

ON CONTINUITY PROPERTY VIA REGULARITY GENERALIZATIONS OF \mathcal{G}^ω -CLOSED SETS IN GRILL TOPOLOGICAL SPACES

Abstract

In this paper, we study the continuity property in grill topological spaces via generalized \mathcal{G}^ω -closed sets and regular generalized \mathcal{G}^ω -closed sets. These notions are generalized \mathcal{G}^ω -continuous functions which is weak form of $g\omega$ -continuous functions, generalized \mathcal{G}^ω -irresolute functions, regular generalized \mathcal{G}^ω -continuous functions which is weak form of $rg\omega$ -continuous functions, regular generalized \mathcal{G}^ω -irresolute functions and investigates some of its properties in grill topological spaces.

Keywords: Grill topological spaces; continuous functions.

1 Introduction

The continuity property is one of the fundamental concepts in point-set topology. In 1982, Hdeib [7] introduced the notion of ω -continuous functions. In 1983, [8] Mashhour, et al. introduced the notions of α -continuity functions. In 1993, Palaniappan and Rao [11] introduced the notion of regular generalized continuous functions. In 2005, Al-Zoubi [3] introduced the notion of generalized ω -continuous functions. In 2007, Al-Omari and Noorani [1] introduced the notion of regular generalized ω -continuous functions. In 2009, [10] Noiri, et al. introduced the class of $\alpha - \omega$ -continuous functions.

For the study of grill topological spaces. In 2011, Al-Omari and Noiri [2] introduced the notion of $\mathcal{G} - \alpha$ -continuous function in grill topological spaces. In 2020, [13] Saif, et al. introduced the notions of \mathcal{G}^ω -open sets and generalized \mathcal{G}^ω -closed sets in grill topological spaces. In 2021, [14] Saif, et al. introduced the notion of g^ω -continuous property in grill topological spaces. In 2021, [16] Saif, et al. introduced the notion of $\mathcal{G}_\omega^\alpha$ -continuous property in grill topological spaces. In 2021, [17] Saif, et al. introduced the notion of \mathcal{G}_{rg}^ω -closed sets in grill topological spaces.

In this paper, we introduce and investigate new notions of generalized \mathcal{G}^ω -closed sets and regular generalized \mathcal{G}^ω -closed sets in grill topological spaces. These notions are called generalized \mathcal{G}^ω -continuous functions and regular generalized \mathcal{G}^ω -continuous functions in grill topological spaces. In Section 3, we introduce and investigates of notions of generalized \mathcal{G}^ω -closed sets called generalized \mathcal{G}^ω -continuous functions, generalized \mathcal{G}^ω -closed functions, generalized \mathcal{G}^ω -irresolute functions and introduce investigates of notion of \mathcal{G}^ω -irresolute functions.

In Section 4, we introduce and investigate new notions of generalized \mathcal{G}^ω -continuous functions, generalized \mathcal{G}^ω -closed functions and generalized \mathcal{G}^ω -irresolute functions called regular generalized \mathcal{G}^ω -continuous functions, regular generalized \mathcal{G}^ω -closed functions and regular generalized \mathcal{G}^ω -irresolute functions, respectively.

2 Preliminaries

Definition 2.1. A function $f : (X, \tau) \rightarrow (Y, \rho)$ of a topological space (X, τ) into a topological space (Y, ρ) is called:

1. *g-continuous function* [4] if $f^{-1}(F)$ is *g*-closed set in (X, τ) for every closed set F of (Y, ρ) ;
2. *ω -continuous function* [7] if $f^{-1}(V)$ is ω -open set in (X, τ) for every open set V of (Y, ρ) ;
3. *$g\omega$ -continuous function* [3] if $f^{-1}(F)$ is $g\omega$ -closed set in (X, τ) for every closed set F of (Y, ρ) ;
4. *$g\omega$ -closed function* [3] if the image of every closed set of (X, τ) is $g\omega$ -closed set in (Y, ρ) ;
5. *$rg\omega$ -continuous function* [1] if $f^{-1}(F)$ is $rg\omega$ -closed set in (X, τ) for every ω -closed set F of (Y, τ) .

Definition 2.2. A function $f : (X, \tau, \mathcal{G}) \rightarrow (Y, \rho)$ of a grill topological space (X, τ, \mathcal{G}) into a topological space (Y, ρ) is called:

1. *\mathcal{G}^ω -continuous function* [14] if $f^{-1}(U)$ is \mathcal{G}^ω -open set in (X, τ, \mathcal{G}) for every open set U in (Y, ρ) ;
2. *\mathcal{G}^ω -closed function* [14] if $f(F)$ is a closed set in (Y, ρ) for every \mathcal{G}^ω -closed set F in (X, τ, \mathcal{G}) ;
3. *$\mathcal{G}_\omega^\alpha$ -continuous function* [16] if $f^{-1}(U)$ is $\mathcal{G}_\omega^\alpha$ -open set in (X, τ, \mathcal{G}) for every open set U in (Y, ρ) ;
4. *$\mathcal{G}_\omega^\alpha$ -closed function* [16] if $f(F)$ is a closed set in Y for every $\mathcal{G}_\omega^\alpha$ -closed set F in (X, τ, \mathcal{G}) .

By $Cl(A)$ and $Int(A)$, we mean the closure set and the interior set of A in topological space (X, τ) , respectively.

A collection \mathcal{G} of subsets of a topological space (X, τ) is said to be a grill [5] on X if \mathcal{G} satisfies the following conditions:

1. $\emptyset \notin \mathcal{G}$;
2. $A \in \mathcal{G}$ and $A \subseteq B$ implies that $B \in \mathcal{G}$;

3. $A, B \subseteq X$ and $A \cup B \in \mathcal{G}$ implies that $A \in \mathcal{G}$ or $B \in \mathcal{G}$.

For a grill \mathcal{G} on a topological space X , an operator from the power set $P(X)$ of X to $P(X)$ was defined in [12] the following manner : For any $A \in P(X)$,

$$\Phi(A) = \{x \in X : U \cap A \in \mathcal{G}, \text{ for each open neighborhood } U \text{ of } x\}.$$

Then the operator $\Psi : P(X) \rightarrow P(X)$, given by $\Psi(A) = A \cup \Phi(A)$, for $A \in P(X)$, was also shown in [12] to be a Kuratowski closure operator, defining a unique topology $\tau_{\mathcal{G}}$ on X such that $\tau \subseteq \tau_{\mathcal{G}}$. This topology defined by:

$$\tau_{\mathcal{G}} = \{U \subseteq X : \Psi(X - U) = X - U\},$$

where $\tau \subseteq \tau_{\mathcal{G}}$ and for any $A \subseteq X$, $\Psi(A) = {}_{\mathcal{G}}Cl(A)$ such that ${}_{\mathcal{G}}Cl(A)$ denotes the set of all closure points of A in a topological space $(X, \tau_{\mathcal{G}})$. The set of all interior points of A in a topological space $(X, \tau_{\mathcal{G}})$ denoted by ${}_{\mathcal{G}}Int(A)$.

Definition 2.3. A subset A of a topological space (X, τ) is called:

1. α -open set [9] if $A \subseteq Int(Cl(Int(A)))$. The complement of α -open set is called α -closed set;
2. ω -open set [6] if for each $x \in A$, there is an open set $U_x \in \tau$ containing x such that $U_x - A$ is a countable set. The complement of ω -open set is called ω -closed set;
3. $\alpha - \omega$ -open set [10] if $A \subseteq Int_{\omega}(Cl(Int_{\omega}(A)))$. The complement of $\alpha - \omega$ -open set is called $\alpha - \omega$ -closed set.

Definition 2.4. A subset A of a grill topological space (X, τ, \mathcal{G}) is called:

1. $\mathcal{G} - \alpha$ -open set [2] if $A \subseteq Int(\Psi(Int(A)))$. The complement of $\mathcal{G} - \alpha$ -open set is called $\mathcal{G} - \alpha$ -closed set;
2. \mathcal{G}^{ω} -open set [13] if $A \subseteq Cl(Int_{\omega}(\Psi(A)))$. The complement of \mathcal{G}^{ω} -open set is called \mathcal{G}^{ω} -closed set;
3. $\mathcal{G}_{\omega}^{\alpha}$ -open set [15] if $A \subseteq Int(\Psi(Int_{\omega}(A)))$. The complement of $\mathcal{G}_{\omega}^{\alpha}$ -open set is called $\mathcal{G}_{\omega}^{\alpha}$ -closed set. The set of all $\mathcal{G}_{\omega}^{\alpha}$ -open sets in X is denoted by $\mathcal{G}_{\omega}^{\alpha}O(X, \tau)$ and the set of all $\mathcal{G}_{\omega}^{\alpha}$ -closed sets in X is denoted by $\mathcal{G}_{\omega}^{\alpha}C(X, \tau)$;
4. \mathcal{G}_g^{ω} -closed set [13] if ${}_{\mathcal{G}^{\omega}}Cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open set of (X, τ, \mathcal{G}) . The complement of \mathcal{G}_g^{ω} -closed set is called \mathcal{G}_g^{ω} -open set;
5. $\mathcal{G}_{rg}^{\omega}$ -closed set [17] if ${}_{\mathcal{G}^{\omega}}Cl(A) \subseteq U$ whenever $A \subseteq U$ and U is a regular-open set of (X, τ, \mathcal{G}) . The complement of $\mathcal{G}_{rg}^{\omega}$ -closed set is called $\mathcal{G}_{rg}^{\omega}$ -open set.

From the definition stated above we obtain the following figure of implications:

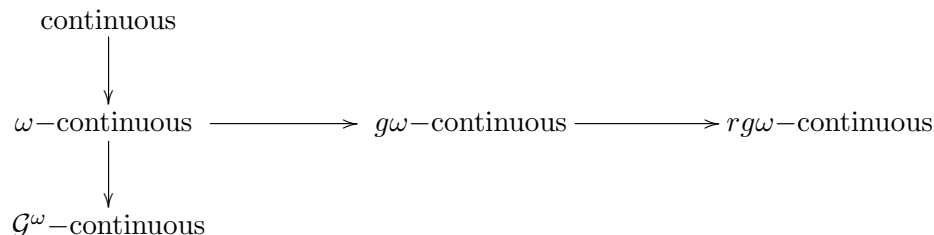


Figure 1:

3 \mathcal{G}_g^ω -Continuous functions

Definition 3.1. A function $f : (X, \tau, \mathcal{G}) \rightarrow (Y, \rho)$ of a grill topological space (X, τ, \mathcal{G}) into a space (Y, ρ) is called \mathcal{G}_g^ω -continuous function if $f^{-1}(U)$ is \mathcal{G}_g^ω -closed set in (X, τ, \mathcal{G}) for every closed set U in (Y, ρ) .

Example 3.2. Any function $f : (X, \tau, \mathcal{G}) \rightarrow (Y, \rho)$ of any countable grill topological space (X, τ, \mathcal{G}) into any space (Y, ρ) is \mathcal{G}_g^ω -continuous.

Definition 3.3. A function $f : (X, \tau, \mathcal{G}) \rightarrow (X, \tau, \mathcal{H})$ of a grill topological space (X, τ, \mathcal{G}) into a grill topological space (Y, τ, \mathcal{H}) is called:

1. \mathcal{G}_g^ω -irresolute if $f^{-1}(U)$ is \mathcal{G}_g^ω -closed set in (X, τ, \mathcal{G}) for every \mathcal{G}_g^ω -closed set in (Y, τ, \mathcal{H}) .
2. \mathcal{G}^ω -irresolute if $f^{-1}(U)$ is \mathcal{G}^ω -closed set in (X, τ, \mathcal{G}) for every \mathcal{G}^ω -closed set in (Y, τ, \mathcal{H}) .

Theorem 3.4. Let $f : (X, \tau, \mathcal{G}) \rightarrow (Y, \rho)$ be a function of a grill topological space (X, τ, \mathcal{G}) into a space (Y, ρ) . Then f is \mathcal{G}_g^ω -continuous function if $f^{-1}(U)$ is \mathcal{G}_g^ω -open set in (X, τ, \mathcal{G}) for every open set U in (Y, ρ) .

Proof. Let f be \mathcal{G}_g^ω -continuous function. Let U be open set in (Y, ρ) then $Y - U$ is closed set in (Y, ρ) . Since f is \mathcal{G}_g^ω -continuous function. Then $f^{-1}(Y - U) = X - f^{-1}(U)$ is \mathcal{G}_g^ω -closed set in a grill topological space (X, τ, \mathcal{G}) . Therefore $f^{-1}(U)$ is \mathcal{G}_g^ω -open set in (X, τ, \mathcal{G}) . Hence f is \mathcal{G}_g^ω -continuous function. \square

It is clear that every \mathcal{G}^ω -continuous function is \mathcal{G}_g^ω -continuous function but the converse of this fact no need to be true.

Example 3.5. Let $f : (\mathbb{R}, \tau, \mathcal{G}) \rightarrow (Y, \rho)$ be a function defined by:

$$f(x) = \begin{cases} 1, & x \in (0, \frac{1}{2}] \\ 2, & x \in \mathbb{R} - (0, \frac{1}{2}] \end{cases}$$

where $Y = \{1, 2\}$,

$$\tau = \{\emptyset, \mathbb{R}, \mathbb{R} - (0, 1)\}, \mathcal{G} = P(\mathbb{R}) - \{\emptyset\} \text{ and } \rho = \{\emptyset, Y, \{1\}\}.$$

The function f is \mathcal{G}_g^ω -continuous, since $f^{-1}(\{2\}) = \mathbb{R} - (0, \frac{1}{2}]$ and $f^{-1}(Y) = \mathbb{R}$ are \mathcal{G}_g^ω -closed sets in $(\mathbb{R}, \tau, \mathcal{G})$. The function f is not \mathcal{G}^ω -continuous function, since $f^{-1}(\{2\}) = \mathbb{R} - (0, \frac{1}{2}]$ is not \mathcal{G}^ω -closed set.

It is clear that every $g\omega$ -continuous function is \mathcal{G}_g^ω -continuous function but the converse of this fact no need to be true.

Example 3.6. Let $f : (X, \tau, \mathcal{G}) \rightarrow (Y, \rho)$ be a function defined by:

$$f(x) = \begin{cases} 1, & x \in X - B \\ 2, & x \in B \end{cases}$$

where X, B and A are uncountable sets, $B \subset A$ and $Y = \{1, 2\}$,

$$\tau = \{\emptyset, X, A\}, \mathcal{G} = P(X) - \{\emptyset\} \text{ and } \rho = \{\emptyset, Y, \{1\}\}.$$

Then the function f is \mathcal{G}_g^ω -continuous but f is not $g\omega$ -continuous. Since $f^{-1}(\{2\}) = B$ and $f^{-1}(Y) = X$ are \mathcal{G}_g^ω -closed sets in (X, τ, \mathcal{G}) but $f^{-1}(\{2\}) = B$ is not $g\omega$ -closed set.

It is clear that every $\mathcal{G}_\omega^\alpha$ -continuous function is \mathcal{G}_g^ω -continuous function but the converse of this fact no need to be true.

Example 3.7. In Example(3.5), we note that f is \mathcal{G}_g^ω -continuous function but the function f is not $\mathcal{G}_\omega^\alpha$ -continuous function. Since $f^{-1}(\{2\}) = \mathbb{R} - (0, \frac{1}{2}]$ is not $\mathcal{G}_\omega^\alpha$ -closed set in (X, τ, \mathcal{G}) .

It is clear that every \mathcal{G}^ω -irresolute function is \mathcal{G}^ω -continuous function but the converse of this fact no need to be true.

Example 3.8. Let $f : (\mathbb{R}, \tau, \mathcal{G}) \rightarrow (\mathbb{R}, \rho, \mathcal{H})$ be the identity function where

$$\tau = \{\emptyset, \mathbb{R}, \mathbb{R} - (0, 1)\}, \mathcal{G} = P(\mathbb{R}) - \{\emptyset\}, \mathcal{H} = P(\mathbb{R}) - \{\emptyset\} \text{ and } \rho = \{\emptyset, \mathbb{R}\}.$$

The function f is \mathcal{G}^ω -continuous function since $f^{-1}(\emptyset) = \emptyset$ and $f^{-1}(\mathbb{R}) = \mathbb{R}$ are \mathcal{G}^ω -closed sets in $(\mathbb{R}, \tau, \mathcal{G})$ but f it is \mathcal{G}^ω -irresolute function. Since $\mathbb{R} - (0, 1)$ is \mathcal{G}^ω -closed set in $(\mathbb{R}, \tau, \mathcal{H})$ but it is not \mathcal{G}^ω -closed set in $(\mathbb{R}, \tau, \mathcal{G})$.

It is clear that every \mathcal{G}_g^ω -irresolute function is \mathcal{G}_g^ω -continuous function but the converse of this fact no need to be true.

Example 3.9. In Example(3.8), we note that f is \mathcal{G}_g^ω -continuous function, since $f^{-1}(\emptyset) = \emptyset$ and $f^{-1}(\mathbb{R}) = \mathbb{R}$ are \mathcal{G}^ω -closed sets in $(\mathbb{R}, \tau, \mathcal{G})$ but the function f is not \mathcal{G}_g^ω -irresolute function, since $\mathbb{R} - (0, 1)$ is \mathcal{G}_g^ω -closed set in $(\mathbb{R}, \tau, \mathcal{H})$ but it is not \mathcal{G}_g^ω -closed set in $(\mathbb{R}, \tau, \mathcal{G})$.

It is clear that every \mathcal{G}^ω -irresolute function is \mathcal{G}_g^ω -irresolute function but the converse of this fact no need to be true.

Example 3.10. In Example(3.5), if $f : (\mathbb{R}, \tau, \mathcal{G}) \rightarrow (\mathbb{R}, \rho, \mathcal{H})$, we note that f is \mathcal{G}_g^ω -irresolute function, but the function f is not \mathcal{G}^ω -irresolute function, since $\mathbb{R} - (0, \frac{1}{2}]$ is \mathcal{G}^ω -closed set in (Y, τ, \mathcal{H}) but $f^{-1}(\{2\}) = \mathbb{R} - (0, \frac{1}{2}]$ is not \mathcal{G}^ω -closed set in (X, τ, \mathcal{G}) .

The following figure is an enlargement of figure(1) and an theorems above we have the following figure:

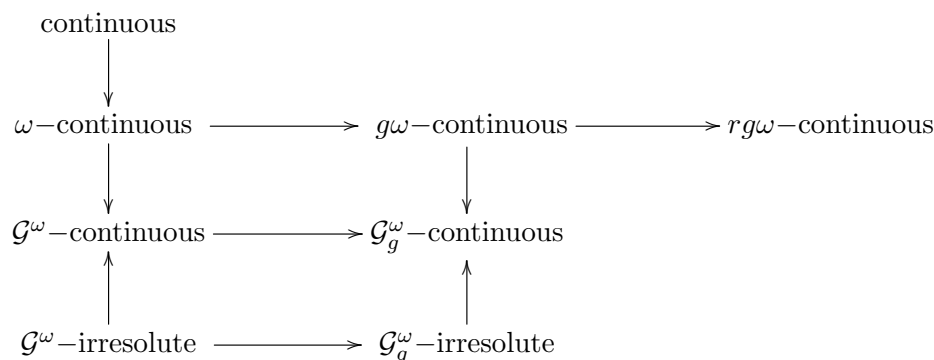


Figure 2:

Theorem 3.11. If $f : (X, \tau, \mathcal{G}) \rightarrow (Y, \rho)$ is \mathcal{G}_g^ω -continuous, then for each $x \in X$ and each open set V in (Y, ρ) with $f(x) \in V$, there exists \mathcal{G}_g^ω -open set U in (X, τ, \mathcal{G}) such that $x \in U$ and $f(U) \subseteq V$.

Proof. Lit f is \mathcal{G}_g^ω -continuous function. Let $x \in X$ and V be any open set in (Y, ρ) containing $f(x)$. Put $U = f^{-1}(V)$. Since f is \mathcal{G}_g^ω -continuous then U is \mathcal{G}_g^ω -open set in (X, τ, \mathcal{G}) such that $x \in U$ and $f(U) \subseteq V$. \square

The converse of the above theorem is not true in general as the following example shows.

Example 3.12. In Example(3.6), if

$$\tau = \{\emptyset, X, B\}, \mathcal{G} = P(X) - \{\emptyset\} \text{ and } \rho = \{\emptyset, Y, \{1\}\}.$$

We note that f is satisfies the property stated in Theorem(3.11). The function f is not \mathcal{G}_g^ω -continuous function. Since $\{2\}$ is closed set in Y but $f^{-1}(\{2\}) = B$ is not \mathcal{G}_g^ω -closed set in (X, τ, \mathcal{G}) .

Theorem 3.13. Let $f : (X, \tau, \mathcal{G}) \rightarrow (Y, \rho)$ be a \mathcal{G}_g^ω -continuous function and let A be a closed subset of (X, τ, \mathcal{G}) . Then, the restriction $f|_A : (A, \tau_A) \rightarrow (Y, \rho)$ is \mathcal{G}_g^ω -continuous.

Proof. Let F be a closed subset of (Y, ρ) . Then $(f|A)^{-1}(F) = f^{-1}(F) \cap A$. Since f is \mathcal{G}_g^ω -Continuous, $f^{-1}(F) \in \mathcal{G}_g^\omega C(X, \tau, \mathcal{G})$ and since It is clear that if $A \in \mathcal{G}_g^\omega C(X, \tau, \mathcal{G})$ and B is closed set in (X, τ, \mathcal{G}) then $A \cap B \in \mathcal{G}_g^\omega C(X, \tau, \mathcal{G})$. Therefore, since It is clear that if $A \in \mathcal{G}_g^\omega C(X, \tau, \mathcal{G})$, then $A \in \mathcal{G}_g^\omega C(Y, \tau_Y)$, $(f|A)^{-1}(F) \in \mathcal{G}_g^\omega C(Y, \tau_Y)$. \square

Theorem 3.14. For a function $f : (X, \tau, \mathcal{G}) \rightarrow (Y, \rho)$, the following holed:

1. f is \mathcal{G}_g^ω -continuous function.
2. For each $x \in X$ and each open set V in (Y, ρ) with $f(x) \in V$, there exists \mathcal{G}_g^ω -open set U in (X, τ, \mathcal{G}) such that $x \in U$ and $f(U) \subseteq V$.

Definition 3.15. A function $f : (X, \tau, \mathcal{G}) \rightarrow (Y, \rho)$ of a grill topological space (X, τ, \mathcal{G}) into a space (Y, ρ) is called:

1. \mathcal{G}_g^ω -closed function if $f(G)$ is a closed set in (Y, ρ) for every \mathcal{G}_g^ω -closed set G in (X, τ, \mathcal{G}) .
2. \mathcal{G}_g^ω -open function if $f(G)$ is a open set in (Y, ρ) for every \mathcal{G}_g^ω -open set G in (X, τ, \mathcal{G}) .

Example 3.16. $f : (\mathbb{R}, \tau, \mathcal{G}) \rightarrow (\mathbb{R}, \rho_D)$ be a any function, with

$$\tau = \{\emptyset, \mathbb{R}, \{1, 3\}\} \text{ and } \mathcal{G} = P(\mathbb{R}) - \{\emptyset\}.$$

Since $\{1, 3\}$ is a countable set then by Remark(??) any subset of X is a both \mathcal{G}_g^ω -open set and \mathcal{G}_g^ω -closed set. Therefore f is a both \mathcal{G}_g^ω -closed function and \mathcal{G}_g^ω -open function.

Remark 3.17. Let (X, τ_I, \mathcal{G}) be a grill topological space with any set X then any subset of X is a both \mathcal{G}_g^ω -open set and \mathcal{G}_g^ω -closed set.

Theorem 3.18. Let $f : (X, \tau_I, \mathcal{G}) \rightarrow (Y, \rho)$ be any function then f is \mathcal{G}_g^ω -continuous function.

Proof. Let $A \subseteq Y$ then $f^{-1}(A)$ is a both \mathcal{G}_g^ω -open set and \mathcal{G}_g^ω -closed set. Therefore f is \mathcal{G}_g^ω -continuous function. \square

Theorem 3.19. Every \mathcal{G}_g^ω -closed function is $\mathcal{G}_\omega^\alpha$ -closed function.

The proof is obvious.

The converse of above theorem no need to be true.

Example 3.20. Let $f : (X, \tau, \mathcal{G}) \rightarrow (Y, \rho)$ be a function defined by:

$$f(x) = \begin{cases} b, & x \in \{2, 3\} \\ a, & x \in \{1\} \end{cases}$$

where $X = \{1, 2, 3\}, Y = \{a, b\}$,

$$\tau = \{\emptyset, X, \{1\}\}, \mathcal{G} = P(X) - \{\emptyset\} \text{ and } \rho = \{\emptyset, Y, \{a\}\}.$$

We note that f is $\mathcal{G}_\omega^\alpha$ -closed function but it is not \mathcal{G}_g^ω -closed function. Since $\{1\}$ is \mathcal{G}_g^ω -closed set in (X, τ, \mathcal{G}) but $f(\{1\}) = \{a\}$ it is not closed set in (Y, ρ) .

Theorem 3.21. Every \mathcal{G}_g^ω -closed function is \mathcal{G}^ω -closed function.

The proof is obvious.

The converse of above theorem no need to be true.

Example 3.22. Let $f : (\mathbb{R}, \tau, \mathcal{G}) \rightarrow (Y, \rho)$ be a function defined by:

$$f(x) = \begin{cases} 1, & x \in B, B \subset (0, 1) \\ 3, & x \in \mathbb{R} - B, x \in (0, 1) \\ 2, & x \in \mathbb{R} - C \end{cases}$$

where $(0, 1) \subset C, Y = \{1, 2, 3\}$,

$$\tau = \{\emptyset, \mathbb{R}, \mathbb{R} - (0, 1)\}, \mathcal{G} = P(\mathbb{R}) - \{\emptyset\} \text{ and } \rho = \{\emptyset, Y, \{1, 3\}, \{2, 3\}, \{3\}\}.$$

We note that f is \mathcal{G}^ω -closed function but it is not \mathcal{G}_g^ω -closed function. Since $\mathbb{R} - B$ is \mathcal{G}_g^ω -closed set in (X, τ, \mathcal{G}) but $f(\mathbb{R} - B) = \{3\}$ it is not closed set in (Y, ρ) .

Theorem 3.23. Let $f : (X, \tau, \mathcal{G}) \rightarrow (Y, \rho)$ and $g : (Y, \rho, \mathcal{H}) \rightarrow (Z, \gamma)$ two functions. Then the following hold:

1. $g \circ f$ is \mathcal{G}_g^ω -continuous function if g is continuous function and f is \mathcal{G}_g^ω -continuous function.
2. $g \circ f$ is \mathcal{G}_g^ω -continuous function if g is g -continuous function and (Y, ρ) is $T_{1|2}$ -space and f is \mathcal{G}_g^ω -continuous function.

Theorem 3.24. Let $f : (X, \tau, \mathcal{G}) \rightarrow (Y, \rho, \mathcal{I})$ and $g : (Y, \rho, \mathcal{H}) \rightarrow (Z, \gamma, \mathcal{K})$ two functions. Then the following hold:

1. $g \circ f$ is \mathcal{G}_g^ω -continuous function if g is \mathcal{G}_g^ω -continuous function and f is \mathcal{G}_g^ω -irresolute function.
2. $g \circ f$ is \mathcal{G}_g^ω -irresolute function if f and g are \mathcal{G}_g^ω -irresolute function.

Theorem 3.25. Let $f : (X, \tau, \mathcal{G}) \rightarrow (Y, \rho)$ and $g : (Y, \rho) \rightarrow (Z, \gamma)$ be two functions. Then $g \circ f$ is \mathcal{G}_g^ω -closed function if g is a closed function and f is \mathcal{G}_g^ω -closed function.

Proof. Let U be \mathcal{G}_g^ω -closed set in (X, τ, \mathcal{G}) . Since f is \mathcal{G}_g^ω -closed function then $f(U)$ is a closed set in Y . Since g is closed function then $g[f(U)] = (g \circ f)(U)$ is closed set in (Z, γ) . That is, $g \circ f$ is \mathcal{G}_g^ω -closed function. \square

Corollary 3.26. Let $f : (X, \tau, \mathcal{G}) \rightarrow (Y, \rho)$ and $g : (Y, \rho) \rightarrow (Z, \gamma)$ be two functions. Then $g \circ f$ is \mathcal{G}_g^ω -open function if g is a open function and f is \mathcal{G}_g^ω -open function.

The following example shows that the composition of two \mathcal{G}_g^ω -continuous functions need not be \mathcal{G}_g^ω -continuous.

Example 3.27. Let $f : (\mathbb{R}, \tau, \mathcal{G}) \rightarrow (Y, \rho)$ and $g : (Y, \rho, \mathcal{H}) \rightarrow (Y, \gamma)$ be two functions defined by:

$$f(x) = \begin{cases} 0, & x \in \mathbb{R} - (0, 1) \\ 1, & x \in (0, 1) \end{cases}$$

And g is identity function. Where $Y = \{0, 1\}$,

$$\tau = \{\emptyset, \mathbb{R}, \mathbb{R} - (0, 1)\}, \mathcal{G} = P(\mathbb{R}) - \{\emptyset\}, \mathcal{H} = P(Y) - \{\emptyset\} \text{ and } \rho = \{\emptyset, Y, \{0\}\}.$$

And $\gamma = \{\emptyset, \{1\}, Y\}$. Then f and g are \mathcal{G}_g^ω -continuous functions but $g \circ f$ is not \mathcal{G}_g^ω -continuous function. Since $\{0\}$ is a closed set in (Y, γ) but $f^{-1}(\{0\}) = \mathbb{R} - (0, 1)$ is not \mathcal{G}_g^ω -closed set in $(\mathbb{R}, \tau, \mathcal{G})$.

The following example shows that the composition of two f is \mathcal{G}_g^ω -continuous function and g is $g\omega$ -continuou function need not be \mathcal{G}_g^ω -continuous.

Example 3.28. In Example(3.27), we note that f is \mathcal{G}_g^ω -continuous function and g is $g\omega$ -continuou function, but $g \circ f$ is not \mathcal{G}_g^ω -continuous function Since $\{0\}$ is a closed set in (Y, γ) but $f^{-1}(\{0\}) = \mathbb{R} - (0, 1)$ is not \mathcal{G}_g^ω -closed set in $(\mathbb{R}, \tau, \mathcal{G})$.

Theorem 3.29. Let $f : (X, \tau, \mathcal{G}) \rightarrow (Y, \rho)$ be a \mathcal{G}_g^ω -closed function and A is g -closed subset in (X, τ, \mathcal{G}) . Then $f(A)$ is $rg\omega$ -closed subset in (Y, ρ) .

Proof. Let f be \mathcal{G}_g^ω -closed function and $A \subseteq X$ suth that A is g -closed set. Since f is \mathcal{G}_g^ω -closed function and A is g -closed set. Since it is clear that every g -closed set is \mathcal{G}_g^ω -closed set then $f(A)$ is a closed set in (Y, ρ) . Since any closed set is $rg\omega$ -closed set. Therefore $f(A)$ is $rg\omega$ -closed set in (Y, ρ) . \square

Theorem 3.30. Let $f : (X, \tau, \mathcal{G}) \rightarrow (Y, \rho)$ be a \mathcal{G}_g^ω -continuous function and (Y, ρ) is $T_{1|2}$ -space and A is g -closed subset in (Y, ρ) . Then $f^{-1}(A)$ is \mathcal{G}_g^ω -closed subset in (X, τ, \mathcal{G}) .

Proof. Let f be \mathcal{G}_g^ω -continuous function and (Y, ρ) is $T_{1|2}$ -space. Let A be g -closed subset in (Y, ρ) . Since (Y, ρ) is $T_{1|2}$ -space, then for every g -closed subset in (Y, ρ) is closed subset in (Y, ρ) . Since f is \mathcal{G}_g^ω -continuous function then for every A closed subset in (Y, ρ) implies $f^{-1}(A)$ is \mathcal{G}_g^ω -closed subset in (X, τ, \mathcal{G}) . \square

4 \mathcal{G}_{rg}^ω -Continuous functions

Definition 4.1. A function $f : (X, \tau, \mathcal{G}) \rightarrow (Y, \rho, \mathcal{H})$ of a grill topological space (X, τ, \mathcal{G}) into a grill topological space (Y, ρ, \mathcal{H}) is called \mathcal{G}_{rg}^ω -continuous function if $f^{-1}(U)$ is \mathcal{G}_{rg}^ω -closed set in (X, τ, \mathcal{G}) for every \mathcal{G}^ω -closed set U in (Y, ρ, \mathcal{H}) .

Definition 4.2. A function $f : (X, \tau, \mathcal{G}) \rightarrow (Y, \rho, \mathcal{H})$ of a grill topological space (X, τ, \mathcal{G}) into a grill topological space (Y, τ, \mathcal{H}) is called \mathcal{G}_{rg}^ω -irresolute if $f^{-1}(U)$ is \mathcal{G}_{rg}^ω -closed set in (X, τ, \mathcal{G}) for every \mathcal{G}_{rg}^ω -closed set in (Y, τ, \mathcal{H}) .

Remark 4.3. Let (X, τ, \mathcal{G}) be any grill topological space with $\tau = \{\emptyset, X, A\}$, $\mathcal{G} = P(X) - \{\emptyset\}$, where A is an uncountable set. Then any subset of (X, τ, \mathcal{G}) is a both \mathcal{G}_{rg}^ω -open set and \mathcal{G}_{rg}^ω -closed set.

Example 4.4. Let $f : (\mathbb{R}, \tau, \mathcal{G}) \rightarrow (\mathbb{R}, \tau_D, \mathcal{H})$ be a any function with

$$\tau = \{\emptyset, \mathbb{R}, (0, 1)\}, \mathcal{H} = P(\mathbb{R}) - \{\emptyset\} \text{ and } \mathcal{G} = P(\mathbb{R}) - \{\emptyset\}.$$

Since every $f^{-1}(U)$ is \mathcal{G}_{rg}^ω -closed set in $(\mathbb{R}, \tau, \mathcal{G})$ for every \mathcal{G}^ω -closed set U in $(\mathbb{R}, \tau_D, \mathcal{H})$. Then f is \mathcal{G}_{rg}^ω -continuous function.

Remark 4.5. Let (X, τ, \mathcal{G}) be any grill topological space with

$$\tau = \{\emptyset, X, A\} \cup \{A \cup B_i : A \subseteq B_i \text{ for every } i \in \mathbb{N}\} \text{ and } \mathcal{G} = P(X) - \{\emptyset\}.$$

Where A be any subset of X . Then any subset of (X, τ, \mathcal{G}) is a both \mathcal{G}_{rg}^ω -open set and \mathcal{G}_{rg}^ω -closed set.

Example 4.6. Let $f : (\mathbb{R}, \tau, \mathcal{G}) \rightarrow (Y, \rho, \mathcal{H})$ be a any function where (Y, ρ, \mathcal{H}) is any grill topological space and

$$\tau = \{\emptyset, \mathbb{R}, (0, 1], (0, 2), (0, 7)\}, \mathcal{H} = P(Y) - \{\emptyset\} \text{ and } \mathcal{G} = P(\mathbb{R}) - \{\emptyset\}.$$

Then f is \mathcal{G}_{rg}^ω -continuous function by Remark (4.5).

It is clear that every $rg\omega$ -continuous function is \mathcal{G}_{rg}^ω -continuous function but the converse of this fact no need to be true.

Example 4.7. Let $f : (\mathbb{R}, \tau, \mathcal{G}) \rightarrow (\mathbb{R}, \rho, \mathcal{H})$ be the identity function with

$$\tau = \{\emptyset, \mathbb{R}, (0, 1], \mathbb{R} - (0, 3)\}, \mathcal{H} = P(\mathbb{R}) - \{\emptyset\}, \mathcal{G} = P(\mathbb{R}) - \{\emptyset\} \text{ and } \rho = \{\emptyset, \mathbb{R}, \mathbb{R} - (0, \frac{1}{2}]\}$$

We note that f is \mathcal{G}_{rg}^ω -continuous function. Since $(0, \frac{1}{2}]$ and \mathbb{R} are closed sets in $(\mathbb{R}, \rho, \mathcal{H})$. Then $f^{-1}((0, \frac{1}{2}]) = (0, \frac{1}{2}]$ and $f^{-1}(\mathbb{R}) = \mathbb{R}$ are \mathcal{G}_{rg}^ω -closed sets but it is not $rg\omega$ -continuous function. Since $(0, \frac{1}{2}]$ is closed set in $(\mathbb{R}, \rho, \mathcal{H})$ but $f^{-1}((0, \frac{1}{2}]) = (0, \frac{1}{2}]$ is not $rg\omega$ -closed set in $(\mathbb{R}, \tau, \mathcal{G})$.

It is clear that every \mathcal{G}_g^ω -continuous function is \mathcal{G}_{rg}^ω -continuous function but the converse of this fact no need to be true.

Example 4.8. Let $f : (\mathbb{R}, \tau, \mathcal{G}) \rightarrow (\mathbb{R}, \rho, \mathcal{H})$ be the identity function with

$$\tau = \{\emptyset, \mathbb{R}, \mathbb{R} - (0, 1)\}, \mathcal{H} = P(\mathbb{R}) - \{\emptyset\}, \mathcal{G} = P(\mathbb{R}) - \{\emptyset\} \text{ and } \rho = \{\emptyset, \mathbb{R}, (0, 1)\}.$$

The function f is \mathcal{G}_{rg}^ω -continuous, since $f^{-1}(\{\mathbb{R} - (0, 1)\}) = \mathbb{R} - (0, 1)$ and $f^{-1}(\mathbb{R}) = \mathbb{R}$ are \mathcal{G}_{rg}^ω -closed sets in $(\mathbb{R}, \tau, \mathcal{G})$. The function f is not \mathcal{G}_g^ω -continuous function, since $f^{-1}(\{\mathbb{R} - (0, 1)\}) = \mathbb{R} - (0, 1)$ is not \mathcal{G}_g^ω -closed set in $(\mathbb{R}, \tau, \mathcal{G})$.

It is clear that every \mathcal{G}_{rg}^ω -irresolute function is \mathcal{G}_{rg}^ω -continuous function but the converse of this fact no need to be true.

Example 4.9. Let $f : (\mathbb{R}, \tau, \mathcal{G}) \rightarrow (\mathbb{R}, \rho_I, \mathcal{H})$ be the identity function with

$$\tau = \{\emptyset, \mathbb{R}, (0, 1), (0, 1], \mathbb{R} - (0, 3), \mathbb{R} - (1, 3), \mathbb{R} - [1, 3)\}, \mathcal{G} = P(\mathbb{R}) - \{\emptyset\} \text{ and } \mathcal{H} = P(\mathbb{R}) - \{\emptyset\}.$$

We note that f is \mathcal{G}_{rg}^ω -continuous function but it is not \mathcal{G}_{rg}^ω -irresolute function. Since $(0, 1)$ is \mathcal{G}_{rg}^ω -closed set in $(\mathbb{R}, \rho_I, \mathcal{H})$ but $f^{-1}((0, 1)) = (0, 1)$ is not \mathcal{G}_{rg}^ω -closed set in $(\mathbb{R}, \tau, \mathcal{G})$.

It is clear that every \mathcal{G}_g^ω -irresolute function is \mathcal{G}_{rg}^ω -irresolute function but the converse of this fact no need to be true.

Example 4.10. Let $f : (\mathbb{R}, \tau, \mathcal{G}) \rightarrow (\mathbb{R}, \rho_I, \mathcal{H})$ be the identity function with

$$\tau = \{\emptyset, \mathbb{R}, \mathbb{R} - (1, 3)\}, \mathcal{H} = P(\mathbb{R}) - \{\emptyset\} \text{ and } \mathcal{G} = P(\mathbb{R}) - \{\emptyset\}.$$

We note that f is \mathcal{G}_{rg}^ω -irresolute function but it is not \mathcal{G}_g^ω -irresolute function. Since the set $\mathbb{R} - (1, 3)$ is \mathcal{G}_g^ω -closed set in $(\mathbb{R}, \rho_I, \mathcal{H})$ but $f^{-1}(\mathbb{R} - (1, 3)) = \mathbb{R} - (1, 3)$ is not \mathcal{G}_g^ω -closed set in $(\mathbb{R}, \tau, \mathcal{G})$.

The following figure is an enlargement of figure(2) and an theorems above we have the following figure:

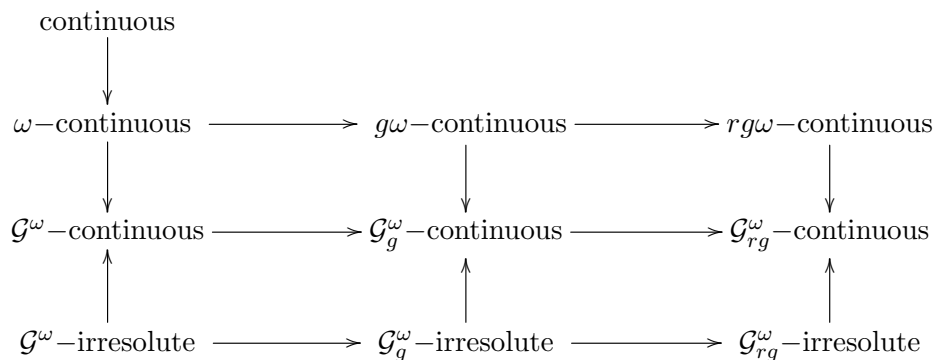


Figure 3:

Definition 4.11. A function $f : (X, \tau, \mathcal{G}) \rightarrow (Y, \rho, \mathcal{H})$ of a grill topological space (X, τ, \mathcal{G}) into a grill topological space (Y, ρ, \mathcal{H}) is called:

1. \mathcal{G}_{rg}^ω -closed function if $f(G)$ is a closed set in (Y, ρ) for every \mathcal{G}_{rg}^ω -closed set G in (X, τ, \mathcal{G}) .
2. \mathcal{G}_{rg}^ω -open function if $f(G)$ is a open set in (Y, ρ) for every \mathcal{G}_{rg}^ω -open set G in (X, τ, \mathcal{G}) .
3. $pre\text{-}\mathcal{G}^\omega$ -closed function if $f(V)$ is \mathcal{G}^ω -closed in (Y, ρ, \mathcal{H}) for every \mathcal{G}^ω -closed subset V of (X, τ, \mathcal{G}) .

Example 4.12. In Example(4.4), we note that f is a both \mathcal{G}_{rg}^ω -closed function and \mathcal{G}_{rg}^ω -open function.

Definition 4.13. A grill topological space (X, τ, \mathcal{G}) is a regular generalized $\mathcal{G}^\omega - T_{1|2}$ -space (simply, $\mathcal{G}_{rg}^\omega - T_{1|2}$ -space) if every \mathcal{G}_{rg}^ω -closed set in (X, τ, \mathcal{G}) is \mathcal{G}^ω -closed set.

Theorem 4.14. Let $f : (X, \tau, \mathcal{G}) \rightarrow (Y, \rho, \mathcal{H})$ be a function of a grill topological space (X, τ, \mathcal{G}) into a grill topological space (Y, ρ, \mathcal{H}) . Then f is \mathcal{G}_{rg}^ω -continuous function if $f^{-1}(U)$ is \mathcal{G}_{rg}^ω -open set in (X, τ, \mathcal{G}) for every \mathcal{G}^ω -open set in (Y, ρ, \mathcal{H}) .

Proof. Let f be \mathcal{G}_g^ω -continuous function. Let U be \mathcal{G}^ω -open set in (Y, ρ, \mathcal{H}) then $Y - U$ is \mathcal{G}^ω -closed set in (Y, ρ, \mathcal{H}) . Since f is \mathcal{G}_{rg}^ω -continuous function. Then $f^{-1}(Y - U) = X - f^{-1}(U)$ is \mathcal{G}_g^ω -closed set in a grill topological space (X, τ, \mathcal{G}) . Therefore $f^{-1}(U)$ is \mathcal{G}_{rg}^ω -open set in (X, τ, \mathcal{G}) . Hence f is \mathcal{G}_{rg}^ω -continuous function. \square

Theorem 4.15. Let $f : (X, \tau, \mathcal{G}) \rightarrow (Y, \rho, \mathcal{I})$ and $g : (Y, \rho, \mathcal{H}) \rightarrow (Z, \gamma, \mathcal{K})$ two functions. Then the following hold:

1. $g \circ f$ is \mathcal{G}_{rg}^ω -continuous function if g is \mathcal{G}^ω -irresolute function and f is \mathcal{G}_{rg}^ω -continuous function.
2. $g \circ f$ is \mathcal{G}_{rg}^ω -continuous function if g is \mathcal{G}_{rg}^ω -continuous function and f is \mathcal{G}_{rg}^ω -irresolute function.
3. $g \circ f$ is \mathcal{G}_{rg}^ω -irresolute function if f and g are \mathcal{G}_{rg}^ω -irresolute function.
4. Let (Y, ρ) be a $\mathcal{G}_{rg}^\omega - T_{1/2}$ -space. Then, $g \circ f$ is \mathcal{G}_{rg}^ω -continuous function if g is \mathcal{G}_{rg}^ω -continuous function and f is \mathcal{G}_{rg}^ω -continuous function.

The following Example shows that the composition of two \mathcal{G}_{rg}^ω -continuous functions need not be \mathcal{G}_{rg}^ω -continuous.

Example 4.16. Let $f : (\mathbb{R}, \tau, \mathcal{G}) \rightarrow (Y, \rho, \mathcal{H})$ and $g : (Y, \rho, \mathcal{H}) \rightarrow (Y, \gamma, \mathcal{K})$ be two functions defined by:

$$f(x) = \begin{cases} 0, & x \in (0, 1) \\ 1, & x \in \mathbb{R} - (0, 1) \end{cases}$$

And g is identity function. Where $Y = \{0, 1\}$,

$$\tau = \{\emptyset, \mathbb{R}, (0, 1), (0, 1], \mathbb{R} - (0, 3), \mathbb{R} - (1, 3), \mathbb{R} - [1, 3)\}, \mathcal{G} = P(\mathbb{R}) - \{\emptyset\}, \mathcal{K} = P(Y) - \{\emptyset\}, \mathcal{H} = P(Y) - \{\emptyset\},$$

$$\rho = \{\emptyset, Y, \{0\}\} \text{ and } \gamma = \{\emptyset, \{1\}, Y\}.$$

Then f and g are \mathcal{G}_{rg}^ω -continuous function but $g \circ f$ is not \mathcal{G}_{rg}^ω -continuous function. Since $\{0\}$ is \mathcal{G}^ω -closed set in (Y, γ, \mathcal{H}) but $f^{-1}(\{0\}) = (0, 1)$ is not \mathcal{G}_{rg}^ω -closed set in $(\mathbb{R}, \tau, \mathcal{G})$.

Theorem 4.17. Let (X, τ) be any door topological space and $f : (X, \tau, \mathcal{G}) \rightarrow (Y, \rho_D, \mathcal{H})$ be any function. Then f is a both \mathcal{G}_{rg}^ω -open function and \mathcal{G}_{rg}^ω -closed function and it is \mathcal{G}_{rg}^ω -continuous function.

Proof. Let (X, τ) be any door topological space. Let $A \subseteq X$, since X is door topological space then A is either open set or closed set. If A is open set of X , it is clear that every open set is \mathcal{G}_{rg}^ω -open set implies A is \mathcal{G}_{rg}^ω -open set. Then $f(A)$ is open set in Y . Therefore f is \mathcal{G}_{rg}^ω -open function. If A is closed set of X , it clear that every closed set is \mathcal{G}_{rg}^ω -closed set implies A is \mathcal{G}_{rg}^ω -closed set. Then $f(A)$ is closed set in Y . Therefore f is \mathcal{G}_{rg}^ω -closed function.

Let $A \subseteq Y$, since X is door topological space then $f^{-1}(A)$ is either open set or closed set in X . If $f^{-1}(A)$ is \mathcal{G}_{rg}^ω -open set, since Y is (Y, ρ_D) then A is open set. If $f^{-1}(A)$ is \mathcal{G}_{rg}^ω -closed set since Y is (Y, ρ_D) then A is closed set. Therefore f is \mathcal{G}_{rg}^ω -continuous function. \square

In Theorem(4.17), if X is a countable set and (Y, ρ) be any topological space then f is \mathcal{G}_{rg}^ω -continuous function.

Theorem 4.18. Let $f : (X, \tau_I, \mathcal{G}) \rightarrow (Y, \rho_D, \mathcal{H})$ be any function. Then f is \mathcal{G}_{rg}^ω -continuous function.

Proof. Let $A \subseteq Y$ and since Y is (Y, ρ_D, \mathcal{H}) then any subset of Y is a both open set and closed set. Now $f^{-1}(A)$ is a both \mathcal{G}_{rg}^ω -open set and \mathcal{G}_{rg}^ω -closed set. Therefore f is \mathcal{G}_{rg}^ω -continuous function by Remark(3.17). \square

Theorem 4.19. Let $f : (X, \tau_I, \mathcal{G}) \rightarrow (Y, \rho, \mathcal{H})$ be any function. Then f is \mathcal{G}_{rg}^ω -continuous function.

Proof. Let A be an open subset of Y . Then $f^{-1}(A)$ is \mathcal{G}_{rg}^ω -open set by Remark(3.17). Let A be a closed subset of Y . Then $f^{-1}(A)$ is \mathcal{G}_{rg}^ω -closed set by Remark(3.17). Therefore f is \mathcal{G}_{rg}^ω -continuous function. \square

Theorem 4.20. Every \mathcal{G}_{rg}^ω -closed function is \mathcal{G}_g^ω -closed function but the converse of this fact no need to be true.

Example 4.21.

Theorem 4.22. A grill topological space (X, τ, \mathcal{G}) is called a regular generalized $\mathcal{G}^\omega - T_{1|2}$ -space (simply $\mathcal{G}_{rg}^\omega - T_{1|2}$ -space) if and only if every \mathcal{G}_{rg}^ω -open set in (X, τ, \mathcal{G}) is \mathcal{G}^ω -open set.

Theorem 4.23. A grill topological space (X, τ, \mathcal{G}) is called a regular generalized $\mathcal{G}^\omega - T_{1|2}$ -space (simply $\mathcal{G}_{rg}^\omega - T_{1|2}$ -space) if and only if every singleton set in (X, τ, \mathcal{G}) is either regular-closed set or \mathcal{G}^ω -open set.

Proof. Suppose that (X, τ, \mathcal{G}) is $\mathcal{G}_{rg}^\omega - T_{1|2}$ -space and $\{x\}$ is not regular-closed subset of X for some $x \in X$. Then $X - \{x\}$ is not regular-open set in X . Hence X is the only regular-open set containing $X - \{x\}$. That is, $X - \{x\}$ is \mathcal{G}_{rg}^ω -closed set in X . Since (X, τ, \mathcal{G}) is $\mathcal{G}_{rg}^\omega - T_{1|2}$ -space then $X - \{x\}$ is \mathcal{G}^ω -closed set in X . That is, $\{x\}$ is a \mathcal{G}^ω -open set in X .

Conversely, suppose that every singleton set in X is either regular-closed or \mathcal{G}^ω -open Let A be any \mathcal{G}_{rg}^ω -closed set in X and $x \in g^\omega Cl(A)$. We show that $x \in A$. By the hypothesis $\{x\}$ is either regular-closed set or \mathcal{G}^ω -open set in X . The set $\{x\}$ is regular-closed set and $x \notin A$ then

$$x \in g^\omega Cl(A) - A \subseteq X - A.$$

Then $\{x\} \subseteq X - A$ and so $A \subseteq X - \{x\}$. Since A is \mathcal{G}_{rg}^ω -closed set in X contained in regular-open set $X - \{x\}$ then $\mathcal{G}^\omega \subseteq X - \{x\}$ and so $\{x\} \subseteq X - \mathcal{G}^\omega Cl(A)$. Therefore

$$\{x\} \subseteq \mathcal{G}^\omega Cl(A) \cap [X - \mathcal{G}^\omega Cl(A)] = \emptyset,$$

and this is a contradiction. Hence $\{x\} \in A$, that is, $\mathcal{G}^\omega Cl(A) = A$ and so A is \mathcal{G}^ω -closed set. If $\{x\}$ is

\mathcal{G}^ω -open set and $\{x\} \in \mathcal{G}^\omega Cl(A)$ then we have $\{x\} \cap A \neq \emptyset$. Hence $\{x\} \in A$, that is, $\mathcal{G}^\omega Cl(A) = A$ and so A is \mathcal{G}^ω -closed set. \square

Theorem 4.24. If $f : (X, \tau, \mathcal{G}) \rightarrow (Y, \rho, \mathcal{H})$ is \mathcal{G}_{rg}^ω -continuous function and X is $\mathcal{G}_{rg}^\omega - T_{1|2}$ -space then f is \mathcal{G}^ω -continuous.

Proof. Let f be any \mathcal{G}_{rg}^ω -continuous function and X is $\mathcal{G}_{rg}^\omega - T_{1|2}$ -space then every \mathcal{G}_{rg}^ω -closed subset of X is \mathcal{G}^ω -closed set. Let A be any closed set in Y . Since any closed set is \mathcal{G}^ω -closed set, f is \mathcal{G}_{rg}^ω -continuous function and $\mathcal{G}_{rg}^\omega - T_{1|2}$ -space then $f^{-1}(A)$ is \mathcal{G}^ω -closed set. Therefore f is \mathcal{G}^ω -continuous function. \square

Theorem 4.25. Let (X, τ, \mathcal{G}) be any grill topological space with a countable set X . Then every \mathcal{G}_{rg}^ω -closed subset of X is \mathcal{G}_g^ω -closed set.

Theorem 4.26. Let $f : (X, \tau, \mathcal{G}) \rightarrow (Y, \rho, \mathcal{H})$ be any function and X is countable set. Then f is \mathcal{G}_{rg}^ω -continuous function and it is \mathcal{G}_g^ω -continuous function.

Theorem 4.27. If $f : (X, \tau, \mathcal{G}) \rightarrow (Y, \rho, \mathcal{H})$ is \mathcal{G}_{rg}^ω -continuous, then for each $x \in X$ and each \mathcal{G}^ω -open set V in (Y, ρ, \mathcal{H}) with $f(x) \in V$, there exists \mathcal{G}_{rg}^ω -open set U in (X, τ, \mathcal{G}) such that $x \in U$ and $f(U) \subseteq V$.

Proof. Let f is \mathcal{G}_{rg}^ω -continuous function. Let $x \in X$ and V be any \mathcal{G}^ω -open set in (Y, ρ, \mathcal{H}) containing $f(x)$. Put $U = f^{-1}(V)$. Since f is \mathcal{G}_{rg}^ω -continuous then U is \mathcal{G}_{rg}^ω -open set in (X, τ, \mathcal{G}) such that $x \in U$ and $f(U) \subseteq V$. \square

The converse of the above theorem is not true in general as the following example shows.

Example 4.28. In Example(4.16), we note that f is satisfies the property stated in Theorem(4.27). The function f is not \mathcal{G}_{rg}^ω -continuous function. Since $\{0\}$ is \mathcal{G}^ω -closed set in Y but $f^{-1}(\{0\}) = (0, 1)$ is not \mathcal{G}_{rg}^ω -closed set in $(\mathbb{R}, \tau, \mathcal{G})$.

Theorem 4.29. Let $f : (X, \tau, \mathcal{G}) \rightarrow (Y, \rho, \mathcal{H})$ be a surjective, \mathcal{G}_{rg}^ω -irresolute, and pre- \mathcal{G}^ω -closed function. if X is $\mathcal{G}_{rg}^\omega - T_{1|2}$ -space, then Y is also an $\mathcal{G}_{rg}^\omega - T_{1|2}$ -space.

Proof. Let A be \mathcal{G}_{rg}^ω -closed subset of Y . Since f is an \mathcal{G}_{rg}^ω -irresolute function, then $f^{-1}(A)$ is an \mathcal{G}_{rg}^ω -closed subset of X . Since X is $\mathcal{G}_{rg}^\omega - T_{1|2}$ -space, then $f^{-1}(A)$ is an \mathcal{G}^ω -closed subset of X . Since f is a pre- \mathcal{G}^ω -closed function, then $f(f^{-1}(A)) = A$ is an \mathcal{G}^ω -closed subset of Y . Therefore Y is also $\mathcal{G}_{rg}^\omega - T_{1|2}$ -space. \square

Theorem 4.30. Let $f : (X, \tau, \mathcal{G}) \rightarrow (Y, \rho, \mathcal{H})$ be ro-preserving and \mathcal{G}^ω -irresolute function, if B is \mathcal{G}_{rg}^ω -closed set in Y , then $f^{-1}(B)$ is \mathcal{G}_{rg}^ω -closed set in X .

Proof. Let U be a regular-open subset of X such that $f^{-1}(B) \subseteq U$. Then $B \subseteq f(U)$ and $f(U)$ is regular-open. Since B is \mathcal{G}_{rg}^ω -closed set, then ${}_{\mathcal{G}^\omega}Cl(A) \subseteq f(U)$ and $f^{-1}({}_{\mathcal{G}^\omega}Cl(B)) \subseteq U$. Since f is \mathcal{G}^ω -irresolute then $f^{-1}({}_{\mathcal{G}^\omega}Cl(B))$ is \mathcal{G}^ω -closed and ${}_{\mathcal{G}^\omega}Cl(f^{-1}({}_{\mathcal{G}^\omega}Cl(B))) = f^{-1}({}_{\mathcal{G}^\omega}Cl(B))$, therefore ${}_{\mathcal{G}^\omega}Cl(f^{-1}(B)) \subseteq {}_{\mathcal{G}^\omega}Cl(f^{-1}({}_{\mathcal{G}^\omega}Cl(B))) \subseteq U$ thus $f^{-1}(B)$ is \mathcal{G}_{rg}^ω -closed set in X . \square

Definition 4.31. A function $f : (X, \tau, \mathcal{G}) \rightarrow (Y, \rho)$ is said to be \mathcal{G}^ω -contra- R -function if for every regular-open subset V of Y , $f^{-1}(V)$ is \mathcal{G}^ω -closed set.

Example 4.32. Let $f : (\mathbb{R}, \tau, \mathcal{G}) \rightarrow (Y, \rho)$ be function defined

$$f(x) = \begin{cases} c, & x \in \mathbb{R} - (0, 1] \\ a, & x \in (0, 1] \end{cases}$$

where $Y = \{a, b, c\}$,

$$\tau = \{\emptyset, \mathbb{R}, \mathbb{R} - (0, 1]\}, \mathcal{G} = P(\mathbb{R}) - \{\emptyset\}, \text{ and } \rho = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}.$$

Then f is \mathcal{G}^ω -contra- R -function. But the function $f(x)$ defined by:

$$f(x) = \begin{cases} b, & x \in \mathbb{R} - (0, 1] \\ a, & x \in (0, 1] \end{cases}$$

is not \mathcal{G}^ω -contra- R -function. Since $\{b\}$ is regular-open in Y and $f^{-1}(\{b\}) = \mathbb{R} - (0, 1]$ is not \mathcal{G}^ω -closed.

Theorem 4.33. Let $f : (X, \tau, \mathcal{G}) \rightarrow (Y, \rho)$ be \mathcal{G}_{rg}^ω -closed function. Then every U is \mathcal{G}_{rg}^ω -closed set in X exists V is closed set in Y such that $f^{-1}(V) \in \mathcal{G}_{rg}^\omega C(X, \tau, \mathcal{G})$.

Proof. Let U be any \mathcal{G}_{rg}^ω -closed set in (X, τ, \mathcal{G}) . Since f is \mathcal{G}_{rg}^ω -closed function then every \mathcal{G}_{rg}^ω -closed set in (X, τ, \mathcal{G}) such that $f(U) = V$ is closed set in (Y, ρ) . Therefore $f^{-1}(V) = U$ is \mathcal{G}_{rg}^ω -closed set. \square

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