# On Some Representations of the Euler-Mascheroni constants

***Original Research Paper***

## Abstract

In this study, we derive a new representation for the Euler-Mascheroni constant and present various expressions for the classical Euler-Mascheroni constant related to the Riemann zeta function. Also, we proved that $\gamma$ is not algebraic if the Schanuel conjecture is true.

*Keywords: Euler-mascheroni constant; Schanuel Conjecture; Reoresenations*

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# 1 Introduction

The Euler-Mascheroni constant or simply the Euler constant, $\gamma$, was first investigated by Euler in 1734 and later by Mascheroni. As a mathematical constant, the Euler-Mascheroni constant appears in the study of special functions such as the gamma and Riemann zeta functions and keeps recurring in number theory and analysis. The Euler-Mascheroni constant is in the class of mathematical constants such as Ludolph's number ($\pi$) and Euler's number ($e$) and has applications in general theory of relativity and quantum theory (Sukenik and Sima (2018)).

It is known that $\pi$ and $e$ are irrational and transcendental but little is known about the algebaric properties of the Euler-Mascheroni constant. An algebraic number is a complex number which is a root of a polynomial with rational coefficients and a transcendental number is a complex number which is not algebraic. It remains as an open problem whether or not the Euler-Mascheroni constant

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is irrational, transcendental or algebraic.
Euler-Mascheroni constant, $\gamma$, is defined as

$$\gamma = \lim_{n\to\infty}\left(-\ln(n) + \sum_{j=1}^{n}\frac{1}{j}\right). \tag{1.1}$$

Now, other representations of the Euler-Mascheroni constant are given by

$$\gamma = \lim_{n\to\infty}\left(H_n - \ln(n)\right), \tag{1.2}$$

$$= \sum_{n=2}^{\infty}\frac{(-1)^n}{n}\zeta(n), \tag{1.3}$$

where $H_n$ is the nth harmonic number and $\zeta(n)$ is the Riemann zeta function.
Euler actually introduced (1.1) and later Lorenzo Mascheroni also studied this constant and gave a numerical approximation for it up to 32 decimal places and this made him to have some prominence as far as the Euler constant is concerned. Lagarias (2013) examined the Euler-Mascheroni constant and described various mathematical developments about this constant and its connection with arithmetic functions, the Riemann hypothesis, random permutations and random matrix products. In an essay on the Euler-Mascheroni constant, Lima (2014) delved into the recurrence of this constant in multiple branches of mathematics and intimated the possibility that, $\gamma$, could be transcendental. He also concluded that based on the computation of $\gamma$ and its exponential, it strongly indicates that they are irrational.

Wu and Bercu (2012) presented a new sequence that converges to the Euler-Mascheroni constant using an approximation of Pade type. They established lower and upper bound estimates between their sequence and the Euler-Mascheroni constant. A brief survey on the history of the Euler-Mascheroni constant, its applications and appearances in various mathematical settings can be found in Dence and Dence (2009).

## 2 Materials and Methods

The Riemann zeta is defined as

$$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}, Re(s) > 1 \tag{2.1}$$

$$= \prod_{p}\left(1 - p^{-s}\right)^{-1}, Re(s) > 1 \tag{2.2}$$

where $p$ is a prime number.
The derivatives of the Riemann zeta function is given by

$$\zeta^k(s) = (-1)^k\sum_{n=1}^{\infty}\frac{\ln^k n}{n^s}. \tag{2.3}$$

The digamma function is given by the logarithmic derivative of the gamma function:

$$\psi(z) = \frac{d\ln\Gamma(z)}{dz} = \frac{\Gamma''(z)}{\Gamma(z)}. \tag{2.4}$$

The digamma function is defined as

$$\psi(z+1)+\gamma=\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+z}\right), z\in\mathbb{C}, \tag{2.5}$$

where $\gamma$ is the Euler-Mascheroni constant.
For $|z|<1$,

$$\ln\Gamma(z+1)=-\gamma z+\sum_{k=2}^{\infty}\frac{(-1)^k\zeta(k)}{k}z^k, \tag{2.6}$$

$$=-\ln(1+z)-(\gamma-1)z+\sum_{k=2}^{\infty}\frac{(-1)^k(\zeta(k)-1)}{k}z^k. \tag{2.7}$$

To expand $\ln\Gamma(z+1)$ in a power series about $z=0$, stirling numbers are utilized. The stirling numbers of the first kind,$s(m,j)$, is defined by the generating function as

$$\ln^j\left(1+\frac{z}{n}\right)=\sum_{n=j}^{\infty}\frac{j!}{m!}s(m,j)\left(\frac{z}{n}\right)^m, \tag{2.8}$$

where $|z|<1$.
Alternatively, stirling numbers of the first kind are also defined as

$$\frac{1}{j}\ln^j(1+t)=\sum_{n=j}^{\infty}s(n,j)\frac{t^n}{n!}, \tag{2.9}$$

or

$$\ln^j(1-t)=\sum_{n=j}^{\infty}\frac{(-1)^n}{n!}s(n,j)t^n. \tag{2.10}$$

From the above, we see that $s(n,1)=(-1)^{n-1}(n-1)!$, $s(n,n)=1$,
$s(n,2)=(-1)^n(n-1)!\sum_{j=1}^{n-1}\frac{1}{j}$ and
$s(n,3)=\frac{1}{2}(-1)^{n+1}\left(H_{n-1}^2-H_{n-1}^{(2)}\right)$,
where $H_{n-1}$ is the $(n-1$th harmonic number and $H_{n-1}^{(2)}$ is a harmonic number of order 2.

Abe-I-kpeng et al (2021) established a generalized gamma function as

$$\Gamma_k(z)=\lim_{n\to\infty}\frac{\prod_{j=1}^{n}\exp\left(\frac{1}{k+1}\ln^{k+1}\left(1+\frac{1}{j}\right)^z\right)}{\exp\left(\frac{1}{k+1}\ln^{k+1}z\right)\prod_{j=1}^{n}\exp\left(\frac{1}{k+1}\ln^{k+1}\left(1+\frac{z}{j}\right)\right)} \tag{2.11}$$

and a functional equation as

$$\Gamma_k(z+1)=\exp\left(\frac{1}{k+1}\ln^{k+1}z\right)\Gamma_k(z), \tag{2.12}$$

where $z\in\mathbb{C}\setminus\mathbb{Z}^-U\{0\}$ and $k\in\mathbb{N}_0$.

Euler's formula for calculating $\zeta(2k)$ is expressed as

$$\zeta(2k)=\frac{(-1)^k(2\pi)^{2k}B_{2k}}{2(2k)!}, \tag{2.13}$$

where $B_{2k}$ are Bernoulli numbers.
An identity for $\zeta(2k+1)$ was established in Abe-I-kpeng et al (2019) as

$$\zeta(2k+1) = \frac{(-1)^{1-k}(2\pi)^{2k+1}}{2(2k+1)!}\int_0^1 B_{2k+1}(t)\cot(\pi t)dt, \tag{2.14}$$

where $k \in \mathbb{N}$.
In Folland (2009), it is observed that if $f_n$ is a sequence in a measuraable function space $L^+$, then by the monotone convergence theorem

$$\int \sum_n f_n = \sum_n f_n \int f_n. \tag{2.15}$$

**Lemma 2.1.** *(Lindemann-Weierstrass)*
*Given a positive integer $n$ and distinct algebraic numbers $\alpha_0, ..., \alpha_n$. The numbers $e^{\alpha_0}, ..., e^{\alpha_n}$ are linearly independent over the set of algebraic numbers, $A$ for $\beta_0, ..., \beta_n \in A$ not all zero.*

Lemma 2.1 implies that

$$\sum_{k=0}^{n} \beta_k e^{\alpha_k} \neq 0. \tag{2.16}$$

**Lemma 2.2.** *(Schanuel Conjecture)*
*Let $z_1..., z_n$ be complex numbers that are linearly independent over the rational numbers $\mathbb{Q}$. Then the extension field $\mathbb{Q}(z_1, ..., z_n, e^{z_1}, ..., e^{z_n})$ has trnscendence degree of at least $n$ where $e^z$ is the complex exponential of $z$.*

The Lemma below was established as a Corollary in Lima (2014).

**Lemma 2.3.** *For any positive integer $n$, the numbers $\ln\pi$ and $\sqrt{n}\pi$ are linaerly independent.*

*Remark* 2.1. By Lemma 2.2 and taking $n = 1$ implies that $\ln\pi$ and $\pi$ are transcendental.

For $|z| < 1$, Abe-I-kpeng et al (2021) established that

$$\ln^{k+1}(j+1) - \ln^{k+1} j = \frac{1}{j}(k+1)\ln^k j + \sum_{m=2}^{\infty}\left(\frac{1}{j}\right)^m \frac{1}{m!}\sum_{n=1}^{m}\frac{(k+1)!}{(k+1-n)!}s(m,n)\ln^{k+1-n} j. \tag{2.17}$$

In Abramowitz and Stegun (1972), the sine function is given as

$$\frac{\sin x}{x} = \prod_{n=1}^{\infty}\left(1 - \frac{x^2}{n^2\pi^2}\right). \tag{2.18}$$

# 3 Results and Discussion

We begin by establishing a new representation for the Euler-Mascheroni constant.

**Theorem 3.1.** *The Euler-Mascheroni constant is given by*

$$\gamma = \ln\pi - 2\sum_{m=2}^{\infty}\frac{\zeta(m)}{m2^m}, \tag{3.1}$$

*Proof.* From (2.11), we have

$$\Gamma_k(z) = \frac{\exp\left(\sum_{j=1}^{\infty} \frac{z}{k+1} \ln^{k+1}\left(1+\frac{1}{j}\right)\right)}{\exp\left(\frac{1}{k+1}\ln^{k+1} z\right) * \exp\left(\sum_{j=1}^{\infty} \frac{1}{k+1} \ln^{k+1}\left(1+\frac{z}{j}\right)\right)}. \tag{3.2}$$

By (2.17), we obtain

$$\Gamma_k(z) = \frac{\exp\left(\sum_{m=2}^{\infty}\left(\frac{z}{m!}\right)\sum_{n=1}^{m} \frac{k!s(m,n)}{(k+1-n)!}\zeta^{(k+1-n)}(m)\right)}{\exp\left(\frac{1}{k+1}\ln^{k+1} z\right) * \exp\left(\sum_{m=2}^{\infty}\left(\frac{z^m}{m!}\right)\sum_{n=1}^{m} \frac{k!s(m,n)}{(k+1-n)!}\zeta^{(k+1-n)}(m)\right)}. \tag{3.3}$$

By letting $z = -\frac{1}{2}$ and $k = 0$ yields

$$\begin{aligned}
\Gamma\left(-\frac{1}{2}\right) &= \frac{\exp\left(-\frac{1}{2}\sum_{m=2}^{\infty}\left(\frac{1}{m!}\right)\sum_{n=1}^{m}\frac{s(m,n)}{(1-n)!}\zeta^{(1-n)}(m)\right)}{\exp\left(\ln\left(\frac{-1}{2}\right)\right)\exp\left(\sum_{m=2}^{\infty}\left(\frac{\left(-\frac{1}{2}\right)^m}{m!}\right)\sum_{n=1}^{m}\frac{s(m,n)}{(1-n)!}\zeta^{(1-n)}(m)\right)}, \\
&= \frac{\exp\left(-\frac{1}{2}\sum_{m=2}^{\infty}\left(\frac{1}{m!}\right)\sum_{n=1}^{m}\frac{s(m,n)}{(1-n)!}\zeta^{(1-n)}(m)\right)}{\exp\left(\ln\left(\frac{i^2}{2}\right)\right)\exp\left(\sum_{m=2}^{\infty}\left(\frac{\left(-\frac{1}{2}\right)^m}{m!}\right)\sum_{n=1}^{m}\frac{s(m,n)}{(1-n)!}\zeta^{(1-n)}(m)\right)}, \\
-2\sqrt{\pi} &= -2\exp\left(\frac{1}{2}\sum_{m=2}^{\infty}\frac{(-1)^m\zeta(m)}{m} + \sum_{m=2}^{\infty}\frac{(-1)^{2m}\zeta(m)}{m * 2^m}\right).
\end{aligned}$$

Further simplifying gives

$$\sqrt{\pi} = \exp\left(\frac{\gamma}{2} + \sum_{m=2}^{\infty}\frac{\zeta(m)}{m2^m}\right). \tag{3.4}$$

By applying logarithm on both sides of (3.4) and making $\gamma$ the subject completes the proof. ☐

*Remark* 3.1. If $k = 0$ and $z = \frac{1}{2}$ in (3.3), we obtain

$$\Gamma\left(\frac{1}{2}\right) = 2\exp\left(-\frac{1}{2}\sum_{m=2}^{\infty}\frac{(-1)^m\zeta(m)}{m} + \sum_{m=2}^{\infty}\frac{(-1)^m\zeta(m)}{m * 2^m}\right). \tag{3.5}$$

Simplifying further gives

$$\sqrt{\pi} = 2\exp\left(\frac{-\gamma}{2} + \sum_{m=2}^{\infty}\frac{(-1)^m\zeta(m)}{m2^m}\right). \tag{3.6}$$

Taking logarithm on both sides of (3.6) and making $\gamma$ the subject gives

$$\gamma = \ln\frac{4}{\pi} + 2\sum_{m=2}^{\infty}\frac{(-1)^m}{m2^m}\zeta(m), \tag{3.7}$$

which is a known representation of the Euler-Mascheroni constant also found in Bonnar (2017).

We observe that the series in (3.1) converges faster than that of the Euler-Mascheroni constant given by (1.3). Since the Euler-Mascheroni constant, as a mathematical constant, keeps reoccuring in the study of special functions and in analysis, we forsee the suitability of the constant $\lambda = \sum_{m=2}^{\infty}\frac{\zeta(m)}{m*2^m}$ in numerical calculations where $\gamma$ converges slowly.
By the use of infinte series calculator (www.wolframalpha.com/widgets), the constant $\lambda$ converged to 0.283757.
Next, we present a new representation of the constant, $e^{\gamma}$, in the corollary below.

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**Corollary 3.2.**

$$e^{\gamma} = \frac{\pi}{e^{2\lambda}}, \tag{3.8}$$

*where* $\gamma$ *is the Euler-Mascheroni constant and* $\lambda = \sum_{m=2}^{\infty} \frac{\zeta(m)}{m*2^m}$

*Proof.* From (3.1), we obtain

$$\pi = e^{\gamma} e^{2\sum_{m=2}^{\infty} \frac{\zeta(m)}{m*2^m}}. \tag{3.9}$$

By making $e^{\gamma}$ the subject in (3.9) completes the proof. □

The constant $e^{\gamma}$ is vital in number theory and relates to the Marten's third theorem in the following form:

$$e^{\gamma} = \lim_{n\to\infty} \frac{1}{\ln p_n} \prod_{i=1}^{n} \frac{p_i}{p_i - 1}, \tag{3.10}$$

where $p_n$ is the $nth$ prime number.
Following Theorem 3.1, we present various representations of the Euler-Mascheroni constant.

**Theorem 3.3.**

$$\gamma = \ln\left(\ln(i)^{-2i}\right) - 2\sum_{m=2}^{\infty} \frac{\zeta(m)}{m2^m}, \tag{3.11}$$

*where* $i^2 = -1.$

*Proof.* From the Euler's identity, we obtain

$$\pi = \ln(i)^{-2i}. \tag{3.12}$$

Substituting (3.12) into (3.1) completes the proof.

□

**Theorem 3.4.**

$$\gamma = \ln\frac{1}{2} - 2\zeta'(0) - 2\sum_{m=2}^{\infty} \frac{\zeta(m)}{m2^m}. \tag{3.13}$$

*Proof.* Choudhury (1995) established that

$$\zeta^{'}(0) = -\frac{1}{2}\ln 2\pi. \tag{3.14}$$

By making $\ln\pi$ the subject, we get

$$\ln\pi = -\ln 2 - \zeta'(0). \tag{3.15}$$

Substituting (3.15) into (3.1) completes the proof. □

**Theorem 3.5.**

$$\gamma = \ln\frac{1}{2} + 2\sum_{m=2}^{\infty} \left(\ln m - \frac{\zeta(m)}{m2^m}\right). \tag{3.16}$$

*Proof.* For $k = 1$ and $z = 0$, (2.3) becomes

$$\zeta'(0) = -\sum_{m=1}^{\infty} \ln m. \tag{3.17}$$

which yields

$$-\frac{1}{2} \ln 2\pi = \sum_{m=1}^{\infty} \ln m. \tag{3.18}$$

Making $\ln \pi$ the subject, we obtain

$$\ln \pi = \ln \frac{1}{2} + \sum_{m=1}^{\infty} \ln m^2. \tag{3.19}$$

Substituting (3.19) into (3.1) completes the proof. ☐

**Theorem 3.6.**

$$\gamma = \ln \sqrt{6} + \frac{1}{2} \ln \prod_p \left(1 - p^{-2}\right)^{-1} - 2 \sum_{m=2}^{\infty} \frac{\zeta(m)}{m 2^m}. \tag{3.20}$$

*Proof.* Substituting $s = 2$ into (2.1) gives

$$\zeta(2) = \prod_p \left(1 - p^{-2}\right)^{-1}, \tag{3.21}$$

and further yields

$$\frac{\pi^2}{6} = \prod_p \left(1 - p^{-2}\right)^{-1}. \tag{3.22}$$

Applying logarithm on both sides of (3.22), we get

$$\ln \pi = \frac{1}{2} \ln 6 + \frac{1}{2} \ln \left( \prod_p \left(1 - p^{-2}\right)^{-1} \right). \tag{3.23}$$

Substituting (3.23) into (3.1) ends the proof. ☐

*Remark* 3.2. Comparing (3.13) and (3.23), we get

$$\zeta^{'}(0) = \ln \left( \frac{\sqrt{\sqrt{6}}}{2\sqrt{3}} \right) - \frac{1}{4} \ln \left( \prod_p \left(1 - p^{-2}\right)^{-1} \right). \tag{3.24}$$

**Theorem 3.7.**

$$\gamma = \ln 2 + \sum_{m=1}^{\infty} \ln \left( \frac{4m^2}{4m^2 - 1} \right) - 2 \sum_{m=2}^{\infty} \frac{\zeta(m)}{m 2^m}. \tag{3.25}$$

*Proof.* By letting $x = \frac{\pi}{2}$ and substituting into (2.18) , we obtain

$$\frac{\pi}{2} = \ln \left( \prod_{m=1}^{\infty} \left( \frac{2m}{2m - 1} \frac{2m}{2m + 1} \right) \right), \tag{3.26}$$

a wallis product formula for $\frac{\pi}{2}$. ☐

Taking logarithm on both sides of (3.26) gives

$$\ln \pi = \ln 2 + \ln \left( \prod_{m=1}^{\infty} \left( \frac{2m}{2m-1} \frac{2m}{2m+1} \right) \right). \tag{3.27}$$

Substituting (3.27) into (3.1) ends the proof.

**Theorem 3.8.** *The constant $\lambda$ is given by*

$$\lambda = \sum_{m=1}^{\infty} \left( \ln m - \ln \left( m - \frac{1}{2} \right) - \frac{1}{2m} \right). \tag{3.28}$$

*Proof.* Integrating (2.5) amd using (2.15), we get

$$\ln \Gamma(z+1) + \gamma z = \sum_{m=1}^{\infty} \left( \frac{z}{m} - \ln(m+z) + \ln(m) \right). \tag{3.29}$$

By letting $z = -\frac{1}{2}$, (3.33) yields

$$\ln \pi - \gamma = 2 \sum_{m=1}^{\infty} \left( -\frac{1}{2m} - \ln \left( m - \frac{1}{2} \right) + \ln m \right). \tag{3.30}$$

By substituting (3.30) into (3.1) completes the proof. ☐

We further give new series representations for the Euler-Mascheroni constant involving Bernoulli numbers, Bernoulli polynomials and generalized clausen functions.

**Theorem 3.9.**

$$\gamma = \ln \pi - \sum_{k=1}^{\infty} \frac{(-1)^k (\pi)^{2k} B_{2k}}{2k(2k)!} + \sum_{k=1}^{\infty} \frac{(-1)^{1-k} (\pi)^{2k+1}}{(2k+1)(2k+1)!} \int_0^1 B_{2k+1}(t) \cot(\pi t) dt, \tag{3.31}$$

*where $B_{2k}$ and $B_{2k+1}(t)$ are Bernoulli numbers and polynomials respectively.*

*Proof.* For even values of $m$, (3.1) becomes

$$\gamma = \ln \pi - 2 \left( \sum_{m=2}^{\infty} \frac{\zeta(m)}{2^m m} - \sum_{m=2}^{\infty} \frac{\zeta(m+1)}{2^{m+1}(m+1)} \right). \tag{3.32}$$

Let $m = 2k$ for $k \in \mathbb{N}$, then we obtain

$$\gamma = \ln \pi - 2 \left( \sum_{k=1}^{\infty} \frac{\zeta(2k)}{2^{2k} 2k} - \sum_{k=1}^{\infty} \frac{\zeta(2k+1)}{2^{2k+1}(2k+1)} \right). \tag{3.33}$$

Substtuting (2.13) and (2.14) into (3.33), the proof is complete. ☐

In Theorem 3.1, $\ln \pi$ is a term in the new representation for the Euler-Mascheroni as well as the constant $\lambda = \sum_{m=2}^{\infty} \frac{\zeta(m)}{m 2^m}$. Understanding the nature of these constants can reveal the nature of the Euler-Mascheroni constant. Already, it is observed that the constant $\lambda$ is algebraic in nature. The transcendentality of $\ln \pi$ is shown in the lemma below.

**Lemma 3.10.** *Let the Schanuel conjecture be true. Then, $\ln \pi$ is transcendental.*

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*Proof.* Assumung Lemma 2.2 and Lemma 2.3 are true. This implies that $\pi$ and $\ln\pi$ are algebraically independent over the set of rational numbers.
Thus, if Lemma 2.2 and Lemma 2.3 are true, then $\ln\pi$ is transcendental. □

**Claim**
$\gamma$ is transcendental.

*Proof.* By (3.1) and Lemma 3.10, the proof is complete. □

*Remark* 3.3. The Euler-Mascheroni constant is not algebraic.

# 4 Conclusions

A new series represenation of the Euler-Mascheroni constant has also been derived and various representations given. We also present a new representation for $e^{\gamma}$ which is useful in number theory and proved that if $\ln\pi$ is transcendental, then the Euler-Mascheroni constant is also transcendental.

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