

Method Article

# Numerical solution of Landau-Lifshitz Equation in the theory of magnetic elastic

## Abstract

In this paper we study Landau-Lifshitz equation which describe a nonlinear magnetic elastic in easy-axis ferromagnets. By means of numerical simulation a numerical solitonic solution to above mentioned equation with SFD and NSFD rules and obtained as a result of solution of Cauchy problem using exact analytical solution to Landau-Lifshitz equation as initial data. Then we obtain the error of each method

*Keywords:* Nonstandard finite difference scheme; Standard finite difference scheme; Landau-Lifshitz equation.

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## 1 Introduction

Investigation of nonlinear properties of magnetoelastic crystals in condensed matter is very important. Most of this work has been done for linear models. It seems, that nonlinear magnetoelastic waves in nonlinear approximation of phonon mode demonstrate a wide range difference their propagation and

interactions in comparison to a linear one. This paper arises to study nonlinear magnetoelastic waves in nonlinear approximation by phonon mode. The paper is organized as follows. In section 2 starting from quantum-mechanical Hamiltonian taking into account modulation of exchange integrals as a result of oscillation of each site of spin chain, using generalized coherent states of  $SU(2)$  group as a basis of trial functions following paper [5], we receive Landau-Lifshitz equation which describes magnetoelastic waves in nonlinear approximation of phonon modes analytically we were able to our best find the solution to the above mentioned system only in linear approximation of phonon mode, which requires application of numerical methods. In the following we discuss a numerical approach, which is not so simple due to singularities arising in the poles of Bloch sphere, as for all types of sigma-models. To avoid this we use a stereographical projection to a complex plane, that corresponds to the projection of  $SO(3)$  space to a coset  $SU(2)/U(1)$  space, due to their isomorphism. In section 3 introduction NSFD Rules and implementation of NSFD for Landau-Lifshitz equation. In section 4 we solve the Cauchy problem using the analytical solution with a linear approximation in phonon mode as a initial condition, which could be considered as a perturbed solitonic solution. As a result evolution of the initial perturbation we receive a stable solitonic solution moving with some speed, and perturbations irradiated to the boundaries, where they are absorbed by so called "black box" boundary conditions. Let us remind, that the Landau-Lifshitz equation could be written, in the isotropic case, in the following form

$$i\hbar S_t = \frac{1}{2} [S, S_{xx}] \quad (1.1)$$

Where we introduce  $S = \begin{pmatrix} S^z & S^- \\ S^+ & -S^z \end{pmatrix} = S \cdot \sigma$ ,

$$\sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma^y = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \text{ are Pauli operators.}$$

This equation (1.1) gives us a macroscopic description of the magnetization dynamic in ferromagnets and represents the equation of motion of the magnetization vector in non-dissipative media. On the other hand, it is well known, that in microscopic level the most popular models for description of magnetic properties of a crystal are spin models of the Heisenberg-Frenkel magnet

$$\hat{H} = - \sum_{i,k} J_{ik} \hat{S}_i \hat{S}_k \quad (1.2)$$

where  $J_{ik}$  are exchange integrals ,  $\hat{S}_j$  are the spin operators of the atom in the  $j$ th site.

## 2 Mathematical Modeling of problem

Let us consider the model of the Heisenberg ferromagnet with single-axis anisotropy in the presence of oscillations of sites of the crystal lattice [5]

$$\hat{H} = \hat{H}_s + \hat{H}_p$$

where

$$\begin{aligned} \hat{H}_s &= - \sum_{j=1}^N \left\{ \frac{J^0}{2} (\hat{s}_j^+ s_{j+1}^- + \hat{s}_j^- s_{j+1}^+) + J^z \hat{s}_j^z s_{j+1}^z \right\} \\ &= -J^0 \sum_{j=1}^N \{ \hat{s}_j s_{j+1} + \Delta \hat{s}_j^z s_{j+1}^z \} \end{aligned}$$

is the spin part of the Hamiltonian,

$$H_p = \sum_{j=1}^N \left\{ \frac{p_j^2}{2m} + \frac{k}{2} (y_{j+1} - y_j)^2 \right\}$$

is the phonon part of the Hamiltonian. Here  $\Delta = \frac{(J^z - J^0)}{J^0}$  is the constant of exchange anisotropy,  $m$  and  $p$  are the mass and momentum of the atom, correspondingly  $|y_{j+1} - y_j|$  is the displacement of the  $j$ th atom from the equilibrium position,  $k$  is the elastic constant,  $j$  is the summation index, here  $s^\pm = s^x \pm is^y$ . Let us pass over to the classical description. In order to make this we average, by use of the  $SU(2)$  generalized coherent states ( $GCS$ ) [6],[7]. Let us remind the reader that  $SU(2)GCS$  in complex parameterization has the form

$$|z\rangle = \prod_j |z_j\rangle = \prod_j (1 + |z_j|^2)^{-k} \exp\{z_j \hat{S}_j^+\} |k, -k\rangle, \quad (2.1)$$

here  $k$  is the number of representation, is the parameter of quasiclassical description. Spin operators averaged by use of the  $SU(2)GCS$  get the following form

$$S_j^+ = \overline{S_j^-} = \langle \hat{S}_j^+ \rangle = \frac{2z_j}{1 + |z_j|^2}, S_j^z = \langle \hat{S}_j^z \rangle = \frac{1 - |z_j|^2}{1 + |z_j|^2} \quad (2.2)$$

Note that the parameterization via more habitual angle variables is possible. In the case the values of the averaged spin operators have the form

$$S = s (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta) \quad (2.3)$$

$$z_j = \tan\left(\frac{\theta}{2}\right) \exp\{i\varphi_j\}, \quad (2.4)$$

where the values of the angle parameters are restricted as  $0 \leq \theta \leq \pi$  and  $0 \leq \varphi \leq 2\pi$ . By use (2.3) we obtained Landau-Lifshitz equation as the following equations [5]

$$\begin{aligned} a_0^2\theta_{xx} - [a_0^2\varphi_x^2 + 2\Delta] \sin\theta\cos\theta + \frac{\hbar}{J^0 s^2} \sin\theta\varphi_t &= 0, \\ a_0^2 (\sin^2\theta\varphi_x)_x - \frac{\hbar}{J^0 s^2} \sin\theta\theta_t &= 0 \end{aligned} \quad (2.5)$$

By use of the vector (2.1) we carry out the averaging procedure of the spin Hamiltonian  $\hat{H}_s$ . We have

$$H = \langle z | \hat{H} | z \rangle = H_s + H_p. \quad (2.6)$$

$$z_{j+1} = z_j + a_0 z_{jx} + \frac{a_0^2}{2} z_{jxx} + \dots$$

$$y_{j+1} = y_j + a_0 y_{jx} + \dots \quad (2.7)$$

Then rewriting Hamiltonian(2.4) in spherical variables we obtain

$$H = -s^2 \sum_{j=1}^N \left\{ \frac{J^0}{2} (S_j^+ S_{j+1}^- + S_j^- S_{j+1}^+) + J^z S_j^z S_{j+1}^z \right\} + H_p. \quad (2.8)$$

Introducing Poisson brackets in the following form

$$\{A, B\} = \int \left\{ -\varepsilon_{ijk} \frac{\delta A}{\delta S^i} \frac{\delta B}{\delta S^j} - \frac{\delta A}{\delta P} \frac{\delta B}{\delta y} + \frac{\delta A}{\delta y} \frac{\delta B}{\delta P} \right\} dx$$

We derive the equations of dynamics of the magnetization vector coupled with the lattice oscillations

$$\hbar S_t^j = \{H, S^j\} = -\varepsilon_{ijk} \frac{\delta H}{\delta S^i} S^k$$

then

$$\begin{aligned} \hbar \vec{S}_t &= s^2 J^0 \frac{a_0^2}{2} (2S_{xx}^y S^z - 2S_{xx}^z S^y) \vec{i} - s^2 J^0 \frac{a_0^2}{2} (2S_{xx}^x S^z - 2S_{xx}^z S^x) \vec{j} \\ &+ s^2 J^0 \frac{a_0^2}{2} (2S_{xx}^x S^y - 2S_{xx}^y S^x) \vec{k} - s^2 J^0 2\Delta (S^z S^y \vec{i} - S^z S^x \vec{j}) \end{aligned} \quad (2.9)$$

we have  $S = (S^x, S^y, S^z)$ ,  $e^z = (0, 0, 1)$  then from (2, 9) obtain

$$\hbar \vec{S}_t + s^2 J^0 a_0^2 [\vec{S} \times \vec{S}_{xx}] + 2s^2 J^0 \Delta S^z [\vec{S} \times \vec{e}^z] = 0 \quad (2.10)$$

or rewriting this equations in matrixs form we obtain

$$i\hbar S_t + s^2 J^0 \frac{a_0^2}{2} [S, S_{xx}] + s^2 J^0 \frac{\Delta}{2} [S, \hat{\sigma}^z] \{S, \sigma^z\} = 0 \quad (2.11)$$

(2, 5), (2, 9), (2, 10), (2, 11) are different form of Landau-Lifshitz equation.

To conduct numerical simulation of magnetoelastic interaction processes it is not convenient the parametrization by spherical variables of by Euler angles, due to the singularities in the poles. In order to prevent that for numerical simulation we use stereographical projection.

**a)** The relationship between the complex parameters  $z$  and  $S^x, S^y, S^z$  by stereographical projection for top plane is given with the following relations

$$S^x = \frac{z + \bar{z}}{1 + z\bar{z}}, S^y = \frac{\bar{z} - z}{1 + z\bar{z}}i, S^z = \frac{1 - z\bar{z}}{1 + z\bar{z}}$$

with use chain rule we have

$$\begin{aligned} S_x^z &= \frac{-2\bar{z}}{(1 + z\bar{z})^2} z_x + \frac{-2z}{(1 + z\bar{z})^2} \bar{z}_x \\ S_x^x &= \frac{1 - \bar{z}^2}{(1 + z\bar{z})^2} z_x + \frac{1 - z^2}{(1 + z\bar{z})^2} \bar{z}_x \\ S_x^y &= \frac{-1 - \bar{z}^2}{(1 + z\bar{z})^2} i z_x + \frac{1 + z^2}{(1 + z\bar{z})^2} i \bar{z}_x \\ S_{xx}^x &= \frac{-2\bar{z}(1 - \bar{z}^2)}{(1 + z\bar{z})^3} (z_x)^2 + \frac{-2z(1 - z^2)}{(1 + z\bar{z})^3} (\bar{z}_x)^2 + \frac{-4z - 4\bar{z}}{(1 + z\bar{z})^3} z_x \bar{z}_x + \frac{1 - \bar{z}^2}{(1 + z\bar{z})^2} z_{xx} + \frac{1 - z^2}{(1 + z\bar{z})^2} \bar{z}_{xx} \end{aligned}$$

$$\begin{aligned}
 S_{xx}^y &= \frac{2\bar{z}(1+\bar{z}^2)}{(1+z\bar{z})^3}i(z_x)^2 + \frac{-2z(1+z^2)}{(1+z\bar{z})^3}i(\bar{z}_x)^2 + \frac{4z-4\bar{z}}{(1+z\bar{z})^3}iz_x\bar{z}_x + \frac{1+z^2}{(1+z\bar{z})^2}i\bar{z}_{xx} + \frac{-1-\bar{z}^2}{(1+z\bar{z})^2}iz_{xx} \\
 S_{xx}^z &= \frac{4\bar{z}^2}{(1+z\bar{z})^3}(z_x)^2 + \frac{4z^2}{(1+z\bar{z})^3}(\bar{z}_x)^2 + \frac{4z\bar{z}-4}{(1+z\bar{z})^3}z_x\bar{z}_x - \frac{2\bar{z}}{(1+z\bar{z})^2}z_{xx} - \frac{2z}{(1+z\bar{z})^2}\bar{z}_{xx} \\
 S_{xx}^y S^z - S_{xx}^z S^y &= \frac{-2z\bar{z}^4 - 2\bar{z}^3 + 2z\bar{z}^2 + 2\bar{z}}{(1+z\bar{z})^4}i(z_x)^2 + \frac{2\bar{z}z^4 + 2z^3 - 2\bar{z}z^2 - 2z}{(1+z\bar{z})^4}i(\bar{z}_x)^2 + \\
 &\quad \frac{z\bar{z}^3 + \bar{z}^2 - z\bar{z} - 1}{(1+z\bar{z})^3}i(z_{xx}) + \frac{-\bar{z}z^3 - z^2 + z\bar{z} + 1}{(1+z\bar{z})^3}i(\bar{z}_{xx}) \\
 S_{xx}^x S^y - S_{xx}^y S^x &= \frac{-4\bar{z}^3z - 4\bar{z}^2}{(1+z\bar{z})^4}i(z_x)^2 + \frac{4\bar{z}z^3 + 4z^2}{(1+z\bar{z})^4}i(\bar{z}_x)^2 + \frac{2\bar{z} + 2z\bar{z}^2}{(1+z\bar{z})^3}iz_{xx} + \frac{-2z - 2\bar{z}z^2}{(1+z\bar{z})^3}i\bar{z}_{xx} \\
 S_{xx}^x S^z - S_{xx}^z S^x &= \frac{-2\bar{z}^4z - 2\bar{z}^3 - 2\bar{z}^2z - 2\bar{z}}{(1+z\bar{z})^4}(z_x)^2 + \frac{-2z^4\bar{z} - 2z^3 - 2z^2\bar{z} - 2z}{(1+z\bar{z})^4}(\bar{z}_x)^2 + \\
 &\quad \frac{z\bar{z}^3 + \bar{z}^2 + z\bar{z} + 1}{(1+z\bar{z})^3}z_{xx} + \frac{\bar{z}z^3 + z^2 + z\bar{z} + 1}{(1+z\bar{z})^3}\bar{z}_{xx}
 \end{aligned}$$

from (2, 9) we have  $\hbar S_t^x = s^2 J^0 a_0^2 (S_{xx}^y S^z - S_{xx}^z S^y) - s^2 J^0 2\Delta S^z S^y$  then

$$\begin{aligned}
 \hbar \left( \frac{1-\bar{z}^2}{(1+z\bar{z})^2}z_t + \frac{1-z^2}{(1+z\bar{z})^2}\bar{z}_t \right) &= s^2 J^0 a_0^2 \left( \frac{-2z\bar{z}^4 - 2\bar{z}^3 + 2z\bar{z}^2 + 2\bar{z}}{(1+z\bar{z})^4}i(z_x)^2 + \right. \\
 &\quad \left. \frac{2\bar{z}z^4 + 2z^3 - 2\bar{z}z^2 - 2z}{(1+z\bar{z})^4}i(\bar{z}_x)^2 + \frac{z\bar{z}^3 + \bar{z}^2 - z\bar{z} - 1}{(1+z\bar{z})^3}i(z_{xx}) + \right. \\
 &\quad \left. \frac{-\bar{z}z^3 - z^2 + z\bar{z} + 1}{(1+z\bar{z})^3}i(\bar{z}_{xx}) \right) - s^2 J^0 2\Delta \frac{\bar{z}-z}{1+z\bar{z}}i \times \frac{1-z\bar{z}}{1+z\bar{z}} \quad (2.12)
 \end{aligned}$$

and also  $\hbar S_t^y = -s^2 J^0 a_0^2 (S_{xx}^x S^z - S_{xx}^z S^x) + s^2 J^0 2\Delta S^z S^x$  then

$$\begin{aligned}
 \hbar \left( \frac{-1-\bar{z}^2}{(1+z\bar{z})^2}iz_t + \frac{1+z^2}{(1+z\bar{z})^2}i\bar{z}_t \right) &= -s^2 J^0 a_0^2 \left( \frac{-2\bar{z}^4z - 2\bar{z}^3 - 2\bar{z}^2z - 2\bar{z}}{(1+z\bar{z})^4}(z_x)^2 + \right. \\
 &\quad \left. \frac{-2z^4\bar{z} - 2z^3 - 2z^2\bar{z} - 2z}{(1+z\bar{z})^4}(\bar{z}_x)^2 + \frac{z\bar{z}^3 + \bar{z}^2 + z\bar{z} + 1}{(1+z\bar{z})^3}z_{xx} + \frac{\bar{z}z^3 + z^2 + z\bar{z} + 1}{(1+z\bar{z})^3}\bar{z}_{xx} \right) \\
 &\quad + s^2 J^0 2\Delta \frac{z+\bar{z}}{1+z\bar{z}} \times \frac{1-z\bar{z}}{1+z\bar{z}}
 \end{aligned}$$

then

$$\begin{aligned} \hbar \left( \frac{-1 - \bar{z}^2}{(1 + z\bar{z})^2} z_t + \frac{1 + z^2}{(1 + z\bar{z})^2} \bar{z}_t \right) &= s^2 J^0 a_0^2 \left( \frac{-2\bar{z}^4 z - 2\bar{z}^3 - 2\bar{z}^2 z - 2\bar{z}}{(1 + z\bar{z})^4} i (z_x)^2 + \right. \\ &\frac{-2z^4 \bar{z} - 2z^3 - 2z^2 \bar{z} - 2z}{(1 + z\bar{z})^4} i (\bar{z}_x)^2 + \frac{z\bar{z}^3 + \bar{z}^2 + z\bar{z} + 1}{(1 + z\bar{z})^3} i z_{xx} + \frac{\bar{z}z^3 + z^2 + z\bar{z} + 1}{(1 + z\bar{z})^3} i \bar{z}_{xx} \left. \right) \\ &\quad - s^2 J^0 2\Delta \frac{z + \bar{z}}{1 + z\bar{z}} i \times \frac{1 - z\bar{z}}{1 + z\bar{z}} \end{aligned} \quad (2.13)$$

and also  $\hbar S_t^z = s^2 J^0 a_0^2 (S_{xx}^x S^y - S_{xx}^y S^x)$  then

$$\begin{aligned} \hbar \left( \frac{-2\bar{z}}{(1 + z\bar{z})^2} z_t + \frac{-2z}{(1 + z\bar{z})^2} \bar{z}_t \right) &= s^2 J^0 a_0^2 \left( \frac{-4\bar{z}^3 z - 4\bar{z}^2}{(1 + z\bar{z})^4} i (z_x)^2 + \frac{4z^3 \bar{z} + 4z^2}{(1 + z\bar{z})^4} i (\bar{z}_x)^2 + \right. \\ &\frac{2\bar{z} + 2z\bar{z}^2}{(1 + z\bar{z})^3} i z_{xx} + \frac{-2z - 2z^2 \bar{z}}{(1 + z\bar{z})^3} i \bar{z}_{xx} \left. \right) \end{aligned} \quad (2.14)$$

we deduce relation (2.12) from relation (2.13) and we obtain

$$\begin{aligned} \hbar \left( \frac{2}{(1 + z\bar{z})^2} z_t + \frac{-2z^2}{(1 + z\bar{z})^2} \bar{z}_t \right) &= s^2 J^0 a_0^2 \left( \frac{4z\bar{z}^2 + 4\bar{z}}{(1 + z\bar{z})^4} i (z_x)^2 + \frac{4\bar{z}z^4 + 4z^3}{(1 + z\bar{z})^4} i (\bar{z}_x)^2 + \right. \\ &\frac{-2}{(1 + z\bar{z})^2} i (z_{xx}) + \frac{-2z^2}{(1 + z\bar{z})^2} i (\bar{z}_{xx}) \left. \right) - s^2 J^0 2\Delta \frac{2z^2 \bar{z} - 2z}{(1 + z\bar{z})^2} i \end{aligned} \quad (2.15)$$

we multiply relation (2.14) in  $-z$  and add with relation (2.15) then we will have

$$\begin{aligned} \hbar \left( \frac{2}{(1 + z\bar{z})} z_t \right) &= s^2 J^0 a_0^2 \left( \frac{4\bar{z}}{(1 + z\bar{z})^2} i (z_x)^2 + \frac{-2}{(1 + z\bar{z})} i z_{xx} \right) - s^2 J^0 2\Delta \frac{2z^2 \bar{z} - 2z}{(1 + z\bar{z})^2} i \\ i\hbar z_t &= s^2 J^0 \left( \frac{-2a_0^2 \bar{z}}{1 + z\bar{z}} z_x^2 + a_0^2 z_{xx} + \Delta \frac{2z(z\bar{z} - 1)}{1 + z\bar{z}} \right) \end{aligned}$$

then the another form of Landau-Lifshitz equation by use stereographical projection for top plane is

$$i\hbar z_t + s^2 J^0 \left( -a_0^2 z_{xx} + \frac{2a_0^2 \bar{z}}{1 + |z|^2} z_x^2 + \Delta \frac{2z(1 - |z|^2)}{1 + |z|^2} \right) = 0$$

then this equation after scale transformation can be written as

$$iz_t - z_{xx} + \frac{2\bar{z}}{1 + |z|^2} z_x^2 + \left( \frac{s^2 J^0}{\omega_0 \hbar} \right)^2 \Delta \frac{2z(1 - |z|^2)}{1 + |z|^2} = 0 \quad (2.16)$$

b) The relationship between the complex parameters  $z$  and  $S^x, S^y, S^z$  by stereographical projection for bottom plane is given with the following relations

$$S^x = \frac{z + \bar{z}}{1 + z\bar{z}}, S^y = \frac{\bar{z} - z}{1 + z\bar{z}} i, S^z = \frac{z\bar{z} - 1}{1 + z\bar{z}}$$

with use chain rule we have

$$\begin{aligned} S_x^z &= \frac{2\bar{z}}{(1 + z\bar{z})^2} z_x + \frac{2z}{(1 + z\bar{z})^2} \bar{z}_x \\ S_{xx}^y S^z - S_{xx}^z S^y &= \frac{2z\bar{z}^4 + 2\bar{z}^3 - 2z\bar{z}^2 - 2\bar{z}}{(1 + z\bar{z})^4} i (z_x)^2 + \frac{-2\bar{z}z^4 - 2z^3 + 2\bar{z}z^2 + 2z}{(1 + z\bar{z})^4} i (\bar{z}_x)^2 + \\ &\quad \frac{-z\bar{z}^3 - \bar{z}^2 + z\bar{z} + 1}{(1 + z\bar{z})^3} i (z_{xx}) + \frac{\bar{z}z^3 + z^2 - z\bar{z} - 1}{(1 + z\bar{z})^3} i (\bar{z}_{xx}) \\ S_{xx}^x S^z - S_{xx}^z S^x &= \frac{2\bar{z}^4 z + 2\bar{z}^3 + 2\bar{z}^2 z + 2\bar{z}}{(1 + z\bar{z})^4} (z_x)^2 + \frac{2z^4 \bar{z} + 2z^3 + 2z^2 \bar{z} + 2z}{(1 + z\bar{z})^4} (\bar{z}_x)^2 + \\ &\quad \frac{-z\bar{z}^3 - \bar{z}^2 - z\bar{z} - 1}{(1 + z\bar{z})^3} z_{xx} + \frac{-\bar{z}z^3 - z^2 - z\bar{z} - 1}{(1 + z\bar{z})^3} \bar{z}_{xx} \end{aligned}$$

then the another form of Landau-Lifshitz equation by stereographical projection for bottom plane is

$$i\hbar z_t + s^2 J^0 \left( a_0^2 z_{xx} - \frac{2a_0^2 \bar{z}}{1 + |z|^2} z_x^2 + \Delta \frac{2z(|z|^2 - 1)}{1 + |z|^2} \right) = 0$$

then this equation after scale transformation can be written as

$$iz_t + z_{xx} - \frac{2\bar{z}}{1 + |z|^2} z_x^2 + \left( \frac{s^2 J^0}{\omega_0 \hbar} \right)^2 \Delta \frac{2z(|z|^2 - 1)}{1 + |z|^2} = 0 \quad (2.17)$$

### 3 Nonstandard Finite Difference Method

#### 3.1) Introduction to NSFD or NFSD Rules

The genesis of nonstandard finite difference (NSFD) modelling procedures began with the 1989 publication of Mickens [2]. Extensions and a summary of the known results up to 1994 are given in Mickens [3]. This class of schemes and their formulation center on two issues. First, how should discrete representations for derivatives be determined, and second, what are the proper forms to be used for nonlinear terms. A brief and first introduction to the discrete derivative issue is discussed in the paper by Mickens et al. [4]. we suppose decay equation[1]

$$\frac{du}{dt} = -\lambda u, u(0) = u_0$$

Note that the usual forward Euler representation for the first-derivative is

$$\frac{du}{dt} \rightarrow \frac{u_{k+1} - u_k}{h}$$

However ,the discrete first-derivative for decay equation is given by the expression

$$\frac{du}{dt} \rightarrow \frac{u_{k+1} - u_k}{\phi}$$

where the denominator function  $\phi$  is

$$\phi = \frac{1 - e^{-\lambda h}}{\lambda}$$

and the *denominator function* satisfies the condition

$$\phi(\lambda, h) = h + O(\lambda h^2)$$

This way of constructing discrete derivatives can be easily extended to partial derivatives. The linear harmonic oscillator is modeled by following second-order ODE . The linear harmonic oscillator is modeled by the following second-order ODE[1]

$$\frac{d^2u}{dt^2} + \omega^2 = 0$$

where  $\omega$  is a real constant. the discrete second-derivative for this equation is given by the expression

$$\frac{d^2u}{dt^2} \rightarrow \frac{u_{k+1} - 2u_k + u_{k-1}}{\frac{4}{\omega^2} \sin^2\left(\frac{h\omega}{2}\right)}$$

where  $h = \Delta t, t_k = hk$ . the "exact" NSFD scheme is

$$\frac{u_{k+1} - 2u_k + u_{k-1}}{\frac{4}{\omega^2} \sin^2\left(\frac{h\omega}{2}\right)} + \omega^2 u_k = 0.$$

This way of constructing discrete derivatives can be easily extended to partial derivatives. In general, the NSFD rules do not lead to a unique discrete model for either ODEs or PDEs. However, this a priori nonuniqueness can often be partially resolved by appeal to various constraints applied to the discrete equations modelling the differential equations. For example, if the ODE or PDE has special solutions, such as fixed points or traveling waves, the requirement that the discrete equations also have these solutions along with the corresponding (linear) stability properties will often force the discrete models to only assume a small set of possible structures.

### 3.2) Implementation of NSFD for Landau-Lifshitz Equation

we use forward and backward and central difference method (explicit method or SFD) for equation (2.16) then

$$i \frac{z_{i,j+1} - z_{i,j-1}}{2\tau} - \frac{z_{i+1,j} - 2z_{i,j} + z_{i-1,j}}{h^2} + \frac{2\bar{z}_{i,j}}{1 + |z_{i,j}|^2} \left( \frac{z_{i+1,j} - z_{i-1,j}}{2h} \right)^2 + \left( \frac{s^2 J^0}{\omega_0 \hbar} \right)^2 \Delta \frac{2z_{i,j}(1 - |z_{i,j}|^2)}{1 + |z_{i,j}|^2} = 0 \quad (3.1)$$

and the NSFD scheme for this equation is

$$i \frac{z_{i,j+1} - z_{i,j-1}}{2\tau} - \frac{z_{i+1,j} - 2z_{i,j} + z_{i-1,j}}{\frac{4}{\left(\frac{s^2 J^0}{\omega_0 \hbar}\right)^2 \Delta \frac{2(-1+|z_{i,j}|^2)}{1+|z_{i,j}|^2}} \sin^2\left(\frac{h\sqrt{\left(\frac{s^2 J^0}{\omega_0 \hbar}\right)^2 \Delta \frac{2(-1+|z_{i,j}|^2)}{1+|z_{i,j}|^2}}}{2}\right)} + \frac{2\bar{z}_{i,j}}{1 + |z_{i,j}|^2} \left( \frac{z_{i+1,j} - z_{i-1,j}}{2h} \right)^2 + \left( \frac{s^2 J^0}{\omega_0 \hbar} \right)^2 \Delta \frac{2z_{i,j}(1 - |z_{i,j}|^2)}{1 + |z_{i,j}|^2} = 0 \quad (3.2)$$

we use forward and backward and central difference method(explicit method) for equation (2.17) then

$$i \frac{z_{i,j+1} - z_{i,j-1}}{2\tau} + \frac{z_{i+1,j} - 2z_{i,j} + z_{i-1,j}}{h^2} - \frac{2\bar{z}_{i,j}}{1 + |z_{i,j}|^2} \left( \frac{z_{i+1,j} - z_{i-1,j}}{2h} \right)^2 + \left( \frac{s^2 J^0}{\omega_0 \hbar} \right)^2 \Delta \frac{2z_{i,j} (|z_{i,j}|^2 - 1)}{1 + |z_{i,j}|^2} = 0 \quad (3.3)$$

and the NSFD scheme for this equation is

$$i \frac{z_{i,j+1} - z_{i,j-1}}{2\tau} + \frac{z_{i+1,j} - 2z_{i,j} + z_{i-1,j}}{\frac{4}{\left( \frac{s^2 J^0}{\omega_0 \hbar} \right)^2 \Delta \frac{2(-1+|z_{i,j}|^2)}{1+|z_{i,j}|^2}} + \frac{h \sqrt{\frac{\left( \frac{s^2 J^0}{\omega_0 \hbar} \right)^2 \Delta \frac{2(-1+|z_{i,j}|^2)}{1+|z_{i,j}|^2}}{2}}} {2} - \frac{2\bar{z}_{i,j}}{1 + |z_{i,j}|^2} \left( \frac{z_{i+1,j} - z_{i-1,j}}{2h} \right)^2 + \left( \frac{s^2 J^0}{\omega_0 \hbar} \right)^2 \Delta \frac{2z_{i,j} (-1 + |z_{i,j}|^2)}{1 + |z_{i,j}|^2} = 0 \quad (3.4)$$

## 4 Numerical Result

We know for magnetic solitons moving with velocity  $v$  exact solution is [5]

$$\tan^2 \frac{\theta}{2} = \frac{\mu^2}{\Omega \cosh^2 \mu \xi - \frac{[\Omega - \Omega_1]}{2}},$$

$$\xi = z - \nu \tau = \sqrt{a_1 \Delta - \omega - \left( \frac{\nu}{2} \right)^2} \quad (4.1)$$

Now for obtain approximation solution at  $h = 0.1, \tau = 0.002, \Delta = 0.1, \nu = 0.6$  at  $T = 10$  we used (4, 1) as initial valu and will have

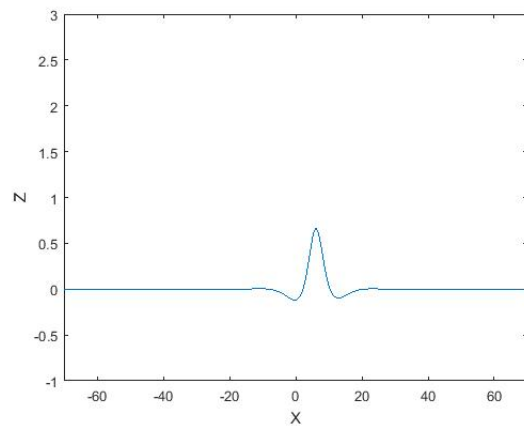


Figure 1: Exact solution of Landau-Lifshitz equation

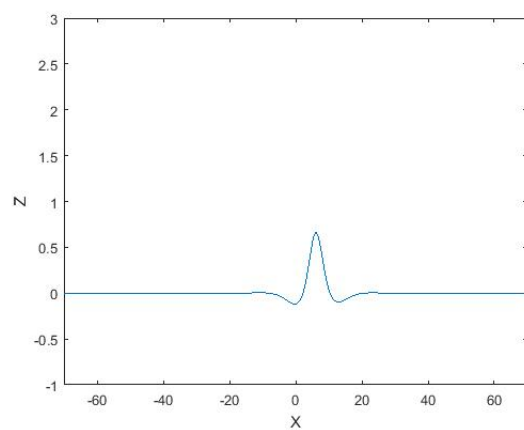


Figure 2: Approximation solution by SFD of Landau-Lifshitz

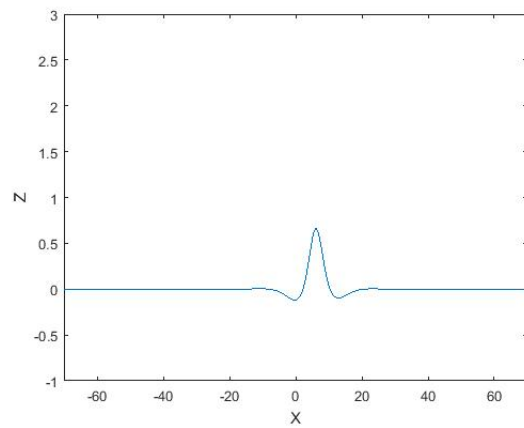


Figure 3: Approximation solution by NSFD of Landau-Lifshitz

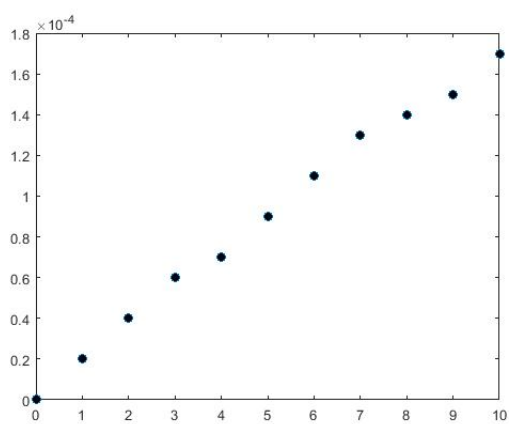


Figure 4: Error of SFD method for  $\nu = 0.6$

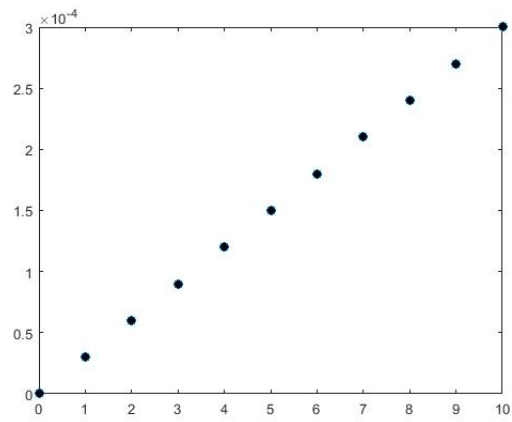


Figure 5:Error of NSFD method for  $\nu = 0.6$

$T$	SFD method	NSFD method
1	$2.0000(-5)$	$3.000(-5)$
2	$4.0000(-5)$	$6.0000(-5)$
3	$6.0000(-5)$	$9.0000(-5)$
4	$7.0000(-5)$	$1.2000(-4)$
5	$9.0000(-5)$	$1.5000(-4)$
6	$1.1000(-4)$	$1.8000(-4)$
7	$1.3000(-4)$	$2.1000(-4)$
8	$1.4000(-4)$	$2.4000(-4)$
9	$1.5000(-4)$	$2.7000(-4)$
10	$1.7000(-4)$	$3.0000(-4)$

Table 1: Maximum absolute errors for  $\nu = 0.6$

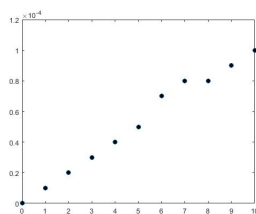


Figure 6:Error of SFD method for  $\nu = 0.8$

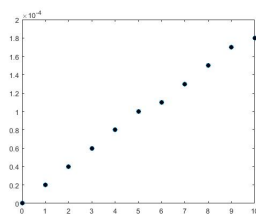


Figure 7:Error of NSFD method for  $\nu = 0.8$

$T$	SFD method	NSFD method
1	1.0000(-5)	2.0000(-5)
2	2.0000(-5)	4.0000(-5)
3	3.0000(-5)	6.0000(-5)
4	4.0000(-5)	8.0000(-5)
5	5.0000(-5)	1.0000(-4)
6	7.0000(-5)	1.1000(-4)
7	8.0000(-5)	1.3000(-4)
8	8.0000(-5)	1.5000(-4)
9	9.0000(-5)	1.7000(-4)
10	1.0000(-4)	1.8000(-4)

Table 2: Maximum absolute errors for  $\nu = 0.8$

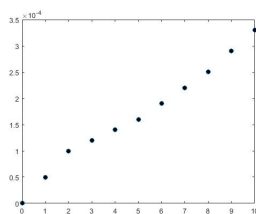


Figure 8:Error of SFD method for  $\nu = 0.3$

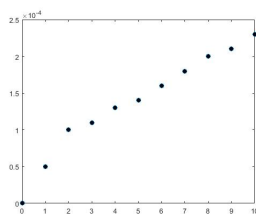


Figure 9:Error of NSFD method for  $\nu = 0.3$

$T$	SFD method	NSFD method
1	$5.0000(-5)$	$5.0000(-5)$
2	$1.0000(-4)$	$1.0000(-4)$
3	$1.2000(-4)$	$1.1000(-4)$
4	$1.4000(-4)$	$1.3000(-4)$
5	$1.6000(-4)$	$1.4000(-4)$
6	$1.9000(-4)$	$1.6000(-4)$
7	$2.2000(-4)$	$1.8000(-4)$
8	$2.5000(-4)$	$2.0000(-4)$
9	$2.9000(-4)$	$2.1000(-4)$
10	$3.3000(-4)$	$2.3000(-4)$

Table 3: Maximum absolute errors for  $\nu = 0.3$

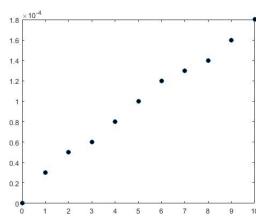


Figure 8:Error of SFD method for  $\nu = 0.1$

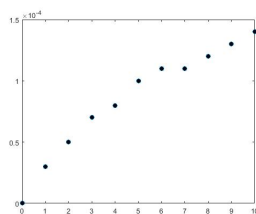


Figure 9:Error of NSFD method for  $\nu = 0.1$

$T$	SFD method	NSFD method
1	530000(-5)	3.0000(-5)
2	5.0000(-5)	5.0000(-5)
3	6.0000(-5)	7.0000(-5)
4	8.0000(-5)	8.0000(-5)
5	1.0000(-4)	1.0000(-4)
6	1.2000(-4)	1.1000(-4)
7	1.3000(-4)	1.1000(-4)
8	1.4000(-4)	1.2000(-4)
9	1.6000(-4)	1.3000(-4)
10	1.8000(-4)	1.4000(-4)

Table 4: Maximum absolute errors for  $\nu = 0.1$ 

## 5 Conclusion

According to numerical examples, it can be concluded that if the velocity is near the 1 the error of SFD method is less than that of the NSFD method, and  $\nu \ll 1$  the error of NSFD method is less than that of the SFD method, but in general, both of these methods have a close error but to increase the accuracy of calculation, we can select kind of method according to the value of velocity.

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