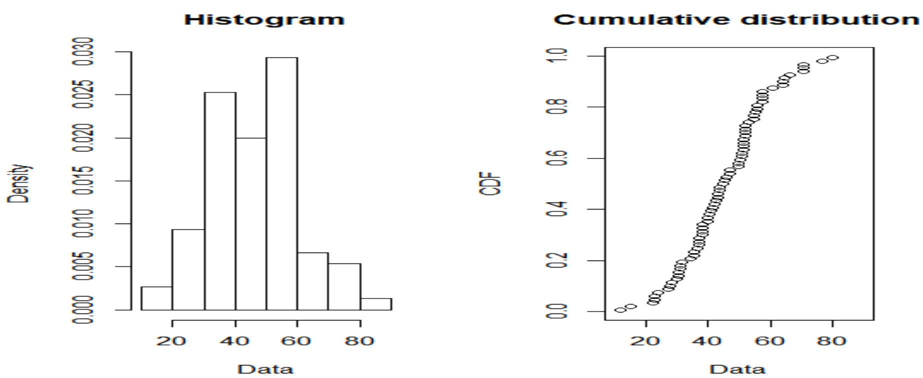


Figure 9: Histogram and Cumulative Distribution Curve for Head Cancer

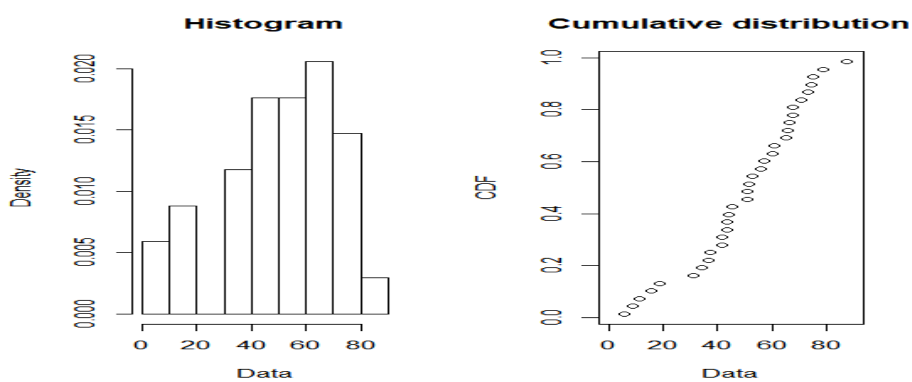


Figure 10: Histogram and Cumulative Distribution Curve for Neck Cancer

Figure 3 shows that there are more married cancer patients than single cancer patients among the patients under study. Figure 4 shows that most head cancer patients were treated with both radiotherapy and chemotherapy, while most neck cancer patients are treated with only radiotherapy. This forms the two groups used for the analysis.

Figure 5 shows that head cancer data is bimodal and negatively skewed, while Figure 6 shows that neck cancer data is positively skewed and has gap. Figures 7 and 8 show the QQ plots of the head and neck cancer datasets respectively. Figures 9 and 10 show the histogram with the CDF curves of the head and neck cancer datasets respectively.

3.2 Data Analysis

The reliability model of Weibull Rayleigh distribution using maximum likelihood estimation (MLE) method are compared with that of Weibull and Rayleigh distribution.

Table 2: Head Cancer Parameter Estimate

Distribution	Parameter	Estimates	Standard Error	*t* value	p-value	-LogL	AIC
WRD	*A*	1.2892	0.1806	7.139	0.0000	152.3029	308.6057
	K	3101.2843	2.4216	12.685	0.0000		
Weibull	*A*	32.350385	1.687846	19.17	0.0000	295.8759	595.7519
	B	0.088700	0.007193	12.33	0.0000		
Rayleigh	σ	53.9107	0.8562	62.97	0.0000	3514.37	7030.749

Table 3: Neck Cancer Parameter Estimate

Distribution	Parameter	Estimates	Standard Error	*t* value	p-value	-LogL	AIC
WRD	*A*	1.7920	0.1515	11.82	0.0000	304.0888	612.1775
	K	2558.4488	1.4848	17.04	0.0000		
Weibull	*A*	30.068891	1.732689	17.35	0.0000	419.4541	842.9083
	B	0.079633	0.006577	12.11	0.0000		
Rayleigh	σ	47.4157	0.5243	90.44	0.0000	6778.51	13559.02

Table 4: Stress-Strength Reliability

Distribution	R
WRD	0.4265
Weibull	0.5269
Rayleigh	0.5000

Table 2 shows that WRD fit the head data than Weibull and Rayleigh distribution using the log likelihood and AIC criterion. This is same for the neck cancer data displayed in Table 3. Table 4 shows the reliability function, which is derived from the methods of treatment from group X and Y. The reliability function of WIR distribution compared favourably with that of Weibull and Rayleigh distributions.

4.0 Conclusion

The conclusion drawn in this work is based on the aim and objectives, and the results of the analysis. The aim of this work is to analysis the stress-strength of WRD and its application to health data.

In this work, the WRD have been fully characterized using all possible functions. The reliability function of its stress-strength has been determined, alongside that of Weibull and Rayleigh distributions. The MLE method was used to estimate the parameters of the reliability function of stress-strength. The stress-strength analysis of WRD was applied to head and neck cancer data collected from LUTH. The results of Weibull-Rayleigh stress-strength analysis was compared to that of Weibull and Rayleigh distribution stress-strength and based on the values of these results, it is concluded that the Weibull Rayleigh distribution is better than Weibull and Rayleigh distributions in reliability function of stress-strength.

The following recommendations are made to support further research in this field of mathematical statistics.

- The reliability stress-strength of convoluted distributions should be determined since they provide better reliability function of stress-strength than the individual distributions.
- The WRD should be applied on data when Weibull and Rayleigh distribution did not provide good fit.
- For the treatment of cancer, both radiotherapy and chemotherapy should be used to treat cancer patients, especially cancer at the head and neck region.

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