UNDER PEER REVIEW

The Solution of the Invariant Subspace Problem. Complex Hilbert space. Part I.External non-Archimedean field ${}^{*}\mathbb{R}_{c}^{\#}$ by Cauchy completion of the internal non-Archimedean field ${}^{*}\mathbb{R}$.

Abstract: The incompleteness of set theory ZFC leads one to look for natural extensions of ZFC in which one can prove statements independent of ZFC which appear to be “true”. One approach has been to add large cardinal axioms. Or, one can investigate second-order expansions like Kelley-Morse class theory, KM or Tarski-Grothendieck set theory TG [1]-[3] It is a non-conservative extension of ZFC and is obtaineed from other axiomatic set theories by the inclusion of Tarski’s axiom which implies the existence of inaccessible cardinals [1].Non-conservative extension of ZFC based on an generalized quantifiers considered in [4].

In this paper we look at a set theory $\mathbf{NC}_{\infty^{\#}}^{\#}$,based on bivalent gyper infinitary logic with restricted Modus Ponens Rule [5]-[8]. In this paper we deal with set theory $\mathbf{NC}_{\infty^{\#}}^{\#}$ based on hyper infinitary logic with Restricted Modus Ponens Rule. Set theory $\mathbf{NC}_{\infty^{\#}}^{\#}$ contains Aczel’s anti-foundation axiom [9].

We present a new approach to the invariant subspace problem for Hilbert spaces. This approach based on nonconservative Extension of the Model Theoretical NSA.

Our main result will be that: if T is a bounded linear operator on an infinite-dimensional complex separable Hilbert space H, it follow that T has a non-trivial closed invariant subspace.Non-conservative extension based on set theory $\mathbf{NC}_{\infty}^{\#}$ of the model theoretical nonstandard analysis [10]-[12] also is considered.

Keywords:Set theory ZFC;Nonconservative extension of ZFC;Internal set theory IST;External set theory HST; A.Robinson model theoretical NSA; Bivalent gyper infinitary logic;Modus ponens rule; Logic with restricted modus ponens rule;internal non-Archimedean field;

1.Introduction

The incompleteness of set theory ZFC leads one to look for natural extensions of ZFC in which one can prove statements independent of ZFC which appear to be “true”. One approach has been to add large cardinal axioms. Or, one can investigate second-order expansions like Kelley-Morse class theory, KM or Tarski-Grothendieck set theory TG [1]-[3].It is a non-conservative extension of ZFC and is obtaineed from other axiomatic set theories by the inclusion of Tarski’s axiom which implies the existence of inaccessible cardinals. Non-conservative extension of ZFC based on an generalized quantifiers considered in [4]. In this paper we look at a set theory $\mathbf{NC}_{\infty^{\#}}^{\#}$,based on bivalent hyper infinitary logic ${}^{2}L_{\infty^{\#}}^{\#}$ with restricted Modus Ponens Rule [5]-[8].Set theory $\mathbf{NC}_{\infty^{\#}}^{\#}$contains Aczel’s anti-foundation axiom [9].
Non-conservative extension based on set theory $\mathbf{NC}_{\infty^{\#}}^{\#}$ of the model theoretical nonstandard analysis [10]-[12] also is considered.

1.1.The Invariant Subspace Problem.Positive classical results.

The problem, in a general form,is stated as follows.**The Invariant Subspace Problem**:
If T is a bounded linear operator on an infinite-dimensional separable Hilbert space H, does it follow that T has a non-trivial closed invariant subspace?
The Invariant Subspace Problem (as it stands today). If T is a bounded linear operator on an infinite-dimensional separable Hilbert space H, does it follow that T has a non-trivial closed invariant subspace?
Sometime during the 1930s John von Neumann proved that compact operators have non-trivial invariant subspaces, but did not publish it. The proof was rediscovered and finally published by N. Aronszajn and K. T. Smith [13] in 1954.
Theorem 1.1. (von Neumann). Every compact operator on H has a non-trivial invariant subspace.
In 1966 Bernstein and Robinson [14] extended the result to the slightly larger class of polynomially compact operators.
Definition 1.1. A linear operator T on a Banach space is said to be polynomially compact if there is a non-zero polynomial $p(t) \in \mathbb{C}[t]$ such that $p(T)$ is compact.
An nonclassical aspect of Bernstein and Robinson’s proof is that it used the relatively new techniques of non-standard analysis, which builds up the foundations of analysis based on a rigorous definition of infinitesimal numbers.Shortly after, the

proof was translated into standard analysis by Halmos [15].
The next major generalization was achieved by Arveson and Feldman [16] in1968.
Definition 1.2.For a bounded linear operator T on $X,$ the uniformly closed algebra generated by T, denoted by $\mathbf{A}(T)$, is defined to be the subspace $[\{I,T,T^2,\ldots\}]$ of $\mathbf{B}(X)$. Alternatively, $\mathbf{A}(T)$ is the smallest closed subspace of $\mathbf{B}(X)$ containing T and I which is closed under function composition.
If T is a bounded operator, then $\mathbf{A}(T)$ can be thought of as the closure of the set of polynomial combinations of T, or the set of all operators which can be norm approximated by polynomial combinations of T.
Theorem 1.1.(Arveson and Feldman [16]). If $T : H \to H$ is a bounded quasinilpotent operator such that $\mathbf{A}(T)$ contains a non-zero compact operator, then T has a non-trivial invariant subspace.
While the techniques of von Neumann and subsequent generalizations yielded many interesting and surprising theorems during the 1950s and 60s, their effectiveness was reaching its limit by the 70s.

1.2.Counterexamples on Banach Spaces.

In 1975 Per Enflo discovered the first example of an operator on a Banach space having only the trivial invariant subspaces. He gave an outline of the proof in 1976 However, his full solution was not submitted until 1981 and did not appear in print until 1987 [17].As Enflo's paper crawled through the publication process, C. J. Read developed a counterexample of his own and submitted it for publication [18]. The paper was of similar length and complexity to Enflo's, however it was published much earlier in 1984.

2.External non-Archimedean field ${}^*\mathbb{R}_c^\#$ by Cauchy completion of the internal non-Archimedean field ${}^*\mathbb{R}$.

2.1.Basic results and definitions.

Definition 2.1. A hyper infinite sequence of hyperreal numbers from ${}^*\mathbb{R}$ is a function $a : \mathbb{N}^\# \to {}^*\mathbb{R}$ from hypernatural numbers $\mathbb{N}^\#$ into the hyperreal numbers ${}^*\mathbb{R}$.
We usually denote such a function by $n \mapsto a_n$, or by $a : n \to a_n$, so the terms in the sequence are written $\{a_1,a_2,a_3,\ldots,a_n\ldots\}$. To refer to the whole hyper infinite sequence, we will write $\{a_n\}_{n=1}^{\infty^\#}$, or $\{a_n\}_{n\in\mathbb{N}^\#}$, or for the sake of brevity simply $\{a_n\}$.
Definition 2.2. Let $\{a_n\}$ be a hyper infinite ${}^*\mathbb{R}$-valued sequence mentioned above. Say that $\{a_n\}$ #-tends to 0 if, given any $\varepsilon > 0, \varepsilon \approx 0$, there is a hypernatural number $N \in \mathbb{N}^\# \backslash \mathbb{N}$, $N = N(\varepsilon)$ such that, after N (i.e.for all $n > N$), $|a_n| \leq \varepsilon$. We denote this symbolically by $a_n \to_\# 0$.
We can also, at this point, define what it means for a hyper infinite ${}^*\mathbb{R}$-valued sequence #-tends to any given number $q \in {}^*\mathbb{R}$: $\{a_n\}$ #-tends to q if the hyper infinite sequence $\{a_n - q\}$ #-tends to 0 i.e., $a_n - q \to_\# 0$.
Definition 2.3. Let $\{a_n\}$ be a hyper infinite ${}^*\mathbb{R}$-valued sequence. We call $\{a_n\}$ a Cauchy hyper infinite ${}^*\mathbb{R}$-valued sequence if the difference between its terms #-tends to 0. To be precise: given any hyperreal number such that $\varepsilon > 0, \varepsilon \approx 0$, there is a hypernatural number $N = N(\varepsilon)$ such that for any $m,n > N, |a_n - a_m| < \varepsilon$.

Theorem 2.1.If $\{a_n\}$ is a #-convergent hyper infinite ${}^*\mathbb{R}$-valued sequence (that is, $a_n \to_\# q$ for some hyperreal number $q \in {}^*\mathbb{R}$), then $\{a_n\}$ is a Cauchy hyper infinite ${}^*\mathbb{R}$-valued sequence.
Proof. We know that $a_n \to_\# q$. Here is a ubiquitous trick: instead of using ε in the definition Definition 3.6.3, start with an arbitrary infinite small $\varepsilon > 0, \varepsilon \approx 0$ and then choose $N \in \mathbb{N}^\# \backslash \mathbb{N}$ so that $|a_n - q| < \varepsilon/2$ when $n > N$. Then if $m, n > N$, we have $|a_n - a_m| = |(a_n - q) - (a_m - q)| \leq |a_n - q| + |a_m - q| < \varepsilon/2 + \varepsilon/2 = \varepsilon$. This shows that $\{a_n\}_{n \in \mathbb{N}^\#}$ is a Cauchy sequence.
Theorem 2.2. If $\{a_n\}$ is a Cauchy hyper infinite ${}^*\mathbb{R}$-valued sequence, then it is bounded or hyper bounded; that is, there is some finite or hyperfinite $M \in {}^*\mathbb{R}$ such that $|a_n| \leq M$ for all $n \in \mathbb{N}^\#$.
Proof.Since $\{a_n\}$ is Cauchy, setting $\varepsilon = 1$ we know that there is some $N \in \mathbb{N}^\#$ such that $|a_m - a_n| < 1$whenever $m, n > N$. Thus, $|a_{N+1} - a_n| < 1$ for $n > N$. We can rewrite this as $a_{N+1} - 1 < a_n < a_{N+1} + 1$. This means that $|a_n|$ is less than the maximum of $|a_{N+1} - 1|$ and $|a_{N+1} + 1|$. So, set M equal to the maximum number in the following list: $\{|a_0|, |a_1|, \ldots, |a_N|, |a_{N+1} - 1|, |a_{N+1} + 1|\}$. Then for any term a_n, if $n \leq N$, then $|a_n|$ appears in the list and so $|a_n| \leq M$; if $n > N$, then
(as shown above) $|a_n|$ is less than at least one of the last two entries in the list, and so $|a_n| \leq M$. Hence, $M \in {}^*\mathbb{R}$ is a bound for the sequence $\{a_n\}$.
Definition 2.4. Let S be a set. A relation $x \sim y$ among pairs of elements of S
is said to be an equivalence relation if the following three properties hold:
Reflexivity: for any $s \in S, s \sim s$.
Symmetry: for any $s, t \in S$, if $s \sim t$ then $t \sim s$.
Transitivity: for any $s, t, r \in S$, if $s \sim t$ and $t \sim r$, then $s \sim r$.
Theorem 2.3. Let S be a set, with an equivalence relation $(\bullet \sim \bullet)$ on pairs of elements. For $s \in S$, denote by $\mathbf{cl}[s]$ the set of all elements in S that are related to s. Then for any $s, t \in S$, either $\mathbf{cl}[s] = \mathbf{cl}[t]$ or $\mathbf{cl}[s]$ and $\mathbf{cl}[t]$ are disjoint.
The hyperreal numbers ${}^*\mathbb{R}_c^\#$ will be constructed as equivalence classes of Cauchy hyper infinite ${}^*\mathbb{R}$-valued sequences. Let $\mathcal{F}_{{}^*\mathbb{R}}$ denote the set of all Cauchy hyper infinite ${}^*\mathbb{R}$-valued sequences of hyperreal numbers. We define the equivalence relation on $\mathcal{F}_{{}^*\mathbb{R}}$.
Definition 2.5. Let $\{a_n\}$ and $\{b_n\}$ be in $\mathcal{F}_{{}^*\mathbb{R}}$. Say they are #-equivalent if $a_n - b_n \to_\# 0$ i.e., if and only if the hyper infinite ${}^*\mathbb{R}$-valued sequence $\{a_n - b_n\}$ tends to 0.
Theorem 2.4.Definition 2.5 yields an equivalence relation on $\mathcal{F}_{{}^*\mathbb{R}}$.
Proof. We need to show that this relation is reflexive, symmetric, and transitive.
Reflexive: $a_n - a_n = 0$, and the hyper infinite sequence all of whose terms are 0 clearly #-converges to 0. So $\{a_n\}$ is related to $\{a_n\}$.
Symmetric: Suppose $\{a_n\}$ is related to $\{b_n\}$, so $a_n - b_n \to_\# 0$.
But $b_n - a_n = -(a_n - b_n)$, and since only the absolute value $|a_n - b_n| = |b_n - a_n|$ comes into play in Definition 3.9.2, it follows that $b_n - a_n \to_\# 0$ as well. Hence, $\{b_n\}$ is related to $\{a_n\}$.
Transitive: Here we will use the $\varepsilon/2$ trick we applied to prove Theorem 3.9.1. Suppose $\{a_n\}$ is related to $\{b_n\}$, and $\{b_n\}$ is related to $\{c_n\}$. This means that $a_n - b_n \to_\# 0$ and $b_n - c_n \to_\# 0$. To be fully precise, let us fix $\varepsilon > 0, \varepsilon \approx 0$; then there exists an $N \in \mathbb{N}^\#$ such that for all $n > N, |a_n - b_n| < \varepsilon/2$; also, there exists an M such that for all $n > M$,

$|b_n - c_n| < \varepsilon/2$. Well, then, as long as n is bigger than both N and M, we have that $|a_n - c_n| = |(a_n - b_n) + (b_n - c_n)| \leq |a_n - b_n| + |b_n - c_n| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$

So, choosing L equal to the max of $N, M,$ we see that given $\varepsilon > 0$ we can always choose L so that for $n > L, |a_n - c_n| < \varepsilon.$ This means that $a_n - c_n \to_{\#} 0$ – i.e. $\{a_n\}$ is related to $\{c_n\}$.

Definition 2.6. The external hyperreal numbers ${}^*\mathbb{R}_c^{\#}$ are the equivalence classes $\mathbf{cl}[\{a_n\}]$ of Cauchy hyper infinite ${}^*\mathbb{R}$-valued sequences of hyperreal numbers, as per Definition 3.9.5. That is, each such equivalence class is an external hyperreal number.

Definition 2.7. Given any hyperreal number $q \in {}^*\mathbb{R}$, define a hyperreal number $q^{\#}$ to be the equivalence class of the hyper infinite ${}^*\mathbb{R}$-valued sequence

$$q^{\#} = ({}^*q, {}^*q, {}^*q, {}^*q, \ldots)$$

consisting entirely of ${}^*q, q \in \mathbb{R}$. So we view ${}^*\mathbb{R}$ as being inside ${}^*\mathbb{R}_c^{\#}$ by thinking of each hyperreal number q as its associated equivalence class $q^{\#}$. It is standard to abuse this notation, and simply refer to the equivalence class as q as well.

Definition 2.8. Let $s, t \in {}^*\mathbb{R}_c^{\#}$, so there are Cauchy hyper infinite ${}^*\mathbb{R}$-valued sequences $\{a_n\}, \{b_n\}$ of hyperreal numbers with $s = \mathbf{cl}[\{a_n\}]$ and $t = \mathbf{cl}[\{b_n\}]$.

(a) Define $s + t$ to be the equivalence class of the sequence $\{a_n + b_n\}$.

(b) Define $s \times t$ to be the equivalence class of the sequence $\{a_n \times b_n\}$.

Theorem 2.5.The operations $+, \times$ in Definition 3.9.8 (a),(b) are well-defined.

Proof. Suppose that $\mathbf{cl}[\{a_n\}] = \mathbf{cl}[\{c_n\}]$ and $\mathbf{cl}[\{b_n\}] = \mathbf{cl}[\{d_n\}]$. Thus means that $a_n - c_n \to_{\#} 0$ and $b_n - d_n \to_{\#} 0$. Then $(a_n + b_n) - (c_n + d_n) = (a_n - c_n) + (b_n - d_n)$. Now, using the familiar $\varepsilon/2$ trick, you can construct a proof that this tends to 0, and so $\mathbf{cl}[\{a_n + b_n\}] = \mathbf{cl}[\{c_n + d_n\}]$.

Multiplication is a little trickier; this is where we will use Theorem 3.9.3. We will also use another ubiquitous technique: adding 0 in the form of $s - s$. Again, suppose that $\mathbf{cl}[(a_n)] = \mathbf{cl}[(c_n)]$ and $\mathbf{cl}[\{b_n\}] = \mathbf{cl}[\{d_n\}]$; we wish to show that $\mathbf{cl}[\{a_n \times b_n\}] = \mathbf{cl}[\{c_n \times d_n\}]$, or, in other words, that $a_n \times b_n - c_n \cdot d_n \to_{\#} 0$. Well, we add and subtract one of the other cross terms, say

$b_n \times c_n$: $a_n \times b_n - c_n \times d_n = a_n \times b_n + (b_n \times c_n - b_n \times c_n) - c_n \times d_n =$

$= (a_n \times b_n - b_n \times c_n) + (b_n \times c_n - c_n \times d_n) = b_n \times (a_n - c_n) + c_n \times (b_n - d_n).$

Hence, we have $|a_n \times b_n - c_n \times d_n| \leq |b_n| \times |a_n - c_n| + |c_n| \times |b_n - d_n|$. Now, from Theorem 2.2, there are numbers M and L such that $|b_n| \leq M$ and $|c_n| \leq L$ for all $n \in \mathbb{N}^{\#}$. Taking some number K which is bigger than both, we have

$|a_n \times b_n - c_n \times d_n| \leq |b_n| \times |a_n - c_n| + |c_n| \times |b_n - d_n| \leq K(|a_n - c_n| + |b_n - d_n|).$

Now, noting that both $a_n - c_n$ and $b_n - d_n$ tend to 0 and using the $\varepsilon/2$ trick (actually, this time we'll want to use $\varepsilon/2K$), we see that $a_n \times b_n - c_n \times d_n \to_{\#} 0.$

Theorem 2.6. Given any hyperreal number $s \in {}^*\mathbb{R}_c^{\#}$, $s \neq 0$, there is a hyperreal number $t \in {}^*\mathbb{R}_c^{\#}$ such that $s \times t = 1$.

Proof. First we must properly understand what the theorem says. The premise is that s is nonzero, which means that s is not in the equivalence class of $\{0, 0, 0, 0, \ldots\}$. In other words, $s = \mathbf{cl}[\{a_n\}]$ where $a_n - 0$ does not #-converge to 0. From this, we are to deduce the existence of a hyperreal number $t = \mathbf{cl}[\{b_n\}]$ such that $s \times t = \mathbf{cl}[\{a_n \times b_n\}]$ is the same equivalence class as $\mathbf{cl}[\{1, 1, 1, 1, \ldots\}]$. Doing so is actually an easy

consequence
of the fact that nonzero hyperreal numbers have multiplicative inverses, but there is a subtle difficulty. Just because s is nonzero (i.e. $\{a_n\}$ does not tend to 0), there's no reason any number of the terms in $\{a_n\}$ can't equal 0. However, it turns out that eventually, $a_n \neq 0$.

That is:

Lemma 2.1. If $\{a_n\}$ is a Cauchy sequence which does not #-tend to 0, then there is an $N \in \mathbb{N}^{\#}$ such that, for $n > N, a_n \neq 0$.

Definition 2.9. Let $s \in {}^{*}\mathbb{R}_c^{\#}$. Say that s is positive if $s \neq 0$, and if $s = \mathbf{cl}[\{a_n\}]$ for some Cauchy sequence of hyperreal numbers such that for some $N \in \mathbb{N}^{\#}, a_n > 0$ for all $n > N$. Given two hyperreal numbers s, t, say that $s > t$ if $s - t$ is positive.

Theorem 2.7. Let $s,t \in {}^{*}\mathbb{R}_c^{\#}$ be hyperreal numbers such that $s > t$, and let $r \in {}^{*}\mathbb{R}_c^{\#}$. Then $s + r > t + r$.

Proof. Let $s = \mathbf{cl}[\{a_n\}], t = \mathbf{cl}[\{b_n\}]$, and $r = \mathbf{cl}[\{c_n\}]$. Since $s > t$ i.e., $s - t > 0$, we know that there is an $N \in \mathbb{N}^{\#}$ such that, for $n > N$, $a_n - b_n > 0$. So $a_n > b_n$ for $n > N$. Now, adding c_n to both sides of this inequality (as we know we can do for hyperreal numbers ${}^{*}\mathbb{R}$), we have $a_n + c_n > b_n + c_n$ for $n > N$, or $(a_n + c_n) - (b_n + c_n) > 0$ for $n > N$. Note also that $(a_n + c_n) - (b_n + c_n) = a_n - b_n$ does not #-converge to 0, by the assumption that $s - t > 0$. Thus, by Definition 2.8, this means that $s + r = \mathbf{cl}[\{a_n + c_n\}] > \mathbf{cl}[\{b_n + c_n\}] = t + r$.

Theorem 2.8. Let $s,t \in {}^{*}\mathbb{R}_c^{\#}$ $s,t > 0$ be hyperreal numbers.Then there is $m \in \mathbb{N}^{\#}$ such that $m \times s > t$.

Proof. Let $s,t > 0$ be hyperreal numbers. We need to find a natural number m so that $m \times s > t$. First, recall that, by m in this context, we mean $\mathbf{cl}[\{m,m,m,m,\ldots\}]$. So, letting $s = \mathbf{cl}[\{a_n\}]$ and $t = \mathbf{cl}[\{b_n\}]$,what we need to show is that there exists m with $\mathbf{cl}[\{m,m,m,m,\ldots\}] \times \mathbf{cl}[\{a_1,a_2,a_3,a_4,\ldots\}] = \mathbf{cl}[\{m \times a_1, m \times a_2, m \times a_3, m \times a_4,\ldots\}] >$
$> \mathbf{cl}[\{b_1,b_2,b_3,b_4,\ldots\}]$.

Now, to say that $\mathbf{cl}[\{m \times a_n\}] > \mathbf{cl}[\{b_n\}]$, or $\mathbf{cl}[\{m \times a_n - b_n\}]$ is positive, is, by Definition 2.9, just to say that there is $N \in \mathbb{N}^{\#}$ such that $m \times a_n - b_n > 0$ for all $n > N$, while $m \times a_n - b_n \nrightarrow_{\#} 0$. To be precise, the first statement is:

There exist $m,N \in \mathbb{N}^{\#}$ so that $m \times a_n > b_n$ for all $n > N$.

To produce a contradiction, we assume this is not the case; assume that

(#) for every m and N, there exists an $n > N$ so that $m \times a_n \leq b_n$.

Now, since $\{b_n\}$ is a Cauchy sequence, by Theorem 2.2 it is hyperbounded - there is a hyperreal number $M \in {}^{*}\mathbb{R}$ such that $b_n \leq M$ for all $n \in \mathbb{N}^{\#}$. Now, by the properties for the hyperreal numbers ${}^{*}\mathbb{R}$, given any hyperreal number such that $\varepsilon > 0, \varepsilon \approx 0$, there is an $m \in \mathbb{N}^{\#}$ such that $M/m < \varepsilon/2$. Fix such an m. Then if $m \times a_n \leq b_n$, we have $a_n \leq b_n/m \leq M/m < \varepsilon/2$.

Now, $\{a_n\}$ is a Cauchy sequence, and so there exists N so that for
$n,k > N, |a_n - a_k| < \varepsilon/2$.

By Asumption (#), we also have an $n > N$ such that $m \times a_n \leq b_n$, which means that $a_n < \varepsilon/2$. But then for every $k > N$, we have that $a_k - a_n < \varepsilon/2$, so $a_k < a_n + \varepsilon/2 < \varepsilon/2 + \varepsilon/2 = \varepsilon$. Hence, $a_k < \varepsilon$ for all $k > N$. This proves that $a_k \rightarrow_{\#} 0$, which by Definition 2.9 contradicts the fact that $\mathbf{cl}[\{a_n\}] = s > 0$.

Thus, there is indeed some $m \in N$ so that $m \times a_n - b_n > 0$ for all sufficiently infinite large $n \in \mathbb{N}^{\#} \backslash \mathbb{N}$. To conclude the proof, we must also show that $m \times a_n - b_n \nrightarrow 0$.

Actually, it is possible that $m \times a_n - b_n \to 0$ (for example if $\{a_n\} = \{1, 1, 1, \dots\}$ and $\{b_n\} = \{m, m, m, \dots\}$). But that's okay: then we can simply choose a larger m. That is: let m be a hypernatural number constructed as above, so that $m \times a_n - b_n > 0$ for all sufficiently large $\in \mathbb{N}^{\#} \backslash \mathbb{N}$. If it happens to be true that $m \times a_n - b_n \nrightarrow 0$, then the proof is complete.

If, on the other hand, it turned out that $m \times a_n - b_n \to 0$, then take instead the integer $m + 1$. Since $s = \mathbf{cl}[\{a_n\}] > 0$, we have $a_n > 0$ for all infinite large n, so $(m+1) \times a_n - b_n = m \times a_n - b_n + a_n > a_n > 0$ for all infinite large n, so $m+1$ works just as well as m did in this regard; and since $m \times a_n - b_n \to 0$, we have $(m+1) \times a_n - b_n = (m \times a_n - b_n) + a_n \nrightarrow 0$ since $s = \mathbf{cl}[\{a_n\}] > 0$ (so $a_n \nrightarrow 0$).

It will be handy to have one more Theorem about how the hyperreals ${}^*\mathbb{R}$ and hyperreals ${}^*\mathbb{R}_c^{\#}$ compare before we proceed. This theorem is known as the density of ${}^*\mathbb{R}$ in ${}^*\mathbb{R}_c^{\#}$, and it follows almost immediately from the construction of the ${}^*\mathbb{R}_c^{\#}$ from ${}^*\mathbb{R}$.

Theorem 2.9. Given any hyperreal number $r \in {}^*\mathbb{R}_c^{\#}$, and any hyperreal number $\varepsilon > 0$, $\varepsilon \approx 0$, there is a hyperreal number $q \in {}^*\mathbb{R}$ such that $|r - q| < \varepsilon$.

Proof. The hyperreal number r is represented by a Cauchy ${}^*\mathbb{R}$-valued sequence $\{a_n\}$. Since this sequence is Cauchy, given $\varepsilon > 0, \varepsilon \approx 0$, there is $N \in \mathbb{N}^{\#}$ so that for all $m, n > N, |a_n - a_m| < \varepsilon$. Picking some fixed $l > N$, we can take the hyperreal number q given by $q = \mathbf{cl}[\{a_l, a_l, a_l, \dots\}]$. Then we have $r - q = \mathbf{cl}[\{a_n - a_l\}_{n \in \mathbb{N}^{\#}}]$, and $q - r = \mathbf{cl}[\{a_l - a_n\}_{n \in \mathbb{N}^{\#}}]$. Now, since $l > N$, we see that for $n > N, a_n - a_l < \varepsilon$ and $a_l - a_n < \varepsilon$, which means by Definition 3.9.9 that $r - q < \varepsilon$ and $q - r < \varepsilon$; hence, $|r - q| < \varepsilon$.

Definition 2.10. Let $S \subsetneq {}^*\mathbb{R}_c^{\#}$ be a non-empty set of hyperreal numbers.

A hyperreal number $x \in {}^*\mathbb{R}_c^{\#}$ is called an upper bound for S if $x \geq s$ for all $s \in S$.

A hyperreal number x is the least upper bound (or supremum $\sup S$) for S if x is an upper bound for S and $x \leq y$ for every upper bound y of S.

Remark 2.1. The order $\leq$ given by Definition 3.6.9 obviously is $\leq$ -incomplete.

Definition 2.11. Let $S \subsetneq {}^*\mathbb{R}_c^{\#}$ be a nonempty subset of ${}^*\mathbb{R}_c^{\#}$. We we will say that:

(1) S is $\leq$ -admissible above if the following conditions are satisfied:

(i) S bounded or hyperbounded above;

(ii) let $A(S)$ be a set $\forall x[x \in A(S) \Leftrightarrow x \geq S]$ then for any $\varepsilon > 0, \varepsilon \approx 0$ there exst $\alpha \in S$ and $\beta \in A(S)$ such that $\beta - \alpha \leq \varepsilon \approx 0$.

(2) S is $\leq$-admissible belov if the following condition are satisfied:

(i) S bounded belov;

(ii) let $L(S)$ be a set $\forall x[x \in L(S) \Leftrightarrow x \leq S]$ then for any $\varepsilon > 0, \varepsilon \approx 0$ there exst $\alpha \in S$ and $\beta \in L(S)$ such that $\alpha - \beta \leq \varepsilon \approx 0$.

Theorem 2.10. (i) Any $\leq$-admissible above subset $S \subset {}^*\mathbb{R}_c^{\#}$ has the least upper bound property.(ii) Any $\leq$-admissible below subset $S \subset {}^*\mathbb{R}_c^{\#}$ has the greatest lower bound property.

Proof. Let $S \subset {}^*\mathbb{R}_c^{\#}$ be a nonempty subset, and let M be an upper bound for S. We are going to construct two sequences of hyperreal numbers, $\{u_n\}$ and $\{l_n\}$. First, since S is nonempty, there is some element $s_0 \in S$. Now, we go through the following hyperinductive procedure to produce numbers $u_0, u_1, u_2, \dots, u_n, \dots$ and $l_1, l_2, l_3, \dots, l_n, \dots$

(i) Set $u_0 = M$ and $l_0 = s$.

(ii) Suppose that we have already defined u_n and l_n. Consider the number $m_n = (u_n + l_n)/2$, the average between u_n and l_n.
(1) If m_n is an upper bound for S, define $u_{n+1} = m_n$ and $l_{n+1} = l_n$.
(2) If m_n is not an upper bound for S, define $u_{n+1} = u_n$ and $l_{n+1} = l_n$.
Remark 2.1.Since $s < M$, it is easy to prove by hyper infinite induction that
(i) $\{u_n\}$ is a non-increasing sequence: $u_{n+1} \leq u_n, n \in \mathbb{N}^{\#}$ and $\{l_n\}$ is a non-decreasing sequence $l_{n+1} \geq l_n, n \in \mathbb{N}^{\#}$, (ii) u_n is an upper bound for S for all $n \in \mathbb{N}^{\#}$ and l_n is never an upper bound for S for any $n \in \mathbb{N}^{\#}$, (iii) $u_n - l_n = 2^{-n}(M - s)$.
This gives us the following lemma.
Lemma 2.2. $\{u_n\}$ and $\{l_n\}$ are Cauchy ${}^*\mathbb{R}$-valued sequences of hyperreal numbers.
Proof. Note that each $l_n \leq M$ for all $n \in \mathbb{N}^{\#}$. Since $\{l_n\}$ is non-decreasing and $u_n - l_n = 2^{-n}(M - s)$, it follows directly that $\{l_n\}$ is Cauchy.
For $\{u_n\}$, we have $u_n \geq s_0$ for all $n \in \mathbb{N}^{\#}$, and so $-u_n \leq -s_0$.
Since $\{u_n\}$ is non-increasing, $\{-u_n\}$ is non-decreasing, and so as above, $\{-u_n\}$ is Cauchy. It is easy to verify that, therefore, $\{u_n\}$ is Cauchy.
The following Lemma shows that $\{u_n\}$ does #-tend to a hyperreal number $u \in {}^*\mathbb{R}_c^{\#}$.
Lemma 2.3. There is a hyperreal number $u \in {}^*\mathbb{R}_c^{\#}$ such that $u_n \to_{\#} u$.
Proof. Fix a term u_n in the sequence $\{u_n\}$. By Theorem 2.9, there is a hyperreal number $q_n \in {}^*\mathbb{R}, n \in \mathbb{N}^{\#}$ such that $|u_n - q_n| < 1/n$. Consider the sequence $\{q_1, q_2, q_3, \ldots, q_n, \ldots\}$ of hyperreal numbers. We will show this sequence is Cauchy. Fix $\varepsilon > 0, \varepsilon \approx 0$. By the Theorem 2.8, we can choose $N \in \mathbb{N}^{\#}$ so that $1/N < \varepsilon/3$. We know, since $\{u_n\}$ is Cauchy, that there is an $M \in \mathbb{N}^{\#}$ such that for $n, m > M$, $|u_n - u_m| < \varepsilon/3$. Then, so long as $n, m > max\{N, M\}$, we have

$$|q_n - q_m| = |(q_n - u_n) + (u_n - u_m) + (u_m - q_m)| \leq$$
$$\leq |q_n - u_n| + |u_n - u_m| + |u_m - q_m| < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.$$

Thus, $\{q_n\}$ is a Cauchy sequence of internal hyperreal numbers, and so it represents the external hyperreal number $u = \mathbf{cl}[\{q_n\}]$. We must show that $u_n - u \to_{\#} 0$, but this is practically built into the definition of u. To be precise, letting q_n^* be the hyperreal number
$\mathbf{cl}[\{q_n, q_n, q_n, \ldots\}]$, we see immediately that $q_n^* - u \to_{\#} 0$ (this is precisely equivalent to the statement that $\{q_n\}$ is Cauchy). But $u_n - q_n^* < 1/n$ by construction; it is easily verify that the assertion that if a sequence $q_n^* \to_{\#} u$ and $u_n - q_n^* \to_{\#} 0$, then $u_n \to_{\#} u$. So $\{u_n\}$, a non-increasing sequence of upper bounds for S, tends to a hyperreal
number u. As you've guessed, u is the least upper bound of our set S. To prove this, we
need one more lemma.
Lemma 2.4. $l_n \to_{\#} u$.
Proof. First, note in the first case above, we have that

$$u_{n+1} - l_{n+1} = m_n - l_n = \frac{u_n + l_n}{2} - l_n = \frac{u_n - l_n}{2}.$$

In the second case, we also have

$$u_{n+1} - l_{n+1} = u_n - m_n = u_n - \frac{u_n + l_n}{2} = \frac{u_n - l_n}{2}.$$

Now, this means that $u_1 - l_1 = \frac{1}{2}(M - s)$, and so $u_2 - l_2 = \frac{1}{2}(u_1 - l_1) = \frac{1}{2^2}(L - s)$, and in general by hyperinfinite induction, $u_n - l_n = 2^{-n}(M - s)$. Since $M > s$ so

$M-s>0$, and since $2^{-n}<1/n$, by the Theorem 2.8, we have for any $\varepsilon>0$ that $2^{-n}(M-s)<\varepsilon$ for all sufficiently large $n\in\mathbb{N}^{\#}$. Thus, $u_n-l_n=2^{-n}(M-s)<\varepsilon$ as well, and so $u_n-l_n\to_{\#}0$. Again, it is easily verify that, since $u_n\to_{\#}u$, we have $l_n\to_{\#}u$ as well.

Remark 2.2.Note that assumption in Theorem 2.10 that S is $\leq$-admissible above subset of $\mathbb{R}_c^{\#}$ is necessarily, othervice Theorem 2.10 is not holds.

Theorem 2.11.(**Generalized Nested Intervals Theorem**)

Let $\{I_n\}_{n\in\mathbb{N}^{\#}}=\{[a_n,b_n]\}_{n\in\mathbb{N}^{\#}},[a_n,b_n]\subset\mathbb{R}_c^{\#}$ be a hyper infinite sequence of closed intervals satisfying each of the following conditions:

(i) $I_1\supseteq I_2\supseteq I_3\supseteq\ldots\supseteq I_n\supseteq\ldots,$

(ii) $b_n-a_n\to_{\#}0$ as $n\to\infty^{\#}$.

Then $\bigcap_{n=1}^{\infty^{\#}}I_n$ consists of exactly one hyperreal number $x\in\mathbb{R}_c^{\#}$. Moreover both sequences $\{a_n\}$ and $\{b_n\}$ #-converge to x.

Proof.Note that: (a) the set $A=\{a_n|n\in\mathbb{N}^{\#}\}$ is hyperbouded above by $b_1$and (b) the set $A=\{a_n|n\in\mathbb{N}^{\#}\}$ is $\leq$-admissible above subset of $\mathbb{R}_c^{\#}$.

By Theorem 3.9.10 there exists $\sup A$. Let $\xi=\sup A$.

Since I_n are nested,for any positive hyperintegers m and n we have $a_m\leq a_{m+n}\leq b_{m+n}\leq b_n$,so that $\xi\leq b_n$ for each $n\in\mathbb{N}^{\#}$.Since we obviously have $a_n\leq\xi$ for each $n\in\mathbb{N}^{\#}$,we have $a_n\leq\xi\leq b_n$ for all $n\in\mathbb{N}^{\#}$,which implies $\xi\in\bigcap_{n=1}^{\infty^{\#}}I_n$.Finally, if $\xi,\eta\in\bigcap_{n=1}^{\infty^{\#}}I_n$, with $\xi\leq\eta$, then we get $0\leq\eta-\xi\leq b_n-a_n$,for all $n\in\mathbb{N}^{\#}$,so that $0\leq\eta-\xi\leq\inf_{n\in\mathbb{N}^{\#}}|b_n-a_n|=0$.

Theorem 2.12.(**Generalized Squeeze Theorem**)

Let $\{a_n\},\{c_n\}$ be two hyper infinite sequences #-converging to L,and $\{b_n\}$ a hyper infinite sequence. If $\forall n\geq K,K\in\mathbb{N}^{\#}$ we have $a_n\leq b_n\leq c_n$, then $\{b_n\}$ also #-converges to L.

Proof. Choose an $\varepsilon>0,\varepsilon\approx0$. By definition of the #-limit,there is an $N_1\in\mathbb{N}^{\#}$ such that for all $n>N_1$ we have $|a_n-L|<\varepsilon$, in other words $L-\varepsilon<a_n<L+\varepsilon$.Similarly, there is an $N_2\in\mathbb{N}^{\#}$ such that for all $n>N_2$ we have $L-\varepsilon<c_n<L+\varepsilon$. Denote $N=\max(N_1,N_2,K)$. Then for $n>N,L-\varepsilon<a_n\leq b_n\leq c_n<L+\varepsilon$, in other words $|b_n-L|<\varepsilon$.Since $\varepsilon>0,\varepsilon\approx0$ was arbitrary, by definition of the #-limit this says that #-$\lim_{n\to\infty^{\#}}b_n=L$.

Theorem 2.13.(**Corollary of the Generalized Squeeze Theorem**).

If #-$\lim_{n\to\infty^{\#}}|a_n|=0$ then #-$\lim_{n\to\infty^{\#}}a_n=0$.

Proof.We know that $-|a_n|\leq a_n\leq|a_n|$.We want to apply the Generalized Squeeze Theorem.We are given that #-$\lim_{n\to\infty^{\#}}|a_n|=0$.This also implies that #-$\lim_{n\to\infty^{\#}}(-|a_n|)=0$.So by the Generalized Squeeze Theorem, #-$\lim_{n\to\infty^{\#}}a_n=0$.

Theorem 2.14. (**Generalized Bolzano-Weierstrass Theorem**)

Every hyperbounded hyper infinite ${}^*\mathbb{R}_c^{\#}$-valued sequence has a #-convergent hyper infinite subsequence.

Proof. Let $\{w_n\}_{n\in\mathbb{N}^{\#}}$ be a hyperbounded hyper infinite sequence. Then, there exists an interval $[a_1,b_1]$ such that $a_1\leq w_n\leq b_1$ for all $n\in\mathbb{N}^{\#}$.

Either $\left[a_1,\frac{a_1+b_1}{2}\right]$ or $\left[\frac{a_1+b_1}{2},b_1\right]$ contains hyper infinitely many terms of $\{w_n\}$. That is, there exists hyper infinitely many n in $\mathbb{N}^{\#}$ such that a_n is in $\left[a_1,\frac{a_1+b_1}{2}\right]$ or there exists hyper infinitely many n in $\mathbb{N}^{\#}$ such that a_n is in $\left[\frac{a_1+b_1}{2},b_1\right]$. If $\left[a_1,\frac{a_1+b_1}{2}\right]$ contains hyper infinitely many terms of $\{w_n\}$, let $[a_2,b_2]=\left[a_1,\frac{a_1+b_1}{2}\right]$. Otherwise, let

$[a_2,b_2] = \left[\frac{a_1+b_1}{2}, b_1\right]$.
Either $\left[a_2, \frac{a_2+b_2}{2}\right]$ or $\left[\frac{a_2+b_2}{2}, b_2\right]$ contains hyper infinitely many terms of $\{w_n\}_{n\in\mathbb{N}^\#}$. If $\left[a_2, \frac{a_2+b_2}{2}\right]$ contains hyper infinitely many terms of $\{w_n\}$, let $[a_3,b_3] = \left[a_2, \frac{a_2+b_2}{2}\right]$. Otherwise, let $[a_3,b_3] = \left[\frac{a_2+b_2}{2}, b_2\right]$. By hyper infinite induction, we can continue this construction and obtain hyper infinite sequence of intervals $\{[a_n,b_n]\}_{n\in\mathbb{N}^\#}$ such that:
(i) for each $n \in \mathbb{N}^\#, [a_n,b_n]$ contains hyper infinitely many terms of $\{w_n\}_{n\in\mathbb{N}^\#}$,
(ii) for each $n \in \mathbb{N}^\#, [a_{n+1},b_{n+1}] \subseteq [a_n,b_n]$ and
(iii) for each $n \in \mathbb{N}^\#, b_{n+1} - a_{n+1} = \frac{1}{2}(b_n - a_n)$.
Then generalized nested intervals theorem implies that the intersection of all of the intervals $[a_n,b_n]$ is a single point w. We will now construct a hyper infinite subsequence of $\{w_n\}_{n\in\mathbb{N}^\#}$ which will #-converge to w.
Since $[a_1,b_1]$ contains hyper infinitely many terms of $\{w_n\}_{n\in\mathbb{N}^\#}$, there exists $k_1 \in \mathbb{N}^\#$ such that w_{k_1} is in $[a_1,b_1]$. Since $[a_2,b_2]$ contains hyper infinitely many terms of $\{w_n\}_{n\in\mathbb{N}^\#}$, there exists $k_2 \in \mathbb{N}^\#, k_2 > k_1,$ such that w_{k_2} is in $[a_2,b_2]$. Since $[a_3,b_3]$ contains hyper infinitely many terms of $\{w_n\}_{n\in\mathbb{N}^\#}$, there exists $k_3 \in \mathbb{N}^\#, k_3 > k_2,$ such that w_{k_3} is in $[a_3,b_3]$. Continuing this process by hyper infinite induction, we obtain hyper infinite sequence $\{w_{k_n}\}_{n\in\mathbb{N}^\#}$ such that $w_{k_n} \in [a_n,b_n]$ for each $n \in \mathbb{N}^\#$. The sequence $\{w_{k_n}\}_{n\in\mathbb{N}^\#}$ is a subsequence of $\{w_n\}_{n\in\mathbb{N}^\#}$ since $k_{n+1} > k_n$ for each $n \in \mathbb{N}^\#$. Since $a_n \to_\# w,$ and $a_n \le w_n \le b_n$ for each $n \in \mathbb{N}^\#$, the squeeze theorem implies that $w_{k_n} \to_\# w$.
Definition 2.12. Let $\{a_n\}$ be a hyperreal sequence i.e.,$a_n \in {}^*\mathbb{R}^\#_c, n \in \mathbb{N}^\#$. Say that $\{a_n\}$ #-tends to 0 if, given any $\varepsilon > 0, \varepsilon \approx 0,$ there is a hypernatural number $N \in \mathbb{N}^\#\backslash\mathbb{N}$, $N = N(\varepsilon)$ such that,for all $n > N$, $|a_n| \le \varepsilon$. We often denote this symbolically by $a_n \to_\# 0$.
We can also, at this point, define what it means for a hyperreal sequence #-tends to a given number $q \in {}^*\mathbb{R}^\#_c$: $\{a_n\}$ #-tends to q if the hyperreal sequence $\{a_n - q\}$ #-tends to 0 i.e., $a_n - q \to_\# 0$.
Definition 2.13. Let $\{a_n\}, n \in \mathbb{N}^\#$ be a hyperreal sequence. We call $\{a_n\}$ a Cauchy hyperreal sequence if the difference between its terms #-tends to 0. To be precise: given any hyperreal number $\varepsilon > 0, \varepsilon \approx 0,$ there is a hypernatural number $N = N(\varepsilon)$ such that for any $m,n > N, |a_n - a_m| < \varepsilon$.
Theorem 2.15. If $\{a_n\}$ is a #-convergent hyperreal sequence (that is, $a_n \to_\# b$ for some hyperreal number $b \in \mathbb{R}^\#_c$), then $\{a_n\}$ is a Cauchy hyperreal sequence.
Theorem 2.16. If $\{a_n\}$ is a Cauchy hyperreal sequence, then it is hyper bounded; that is, there is some $M \in \mathbb{R}^\#_c$ such that $|a_n| \le M$ for all $n \in \mathbb{N}^\#$.
Theorem 2.17. Any Cauchy hyperreal sequence $\{a_n\}$ has a #-limit in ${}^*\mathbb{R}^\#_c$ i.e., there exists $b \in {}^*\mathbb{R}^\#_c$ such that $a_n \to_\# b$.
Proof.By Definition 2.13 given $\varepsilon > 0, \varepsilon \approx 0,$ there is a hypernatural number $N = N(\varepsilon)$ such that for any $n, n' > N$,

$$|a_n - a_{n'}| < \varepsilon. \tag{2.1}$$

From (2.1) for any $n, n' > N$ we get

$$a_{n'} - \varepsilon < a_n < a_n + \varepsilon. \tag{2.2}$$

The generalized Bolzano-Weierstrass theorem implies there is a #-convergent hyper infinite subsequence $\{a_{n_k}\} \subset \{a_n\}$ such that $a_{n_k} \to_\# b$ for some hyperreal number $b \in {}^*\mathbb{R}^\#_c$. Let us show that the sequence $\{a_n\}$ also #-convergent to this $b \in {}^*\mathbb{R}^\#_c$.

We can choose $k \in \mathbb{N}^{\#}$ so large that $n_k > N$ and

$$|a_{n_k} - b| < \varepsilon. \quad (2.3)$$

We choose now in (2.1) $n' = n_k$ and therefore

$$|a_n - a_{n_k}| < \varepsilon. \quad (2.4)$$

From (2.3) and (2.4) for any $n > N$ we get

$$|(a_{n_k} - b) + (a_n - a_{n_k})| = |a_n - b| < 2\varepsilon. \quad (2.5)$$

Thus $a_n \to_{\#} b$ as well.

Remark 2.3. Note that there exist canonical natural embedings

$$\mathbb{R} \hookrightarrow {}^*\mathbb{R} \hookrightarrow {}^*\mathbb{R}_c^{\#}. \quad (2.6)$$

2.2.The Extended Hyperreal Number System ${}^*\hat{\mathbb{R}}_c^{\#}$

Definition 2.14.(a) A set $S \subset \mathbb{N}^{\#}$ is hyperfinite if $\mathbf{card}(S) = \mathbf{card}(\{x|0 \leq x \leq n\})$, $n \in \mathbb{N}^{\#}\backslash\mathbb{N}$. (b) A set $S \subseteq \mathbb{N}^{\#}$ is hyper infinite if $\mathbf{card}(S) = \mathbf{card}(\mathbb{N}^{\#})$.

Notation 2.1. If F is an arbitrary collection of subsets of ${}^*\mathbb{R}_c^{\#}$, then $\cup\{S|S \in F\}$is the set of all elements that are members of at least one of the sets in F , and $\cap\{S|S \in F\}$ is the set of all elements that are members of every set in F. The union and intersection of finitely or hyperfinitely many sets $S_k, 0 \leq k \leq n \in \mathbb{N}^{\#}$ are also written as $\cup_{k=0}^{n} S_k$ and $\cap_{k=0}^{n} S_k$. The union and intersection of an hyperinfinite sequence $S_k, k \in \mathbb{N}^{\#}$ of sets are written as $\cup_{k=0}^{\infty^{\#}} S$ or $\cup_{n\in\mathbb{N}^{\#}} S$ and $\cap_{k=0}^{\infty^{\#}} S$ or $\cap_{n\in\mathbb{N}^{\#}} S$ correspondingly.

A nonempty set S of hyperreal numbers ${}^*\mathbb{R}_c^{\#}$ is unbounded above if it has no hyperfinite upper bound, or unbounded below if it has no hyperfinite lower bound. It is convenient to adjoin to the hyperreal number system two points, $+\infty^{\#}$ (which we also write more simply as $\infty^{\#}$) and $-\infty^{\#}$,and to define the order relationships between them and any hyperreal number $x \in {}^*\mathbb{R}_c^{\#}$ by $-\infty^{\#} < x < \infty^{\#}$.

We call $-\infty^{\#}$ and $\infty^{\#}$ points at hyperinfinity. If S is a nonempty set of hyperreals, we write $\sup S = \infty^{\#}$ to indicate that S is unbounded above, and $\inf S = -\infty^{\#}$ to indicate that S is unbounded below.

2.3. #-Open and #-Closed Sets on ${}^*\hat{\mathbb{R}}_c^{\#}$.

Definition 2.15. If a and b are in the extended hyperreals and $a < b$, then the open interval (a,b) is defined by $(a,b) \triangleq \{x|a < x < b\}$. :

The open intervals $(a,+\infty^{\#})$ and $(-\infty^{\#},b)$ are semi-hyperinfinite if a and b are finite or hyperfinite, and $(-\infty^{\#},\infty^{\#})$ is the entire hyperreal line.

If $-\infty^{\#} < a < b < \infty^{\#}$, the set $[a,b] \triangleq \{x|a \leq x \leq b\}$ is #-closed, since its complement is the union of the #-open sets $(-\infty^{\#},a)$ and $(b,\infty^{\#})$. We say that $[a,b]$ is a #-closed interval. Semi-hyper infinite #-closed intervals are sets of the form $[a,\infty) = \{x|a \leq x\}$ and $(-\infty^{\#},a] = \{x|x \leq a\}$,where a is finite or hyperfinite. They are #-closed sets, since their complements are the #-open intervals $(-\infty^{\#},a)$ and $(a,\infty^{\#})$,respectively.

Definition 2.16. If $x_0 \in \mathbb{R}_c^{\#}$ is a hyperreal number and $\varepsilon > 0, \varepsilon \approx 0$ then the open interval $(x_0 - \varepsilon, x_0 + \varepsilon)$ is an #-neighborhood of x_0. If a set $S \subset {}^*\mathbb{R}_c^{\#}$ contains an #-neighborhood of x_0, then S is a #-neighborhood of x_0, and x_0 is an #-interior point of S.

The set of #-interior points of S is the #-interior of S, denoted by #-$Int(S)$.

(i) If every point of S is an #-interior point (that is, $S =$ #-$Int(S)$), then S is #-open.

(ii) A set S is #-closed if $S^c = {}^*\mathbb{R}_c^{\#} \backslash S$ is #-open.

Example 2.1. An open interval (a,b) is an #-open set, because if $x_0 \in (a,b)$ and $\varepsilon \leq \min\{x_0 - a; b - x_0\}$, then $(x_0 - \varepsilon, x_0 + \varepsilon) \subset (a,b)$

Remark 2.4.The entire hyperline ${}^*\hat{\mathbb{R}}_c^{\#} = (-\infty^{\#}, \infty^{\#})$ is #-open, and therefore $\varnothing$ is #-closed.However, $\varnothing$ is also #-open, for to deny this is to say that $\varnothing$ contains a point that is not an #-interior point, which is absurd because $\varnothing$ contains no points. Since $\varnothing$ is #-open, ${}^*\hat{\mathbb{R}}_c^{\#}$ is #-closed. Thus, ${}^*\hat{\mathbb{R}}_c^{\#}$ and $\varnothing$ are both #-open and #-closed.

Remark 2.5.They are not the only subsets of ${}^*\hat{\mathbb{R}}_c^{\#}$ with this property.

Definition 2.17.A deleted #-neighborhood of a point x_0 is a set that contains every point of some #-neighborhood of x_0 except for x_0 itself. For example, $S = \{x | 0 < |x - x_0| < \varepsilon\}$,where $\varepsilon \approx 0$, is a deleted #-neighborhood of x_0. We also say that it is a deleted ε-#-neighborhood of x_0.

Theorem 2.18.(a) The union of #-open sets is #-open:

(b) The #-intersection of #-closed sets is #-closed:

These statements apply to arbitrary collections, hyperfinite or hyperinfinite, of #-open and #-closed sets.

Proof (a) Let L be a collection of #-open sets and $S = \cup\{G | G \in L\}$.

If $x_0 \in S$, then $x_0 \in G_0$ for some G_0 in L, and since G_0 is #-open, it contains some ε-#-neighborhood of x_0. Since $G_0 \subset S$, this ε-#–neighborhood is in S, which is consequently a #-neighborhood of x_0. Thus, S is a #-neighborhood of each of its points, and therefore #-open, by definition.

(b) Let F be a collection of #-closed sets and $T = \cap\{H | H \in F\}$. Then $T^c = \cup\{H^c | H \in F\}$ and, since each H^c is #-open, T^c is #-open, from (a). Therefore, T is #-closed, by definition.

Example 2.2. If $-\infty^{\#} < a < b < \infty^{\#}$, the set $[a,b] = \{x | a \leq x \leq b\}$ is #-closed, since its complement is the union of the #-open sets $(-\infty^{\#} a)$ and $(b, \infty^{\#})$. We say that $[a,b]$ is a #-closed interval. The set $[a,b) = \{x | a \leq x < b\}$ is a half-#-closed or half-#-open interval if $-\infty^{\#} < a < b < \infty^{\#}$, as is $(a,b] = \{x | a < x \leq b\}$ however, neither of these sets is #-open or #-closed. Semi-infinite #-closed intervals are sets of the form $[a, \infty^{\#}) = \{x | a \leq x\}$ and $(-\infty^{\#}, a] = \{x | x \leq a\}$,where a is hyperfinite. They are #-closed sets, since their complements are the #-open intervals $(-\infty^{\#}, a)$ and $(a, \infty^{\#})$,respectively.

Definition 2.18. Let S be a subset of $\hat{\mathbb{R}}_c^{\#} = (-\infty^{\#}, \infty^{\#})$. Then

(a) x_0 is a #-limit point of S if every deleted #-neighborhood of x_0 contains a point of S.

(b) x_0 is a boundary point of S if every #-neighborhood of x_0 contains at least one point in S and one not in S. The set of #-boundary points of S is the #-boundary of S, denoted by #-∂S. The #-closure of S, denoted by #-$\bar{S}$, is $S \cup$ #-∂S.

(c) x_0 is an #-isolated point of S if $x_0 \in S$ and there is a #-neighborhood of x_0 that contains no other point of S.

(d) x_0 is #-exterior to S if x_0 is in the #-interior of S^c. The collection of such points is the #-exterior of S.

Theorem 2.19. A set S is #-closed if and only if no point of S^c is a #-limit point of S.
Proof. Suppose that S is #-closed and $x_0 \in S^c$. Since S^c is #-open, there is a #-neighborhood of x_0 that is contained in S^c and therefore contains no points of S. Hence, x_0 cannot be a #-limit point of S. For the converse, if no point of S^c is a #-limit point of S then every point in S^c must have a #-neighborhood contained in S^c. Therefore, S^c is #-open and S is #-closed.
Corollary 2.1.A set S is #-closed if and only if it contains all its #-limit points.
If S is #-closed and hyper bounded, then $\inf(S)$ and $\sup(S)$ are both in S.
Proposition 2.1. If S is #-closed and hyper bounded, then $\inf(S)$ and $\sup(S)$ are both in S. .

2.4. #-Open Coverings

Definition 2.19.A collection H of #-open sets of $\mathbb{R}_c^{\#}$ is an #-open covering of a set S if every point in S is contained in a set H belonging to H; that is, if $S \subset \cup\{F|F \in H\}$.
Definition 3.10.20.A set $S \subset \mathbb{R}_c^{\#}$ is called #-compact (or hyper compact) if each of its #-open covers has a hyperfinite subcover. .
Theorem 3.10.20.(**Generalized Heine–Borel Theorem**) If H is an #-open covering of a #-closed and hyper bounded subset S of the hyperreal line ${}^*\mathbb{R}_c^{\#}$ (or of the ${}^*\mathbb{R}_c^{\#n}, n \in \mathbb{N}^{\#}$)
then S has an #-open covering $\widetilde{H}$ consisting of hyper finite many #-open sets belonging to H.
Proof. If a set S in ${}^*\mathbb{R}_c^{\#n}$ is hyper bounded, then it can be enclosed within an n-box $T_0 = [-a,a]^n$ where $a > 0$. By the property above, it is enough to show that T_0 is #-compact.
Assume, by way of contradiction, that T_0 is not #-compact. Then there exists an hyper infinite open cover $C_{\infty^{\#}}$ of T_0 that does not admit any hyperfinite subcover. Through bisection of each of the sides of T_0, the box T_0 can be broken up into $2n$ sub n-boxes, each of which has diameter equal to half the diameter of T_0. Then at least one of the $2n$ sections of T_0 must require an hyper infinite subcover of $C_{\infty^{\#}}$, otherwise $C_{\infty^{\#}}$ itself would have a hyperfinite subcover, by uniting together the hyperfinite covers of the sections. Call this section T_1.Likewise, the sides of T_1 can be bisected, yielding 2^n sections of T_1, at least one of which must require an hyper infinite subcover of $C_{\infty^{\#}}$. Continuing in like manner yields a decreasing hyper infinite sequence of nested n-boxes:
$T_0 \supset T_1 \supset T_2 \supset \ldots \supset T_k \supset \ldots, k \in \mathbb{N}^{\#}$, where the side length of T_k is $(2a)/2^k$, which #-converges to 0 as k tends to hyper infinity, $k \to \infty^{\#}$. Let us define a hyper infinite sequence $\{x_k\}_{k \in \mathbb{N}^{\#}}$ such that each $x_k : x_k \in T_k$. This hyper infinite sequence is Cauchy, so it must #-converge to some #-limit L. Since each $T_k$is #-closed, and for each k the sequence $\{x_k\}_{k \in \mathbb{N}^{\#}}$ is eventually always inside T_k, we see that $L \in T_k$ for each $k \in \mathbb{N}^{\#}$. Since $C_{\infty^{\#}}$ covers T_0, then it has some member $U \in C_{\infty^{\#}}$ such that $L \in U$. Since U is open, there is an n-ball $B(L) \subseteq U$. For large enough k, one has $T_k \subseteq B(L) \subseteq U$, but then the hyper infinite number of members of $C_{\infty^{\#}}$ needed to cover T_k can be replaced by just one: U, a contradiction.Thus, T_0 is #-compact. Since S is #-closed and a subset of the #-compact set T_0, then S is also #-compact.

As an application of the Generalized Heine–Borel theorem, we give a short proof of the
Generalized Bolzano–Weierstrass Theorem.

Theorem 3.10.21.(Generalized Bolzano–Weierstrass Theorem) Every hyper bounded hyper infinite set $S \subset {}^*\mathbb{R}_c^\#$ has at least one #-limit point.

Proof. We will show that a hyper bounded nonempty set without a #-limit point can contain only finite or a hyper finite number of points. If S has no #-limit points, then S is #-closed (Theorem 9.) and every point $x \in S$ has an #-open neighborhood N_x that contains no point of S other than x. The collection $H = \{N_x | x \in S\}$ is an #-open covering for S. Since S is also hyper bounded, Theorem 3.10.20 implies that S can be covered by finite or a hyper finite collection of sets from H, say $N_{x_1},\ldots,N_{x_n}, n \in \mathbb{N}^\#$. Since these sets contain only $x_1,\ldots,x_n$ from S, it follows that $S = \{x_k\}_{1 \leq k \leq n}, n \in \mathbb{N}^\#$.

3.External hyperfinite sum of the ${}^*\mathbb{R}_c^\#$- valued hyperfinite sequences.Main properties.

Theorem 3.1. Let $\{a_i\}_{i=1}^n$and $\{b_i\}_{i=1}^n$be ${}^*\mathbb{R}_c^\#$- valued hyperfinite sequences.The following equalities holds for any $n, k_1, l_1 \in \mathbb{N}^\# \backslash \mathbb{N}$:

(1)

$$b \times \left(Ext\text{-}\sum_{i=0}^{n} a_i \right) = Ext\text{-}\sum_{i=0}^{n} b \times a_i \tag{3.1}$$

(2)

$$Ext\text{-}\sum_{i=0}^{n} a_i \pm Ext\text{-}\sum_{i=0}^{n} b_i = Ext\text{-}\sum_{i=0}^{n} (a_i \pm b_i) \tag{3.2}$$

(3)

$$Ext\text{-}\sum_{i=0}^{n} a_i = Ext\text{-}\sum_{i=0}^{j} a_i + Ext\text{-}\sum_{i=j+1}^{n} a_i \tag{3.3}$$

(4)

$$Ext\text{-}\sum_{i=k_0}^{k_1} \left(Ext\text{-}\sum_{j=l_0}^{l_1} a_{ij} \right) = Ext\text{-}\sum_{j=l_0}^{l_1} \left(Ext\text{-}\sum_{i=k_0}^{k_1} a_{ij} \right) \tag{3.4}$$

(5)

$$\left(Ext\text{-}\sum_{i=0}^{n} a_i \right) \times \left(Ext\text{-}\sum_{j=0}^{n} b_j \right) = Ext\text{-}\sum_{i=0}^{n} \left(Ext\text{-}\sum_{j=0}^{n} a_i \times b_j \right) \tag{3.5}$$

(6)

$$\left(Ext\text{-}\prod_{i=0}^{n} a_i \right) \times \left(Ext\text{-}\prod_{i=0}^{n} b_i \right) = Ext\text{-}\prod_{i=0}^{n} a_i \times b_i \tag{3.11.6}$$

(7)

$$\left(Ext\text{-}\prod_{i=0}^{n} a_i \right)^m = Ext\text{-}\prod_{i=0}^{n} a_i^m \tag{3.7}$$

Proof. Imediately by hyper infinite induction principle.

Theorem 3.2. Let $\{a_i\}_{i=1}^n$and $\{b_i\}_{i=1}^n$be ${}^*\mathbb{R}_c^\#$- valued hyperfinite sequences.
Suppose that $a_i \leq b_i, 1 \leq i \leq n$ then the following equalities holds for any $n \in \mathbb{N}^\#$:

$$Ext\text{-}\sum_{i=0}^{n} a_i \leq Ext\text{-}\sum_{i=0}^{n} b_i \tag{3.8}$$

Theorem 3.3. Let $\{a_i\}_{i=1}^{n}$and $\{b_i\}_{i=1}^{n}$be ${}^*\mathbb{R}_c^\#$- valued hyperfinite sequences.Then the following equalities holds for any $n \in \mathbb{N}^\#$:

$$\left(Ext\text{-}\sum_{i=0}^{n} a_i \times b_i\right)^2 \leq \left(Ext\text{-}\sum_{i=0}^{n} a_i^2\right)\left(Ext\text{-}\sum_{i=0}^{n} b_i^2\right). \tag{3.9}$$

4.External countable sum $Ext\text{-}\sum_{n\in\mathbb{N}} a_n$ from external hyperfinite sum.

Definition 4.1. Let $\{a_n\}_{n\in\mathbb{N}}$ be ${}^*\mathbb{R}$-valued countable sequence. Let $\{a_n\}_{n=1}^{m}$ be any ${}^*\mathbb{R}$-valued hyperfinite sequence with $m \in {}^*\mathbb{N}\backslash\mathbb{N}$ and such that $a_n = 0$ if $n \in \mathbb{N}^\#\backslash\mathbb{N}$. Then we define external sum of the countable sequence $\{a_n\}_{n\in\mathbb{N}}$ (or ω-sum) as the following hyperfinite sum

$$Ext\text{-}\sum_{n=1}^{m} a_n \in {}^*\mathbb{R} \tag{4.1}$$

and denote such sum by the symbol

$$Ext\text{-}\sum_{n\in\mathbb{N}} a_n \tag{4.2}$$

or by the symbol

$$Ext\text{-}\sum_{n=k}^{\omega} a_n. \tag{4.3}$$

Remark 4.1. Let $\{a_n\}_{n\in\mathbb{N}}$ be $\mathbb{R}$-valued countable sequence. Note that: (i) for canonical summation we always apply standard notation

$$\sum_{n=k}^{\infty} a_n. \tag{4.4}$$

(ii) the countable summ (ω-sum) (4.3) in contrast with (4.4) abviously always exists even if a series (4.3) diverges absolutely i.e., $\sum_{n=k}^{\infty}|a_n| = \infty$.

Definition 4.2.[5].(i) Let U be a free ultrafilters on $\mathbb{N}$ and introduce an equivalence relation on sequences in $\mathbb{R}^\mathbb{N}$ as $f_1 \sim_U f_2$ iff $\{i \in \mathbb{N} | f_1(i) = f_2(i)\} \in U$.
(ii) $\mathbb{R}^\mathbb{N}$ divided out by the equivalence relation $\sim_U$ gives us the nonstandard extension ${}^*\mathbb{R}$, the hyperreals; in symbols, ${}^*\mathbb{R} = \mathbb{R}^\mathbb{N}/\sim_U$ and similarly $\mathbb{N}^\mathbb{N}$ divided out by the equivalence relation $\sim_U$ gives us the nonstandard extension ${}^*\mathbb{N}$, the hyperintegers; in symbols,${}^*\mathbb{N} = \mathbb{N}^\mathbb{N}/\sim_U$.

Abbreviation 4.1.If $f \in \mathbb{R}^\mathbb{N}$, we denote its image in ${}^*\mathbb{R}$ by

$$\mathbf{cl}(f), \tag{4.5}$$

i.e., $\mathbf{cl}(f) = \{g \in \mathbb{R}^\mathbb{N} | g \sim_U f\}$.

Assumption 4.1.We assume now that there is an embedding ${}^*\mathbb{N} \hookrightarrow \mathbb{N}^\#$.

Remark 4.2.For any real number $r \in \mathbb{R}$ let $\mathbf{r}$ denote the constant function $\mathbf{r} : \mathbb{N} \to \mathbb{R}$ with value r, i.e.,$\mathbf{r}(n) = r$, for all $n \in \mathbb{N}$. We then have a natural embedding $* : \mathbb{R} \to {}^*\mathbb{R}$ by setting ${}^*r = \mathbf{cl}(\mathbf{r})$ for all $r \in \mathbb{R}$.

Example 4.1. Let ${}^*\mathbf{1}(n) : \mathbb{N} \to {}^*\mathbb{R}$ be the constant ${}^*\mathbb{R}$-valued function with value ^{*1}, i.e., ${}^*\mathbf{1}(n) = {}^*1$, for all $n \in \mathbb{N}$ and ${}^*\mathbf{1}(n) = {}^*0$, for all $n \in \mathbb{N}^\#\backslash\mathbb{N}$.
The ω-sum $Ext\text{-}\sum_{n\in\mathbb{N}} {}^*\mathbf{1}(n) \in {}^*\mathbb{R}\backslash\mathbb{R}$ exists by Theorem 3.1.
Let ${}^*\mathbf{1}^\#(n) : \mathbb{N} \to {}^*\mathbb{R}$ be the constant ${}^*\mathbb{R}$-valued function with value ^{*1}, i.e., ${}^*\mathbf{1}^\#(n) = {}^*1$, for all $n \in \mathbb{N}^\#$. The hyperfinite sum

$$Ext\text{-}\sum_{n=1}^{\nu} {}^*\mathbf{1}^\#(n) \in {}^*\mathbb{R}\backslash\mathbb{R}, \nu \in \mathbb{N}^\#\backslash\mathbb{N} \qquad (4.6)$$

exists for all $\nu \in \mathbb{N}^\#\backslash\mathbb{N}$ by Theorem 3.1.
We denote the value of ω-sum $Ext\text{-}\sum_{n\in\mathbb{N}} {}^*\mathbf{1}(n)$ by $\widehat{\omega}$. Note that

$$\widehat{\omega} \neq \mathbf{cl}(1,2,\ldots,n,\ldots) = \widetilde{\omega}, \qquad (4.7)$$

since $\widehat{\omega} = Ext\text{-}\sum_{n\in\mathbb{N}} {}^*\mathbf{1}(n) = Ext\text{-}\sum_{n=1}^{\widetilde{\omega}} {}^*\mathbf{1}(n) < Ext\text{-}\sum_{n=1}^{\widetilde{\omega}} {}^*\mathbf{1}^\#(n) = \widetilde{\omega}$. Note that the inequality $Ext\text{-}\sum_{n=1}^{\widetilde{\omega}} {}^*\mathbf{1}(n) < Ext\text{-}\sum_{n=1}^{\widetilde{\omega}} {}^*\mathbf{1}^\#(n)$ holds by Theorem 3.2.

Example 4.2. The ω-sum $Ext\text{-}\sum_{n=1}^{\omega} \frac{1}{n} \in {}^*\mathbb{R}\backslash\mathbb{R}$ exists by Theorem 3.1, however $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$.

Theorem 4.1. Let $Ext\text{-}\sum_{n=k}^{\omega} a_n = A$ and $Ext\text{-}\sum_{n=k}^{\omega} b_n = B$, where $A, B, C \in {}^*\mathbb{R}$. Then

(1)

$$Ext\text{-}\sum_{n=k}^{\omega} C \times a_n = C \times \left(Ext\text{-}\sum_{n=k}^{\omega} a_n \right) \qquad (4.8)$$

(2)

$$Ext\text{-}\sum_{n=k}^{\omega} (a_n \pm b_n) = A \pm B. \qquad (4.9)$$

(3)

$$Ext\text{-}\sum_{i=0}^{\omega} a_i = Ext\text{-}\sum_{i=0}^{j} a_i + Ext\text{-}\sum_{i=j+1}^{\omega} a_i \qquad (4.10)$$

Proof. It follows directly from Theorem 3.11.1 by Definition 3.12.1.

Example 4.2. Consider the ω-sum

$$S_\omega(r) = Ext\text{-}\sum_{n=0}^{\omega} r^n, -1 < r < 1. \qquad (4.11)$$

The ω-sum $Ext\text{-}\sum_{n=0}^{\omega} {}^*r^{-n} \in {}^*\mathbb{R}\backslash\mathbb{R}$ exists by Theorem 3.3.1. It follows from (3.12.11)

$$S_\omega(r) = 1 + Ext\text{-}\sum_{n=1}^{\omega} r^n = 1 + r\left(Ext\text{-}\sum_{n=0}^{\omega} r^n \right) = 1 + rS_\omega(r) \qquad (4.12)$$

Thus

$$S_\omega(r) \equiv \frac{1}{1-r}. \qquad (4.13)$$

Remark 4.3. Note that for $|r| < 1$

$$S_\omega(r) = Ext\text{-}\sum_{n=0}^{\omega} r^n = S_\infty(r) = \sum_{n=0}^{\infty} r^n \tag{4.14}$$

since as we know for $|r| < 1$

$$S_\infty(r) = \lim_{n\to\infty} \sum_{n=0}^{n} r^n = \sum_{n=0}^{\infty} r^n = \frac{1}{1-r}. \tag{4.15}$$

Definition 4.2.[5]. An element $x \in {}^*\mathbb{R}$ is called finite if $|x| < r$ for some $r \in \mathbb{Q}, r > 0$.
Abbreviation 3.12.2.For $x \in {}^*\mathbb{R}$ we abbreviate $x \in {}^*\mathbb{R}_{\mathbf{fin}}$ if x is finite.
Remark 4.4.[5]. Let $x \in \mathbb{Q}^\#$ be finite. Let $D_1,$ be the set of $r \in \mathbb{Q}$ such that $r < x$ and D_2 the set of $r' \in \mathbb{Q}$ such that $x < r'$. The pair (D_1, D_2) forms a Dedekind cut in $\mathbb{R}$, hence determines a unique $r_0 \in \mathbb{R}_d$. A simple argument shows that $|x - r_0|$ is infinitesimal,i.e., $|x - r_0| \approx 0$.
Definition 4.3.[5].This unique r_0 is called the standard part of x and is denoted by

$${}^\circ x. \tag{4.16}$$

Definition 4.4. An element $x \in {}^*\mathbb{R}_{\mathbf{fin}}$ is called standard if

$$x = {}^\circ x. \tag{4.17}$$

Abbreviation 4.2.For $x \in {}^*\mathbb{R}$ we abbreviate $x \in {}^*\mathbb{R}_{\mathbf{st}}$ if x is standard.
Theorem 4.4.[5]. If $x \in \mathbb{R}$, then ${}^\circ x = x$; if $x, y \in {}^*\mathbb{R}_{\mathbf{fin}}$ are both finite, then

$${}^\circ(x+y) = {}^\circ(x) + {}^\circ(y), {}^\circ(x-y) = {}^\circ(x) - {}^\circ(y). \tag{4.18}$$

Definition 4.5.Let $\{a_i\}_{i=0}^{\infty}$ be countable ${}^*\mathbb{R}_{\mathbf{fin}}$-valued sequence. We say that a sequence $\{a_i\}_{i=0}^{\infty}$ converges to the standard limit $a \in {}^*\mathbb{R}_{\mathbf{fin}}$ and abbreviate $a = \text{st-}\lim_{i\to\infty} a_i$ if for every $\epsilon > 0, \epsilon \not\approx 0$ there is an integer $N \in \mathbb{N}$ such that $|a_i - a| < \epsilon$ if $i \geq N$.
Theorem 4.5. Let $\{a_i\}_{i=0}^{n}$, $n \in \mathbb{N}^\# \backslash \mathbb{N}$ be a hyperfinite ${}^*\mathbb{R}_{\mathbf{fin}}$-valued sequence such that: (i) ${}^\circ a_i = a_i$ for any $i \leq n$ and (ii) for any $m \leq n$: $Ext\text{-}\sum_{i=0}^{m} |a_i| < \mu \in {}^*\mathbb{R}_{\mathbf{fin}}$, then

$${}^\circ\left(Ext\text{-}\sum_{i=0}^{n} a_i\right) = Ext\text{-}\sum_{i=0}^{n} a_i. \tag{4.19}$$

Proof. From Eq.(4.18) by the condition (ii) and hyper infinite induction we get

$${}^\circ\left(Ext\text{-}\sum_{i=0}^{n} a_i\right) = Ext\text{-}\sum_{i=0}^{n} {}^\circ a_i. \tag{4.20}$$

From Eq.(4.20) by the condition (i) we obtain Eq.(4.19).
Theorem 3.12.6. Let $\{a_i\}_{i\in\mathbb{N}}$ be a countable ${}^*\mathbb{R}_{\mathbf{st}}$-valued sequence, i.e., ${}^\circ a_i = a_i \in {}^*\mathbb{R}_{\mathbf{st}}$ for any $i \in \mathbb{N}$. Assume that: (i) $\sum_{i=0}^{\infty} |a_i| < \infty$ and therefore there exists $\text{st-}\lim_{m\to\infty} \sum_{i=0}^{m} a_i = \sum_{i=0}^{\infty} a_i$; (ii) $Ext\text{-}\sum_{i=0}^{\omega} |a_i| < \infty$ and (iii) $\text{st-}\lim_{k\to\infty} \sum_{i=k}^{\omega} |a_i| = 0$. Then

$${}^\circ\left(Ext\text{-}\sum_{i=0}^{\omega} a_i\right) \equiv Ext\text{-}\sum_{i=0}^{\omega} a_i. \tag{4.21}$$

and

$$Ext\text{-}\sum_{i=0}^{\omega} a_i = \sum_{i=0}^{\infty} a_i. \qquad (4.22)$$

From (4.22) it follows directly

$$\lim_{m\to\infty}\left(Ext\text{-}\sum_{i=m}^{\omega} a_i\right) = 0 \qquad (4.22')$$

Proof. The Eq.(4.21) follows directly from Eq.(4.19) and Definition 4.1.
From the Eq.(3.12.10) we get

$$Ext\text{-}\sum_{i=0}^{\omega} a_i - \sum_{i=0}^{k} a_i = Ext\text{-}\sum_{i=k}^{\omega} a_i. \qquad (4.23)$$

From the Eq.(3.12.23)

$$\left|Ext\text{-}\sum_{i=0}^{\omega} a_i - \sum_{i=0}^{k} a_i\right| = \left|Ext\text{-}\sum_{i=k}^{\omega} a_i\right| \leq \sum_{i=k}^{\omega}|a_i|. \qquad (4.24)$$

From the Eq.(3.12.24) by condition (ii) we get

$$\text{st-}\lim_{k\to\infty}\left|Ext\text{-}\sum_{i=0}^{\omega} a_i - \sum_{i=0}^{k} a_i\right| \leq \text{st-}\lim_{k\to\infty}\sum_{i=k}^{\omega}|a_i| = 0. \qquad (4.25)$$

It follows from the Eq.(3.12.25)

$$Ext\text{-}\sum_{i=0}^{\omega} a_i = \text{st-}\lim_{k\to\infty}\sum_{i=0}^{k} a_i = \sum_{i=0}^{\infty} a_i \qquad (4.26)$$

and therefore the equality (3.12.22) also holds. Assum that the equality (3.12.22) holds. Then from (3.12.22) one obtains for any $m \in \mathbb{N}$

$$Ext\text{-}\sum_{i=m}^{\omega} a_i = \sum_{i=m}^{\infty} a_i \qquad (4.26')$$

and therefore

$$\text{st-}\lim_{m\to\infty}\left(Ext\text{-}\sum_{i=m}^{\omega} a_i\right) = \text{st-}\lim_{m\to\infty}\sum_{i=m}^{\infty} a_i = 0.$$

Example 3.12.2. Let $\rho : \mathbb{N} \to {}^*\mathbb{R}$ be the ${}^*\mathbb{R}$-valued function such that $\rho(n) = {}^*r^n$, $|r| < 1$, for all $n \in \mathbb{N}$ and $\rho(n) = {}^*0$, for all $n \in \mathbb{N}^\#\backslash\mathbb{N}$. The ω-sum $S_\omega(r) = Ext\text{-}\sum_{n=0}^{\omega}{}^*r^n \in {}^*\mathbb{R}\backslash\mathbb{R}$ exists by Theorem 3.1 and by Theorem 4.6 we obtain $S_\omega(r) = S_\infty(r) = \sum_{n=0}^{\infty} r^n = (1-r)^{-1}$ the same result as obtained above by direct calculation (4.14), see Remark 4.3.

Remark 4.4. Note that in general case the conditions (i) $\sum_{i=0}^{\infty}|a_i| < \infty$ and (ii) $Ext\text{-}\sum_{i=0}^{\omega}|a_i| < \infty$ are not imply the condition (iii), but without condition (iii) the equality (4.22) obviously is not holds.

Theorem 4.7. Let $\{a_i\}_{i\in\mathbb{N}}$ be a countable ${}^*\mathbb{R}_{\mathbf{st}}$-valued sequence, i.e., ${}^\circ a_i = a_i \in {}^*\mathbb{R}_{\mathbf{st}}$ for any $i \in \mathbb{N}$. Assume that:(i) $a_i > 0$ for $i \geq m$ and (ii) st-$\overline{\lim}_{i\to\infty}\frac{a_{n+1}}{a_n} < {}^*1$. Then st-$\lim_{k\to\infty}\sum_{i=k}^{\omega}|a_i| = 0$ and therefore

$$Ext\text{-}\sum_{i=0}^{\omega} a_i = \sum_{i=0}^{\infty} a_i \,. \tag{4.27}$$

Proof. Note that if st-$\overline{\lim}_{i\to\infty}(a_{n+1}/a_n) < {}^*1$, there is a number $r \in {}^*\mathbb{R}_{\mathbf{st}}$ such that $0 < r < {}^*1$ and $a_{n+1}/a_n \leq r$ for $n \geq N$. Thus we obtain $a_{N+1} \leq ra_N, a_{N+2} \leq ra_{N+1} \leq r^2 a_N$, $\ldots,\ a_{N+k} \leq r^k a_N, \ldots$ and therefore

$$Ext\text{-}\sum_{i=N+k}^{\omega} a_i \leq Ext\text{-}\sum_{i=k}^{\omega} r^i a_N = r^k a_N\left(Ext\text{-}\sum_{i=0}^{\omega} r^i\right) = \frac{r^k a_N}{1-r}. \tag{4.28}$$

It follows from (4.22) st-$\lim_{k\to\infty}\left(Ext\text{-}\sum_{i=N+k}^{\omega} a_i\right)$ =st-$\lim_{k\to\infty}\frac{r^k a_N}{1-r} = 0$ and by Theorem 4.6 the equality (4.27) holds.

Theorem 4.8. Let $\{a_i\}_{i\in\mathbb{N}}$ be a countable ${}^*\mathbb{R}_{\mathbf{st}}$-valued sequence, i.e., ${}^\circ a_i = a_i \in {}^*\mathbb{R}_{\mathbf{st}}$ for any $i \in \mathbb{N}$. Assume that:(i) $a_i > 0$ for $i \geq m$ and (ii) st-$\overline{\lim}_{i\to\infty}\left(a_i^{1/i}\right) < {}^*1$. Then st-$\lim_{k\to\infty}\sum_{i=k}^{\omega}|a_i| = 0$ and therefore

$$Ext\text{-}\sum_{i=0}^{\omega} a_i = \sum_{i=0}^{\infty} a_i \,. \tag{4.29}$$

Theorem 4.9. Let $\{a_i\}_{i=1}^{n}$ and $\{b_i\}_{i=1}^{n}$ be ${}^*\mathbb{R}$- valued hyperfinite sequences such that $Ext\text{-}\sum_{i=1}^{n} a_i^2 = A \in {}^*\mathbb{R}_{\mathbf{fin}}$ and $Ext\text{-}\sum_{i=1}^{n} b_i^2 = B \in {}^*\mathbb{R}_{\mathbf{fin}}$. Then the following inequality holds

$$\left(Ext\text{-}\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(Ext\text{-}\sum_{i=1}^{n} a_i^2\right)\left(Ext\text{-}\sum_{i=1}^{n} b_i^2\right). \tag{4.30}$$

Proof. The inequality can be proved using only elementary algebra in this case. Consider
the following quadratic polynomial in $x \in {}^*\mathbb{R}$

$$0 \leq Ext\text{-}\sum_{i=1}^{n}(a_i x + b_i)^2 = \left(Ext\text{-}\sum_{i=1}^{n} a_i^2\right)x^2 + 2\left(Ext\text{-}\sum_{i=1}^{n} a_i b_i\right) + Ext\text{-}\sum_{i=1}^{n} b_i^2 \tag{4.31}$$

Since this polynomial is nonnegative, it has at most one real root for x, hence its discriminant is less than or equal to zero. That is,

$$\left(Ext\text{-}\sum_{i=1}^{n} a_i b_i\right)^2 - \left(Ext\text{-}\sum_{i=1}^{n} a_i^2\right)\left(Ext\text{-}\sum_{i=1}^{n} b_i^2\right) \leq 0. \tag{4.32}$$

which yields (4.30).

Theorem 4.10. Let $\{a_i\}_{i=1}^{\infty}$ and $\{b_i\}_{i=1}^{\infty}$ be ${}^*\mathbb{R}$- valued countable sequences such that $Ext\text{-}\sum_{i=1}^{\omega} a_i^2 = A \in {}^*\mathbb{R}_{\mathbf{fin}}$ and $Ext\text{-}\sum_{i=1}^{\omega} b_i^2 = B \in {}^*\mathbb{R}_{\mathbf{fin}}$. Then the following inequality holds

$$\left(Ext\text{-}\sum_{i=1}^{\omega} a_i b_i\right)^2 \leq \left(Ext\text{-}\sum_{i=1}^{\omega} a_i^2\right)\left(Ext\text{-}\sum_{i=1}^{\omega} b_i^2\right). \tag{4.33}$$

Proof.It follows from Theorem 2.9 by Definition 2.1.

5.External hyperfinite matrices and determinants

5.1. Definitions and notations

A rectangular external hyperfinite array of ordered elements wich is hyperreal numbers from external field $\mathbb{R}_c^{\#}$ or field $\mathbb{C}_c^{\#} = \mathbb{R}_c^{\#} + i\mathbb{R}_c^{\#}$, is known as hyperfinite $\mathbb{R}_c^{\#}$-valued (or $\mathbb{C}_c^{\#}$-valued) matrix.

The literal form of a hyperfinite external matrix in general is written symbolically as

$$\left\| \begin{matrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{matrix} \right\| \quad (5.1)$$

where $a_{ij} \in {}^*\mathbb{R}_c^{\#}; 1 \leq i \leq m, 1 \leq j \leq n; m \in \mathbb{N}^{\#} \backslash \mathbb{N}$.

We use boldface type to represent a matrix, and we enclose the array itself in square brackets. The horizontal lines are called rows and the vertical lines are called columns. Each element is associated with its location in the matrix. Thus the element a_{ij} is defined as the element located in the i-th row and the j-th column. Using this notation, we may also use the notation $[a_{ij}]_{m\times n}$ to identify a matrix of order $m \times n$, i.e. a matrix having m rows (the number of rows is given first) and n columns. Some frequently used

matrices have special names. A matrix of one column but any number of rows is known

as a column matrix or a column vector. Frequently, for such a matrix, only a single subscript is used for the elements of the array. Another type of matrix which is given a special name is one which contains only a single row. This is called a row matrix, or a row vector. A matrix which has the same number of rows and columns, i.e. $m = n$, is a square matrix of order $(n \times n)$ or just of order $n \in \mathbb{N}^{\#} \backslash \mathbb{N}$. The main or prin-ciple diagonal of a square matrix consists of the elements $a_{11}, a_{22}, \ldots, a_{nn}$. A square matrix in which all

elements except those of the principal diagonal are zero is known as a diagonal matrix.

If, in addition, all elements of a diagonal matrix are unity, the matrix is known as a unit or identity matrix, denotet by $\mathbf{U}$ or $\mathbf{1}$. If all elements of a matrix are zero, $a_{ij} = 0$, the matrix is called a zero matrix, $\mathbf{0}$. A subclass of a square matrix which is frequently encountered in circuit analysis is a symmetric matrix. The elements of such a matrix satisfy the equality $a_{ij} = a_{ji}$ for all values of i and j, or in other words, this matrix is symmetrical about the main diagonal.

Let $\mathbf{A}_{\omega} = [a_{ij}]$ be a countable matrix, where $a_{ij} \in {}^*\mathbb{R}_c^{\#}; i,j \in \mathbb{N}$. The literal form of a countable matrix in general is written symbolically as

$$\mathbf{A}_{\omega} = \left\| \begin{array}{ccccc} a_{11} & a_{12} & \cdots & a_{1n} & \cdots \\ a_{21} & a_{22} & \cdots & a_{2n} & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdots \\ a_{i1} & a_{i2} & \cdots & a_{ii} & \cdots \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} & \cdots \\ \cdot & \cdot & \cdots & \cdot & \cdots \end{array} \right\| \quad (5.2)$$

Remark 5.1. Note there is canonical embeding $\mathbf{A}_{\omega} \hookrightarrow \mathbf{A}_{\omega,n}$, where $\mathbf{A}_{\omega,n}$ is hyperfinite external matrix of the following literal form

$$\mathbf{A}_{\omega,n} = \left\| \begin{array}{cccccccc} a_{11} & a_{12} & \cdots & a_{1n} & \cdots & 0 & 0 & \cdots \\ a_{21} & a_{22} & \cdots & a_{2n} & \cdots & 0 & 0 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdots & 0 & 0 & \cdots \\ a_{i1} & a_{i2} & \cdots & a_{ii} & \cdots & 0 & 0 & \cdots \\ \cdot & \cdot & \cdots & \cdot & \cdots & 0 & 0 & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & \cdots & 0 & 0 & \cdots \\ \cdot & \cdot & \cdots & \cdot & \cdots & 0 & 1 & \cdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 1 \end{array} \right\| \quad (5.3)$$

where $a_{nn} = 1$ for all $n \in \mathbb{N}^{\#} \backslash \mathbb{N}$ and $a_{mn} = 0$ for all $m \neq n,\ m,n \in \mathbb{N}^{\#} \backslash \mathbb{N}$.

Matrix equality

Two matrixes are equal if and only if (1) they are of the same order, and (2) each element of one matrix is equal to its associated (placed in the row of the same number and the colunm of the same number) element in the other matrix. Thus, for two matrices,
$\mathbf{A}$ and $\mathbf{B}$, of the same order and with elements a_{ij} and b_{ij} respectively, if $\mathbf{A} = \mathbf{B}$, then all
the elements have to be equal, i.e. $a_{ij} = b_{ij}$ for all values of i and j.

5.2.Addition and subtraction of external hyperfinite matrices.

If two external hyperfinite matrices $\mathbf{A}$ and $\mathbf{B}$ are of the same order, i.e. have the same hyperfinite number of rows and the same hyperfinite number of columns, we may determine their sum by adding the corresponding elements. Thus if the elements of $\mathbf{A}$ are a_{ij} and those of $\mathbf{B}$ are b_{ij}, then the elements of the resulting matrix $\mathbf{C} = \mathbf{A} + \mathbf{B}$ are

$$c_{ij} = a_{ij} + b_{ij} \quad (5.4)$$

Clearly $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$ for hyperfinite matrices.Subtraction is similarly defined, i.e. $\mathbf{C} = \mathbf{A} - \mathbf{B}$ are

$$c_{ij} = a_{ij} - b_{ij}. \quad (5.5)$$

5.3. Multiplication by a scalar

The multiplication of external hyperfinite matrix $\mathbf{A}$ by a scalar $k \in \mathbb{R}_c^{\#}$ or $k \in \mathbb{C}_c^{\#}$ means that every element of the matrix $\mathbf{A}$ is multiplied by the scalar. Thus, if k is a scalar and $\mathbf{A}$ is external hyperfinite matrix with elements a_{ij}, the elements of the matrix $k\mathbf{A}$ are ka_{ij} :

$$k\mathbf{A} = \begin{bmatrix} ka_{11} & ka_{12} & \cdots & ka_{1n} \\ ka_{21} & ka_{22} & \cdots & ka_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ ka_{m1} & ka_{m2} & \cdots & ka_{mn} \end{bmatrix} \quad (5.6)$$

5.4.Multiplication of the external hyperfinite matrices.

For the case where $\mathbf{A}$ is an n-th-order square matrix and $\mathbf{Y}$ and $\mathbf{X}$ are column matrices with n rows, the elements of the resulting matrix $\mathbf{Y} = \mathbf{AX}$ is defined by the relation

$$y_i = Ext\text{-}\sum_{k=1}^{n} a_{ik}x_k, \quad (5.7)$$

where $1 \leq i \leq n$.

The multiplication of two external hyperfinite matrices $\mathbf{A}$ and $\mathbf{B}$ is defined only if the number of columns of $\mathbf{A}$ is equal to the number of rows of $\mathbf{B}$. If $\mathbf{A}$ is of order $(m \times n)$ and $\mathbf{B}$ is of order $(n \times p)$ (such a pair of matrices is said to be conform able for multiplication), then the product $\mathbf{A} \cdot \mathbf{B}$ is a matrix $\mathbf{C}$ of order $(m \times p)$

$$\mathbf{A}_{(m\times n)} \cdot \mathbf{B}_{(m\times p)} = \mathbf{C}_{(m\times p)} \quad (5.8)$$

The elements of $\mathbf{C}$ are found from the elements of $\mathbf{A}$ and $\mathbf{B}$ by multiplying the i-th row elements of $\mathbf{A}$ and the corresponding j-th column elements of $\mathbf{B}$ and summing these products to give c_{ij}

$$c_{ij} = Ext\text{-}\sum_{k=1}^{n} a_{ik}b_{kj}, \quad (5.9)$$

where $1 \leq i \leq m, 1 \leq j \leq p$.

5.5.The Determinant of the external hyperfinite matrices.

Suppose we are given a square hyperfinite matrix $\mathbf{A}$, i.e., an array of n^2 hyper real numbers

$$\mathbf{A} = \left\| \begin{matrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{m2} & \cdots & a_{nn} \end{matrix} \right\| \quad (5.10)$$

where $a_{ij} \in {}^*\mathbb{R}_c^{\#}; 1 \leq i \leq n, 1 \leq j \leq n, n \in \mathbb{N}^{\#}\backslash\mathbb{N}$. The number of rows and columns of the matrix (5.10) is called its order. The numbers a_{ij} are called the elements of the matrix. The first index indicates the row and the second index the column in which a_{ij} appears. The elements $a_{ii}, 1 \leq i \leq n$ form the principal diagonal of the matrix.

Consider any product of n elements which appear in different rows and different columns of the matrix (5.10), i.e., a product containing jast one element from each row and each column. Such external product can be written in the form

$$Ext\text{-}\prod_{m=1}^{n} a_{\alpha_m m} = Ext\text{-}(a_{\alpha_1 1} \times a_{\alpha_2 2} \times \ldots \times a_{\alpha_n n}). \qquad (5.11)$$

Actually, for the first factor we can always choose the element appearing in the first column of the matrix (5.10); then, if we denote by oq the number of the row in which the element appears, the indices of the element will be $\alpha_1, 1.$ Similarly, for the second factor we can choose the element appearing in the second column; then its indices will be $\alpha_2, 2$, where α_2 is the number of the row in which the element appears, and so on. Thus, the indices $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the numbers of the rows in which the factors of the product (5.11) appear, when we agree to write the column indices in increasing order.

Definition 5.1. A function F is said to be a permutation of a set S if it is one-to-one and $\mathbf{dom}(F) = \mathbf{range}(F) = S.$

Definition 5.2. Let $[1,n]$ a set $\{k|k \in \mathbb{N}^{\#} \wedge (1 \leq k \leq n)\}.$

Since, by hypothesis, the elements $a_{\alpha_1 1}, a_{\alpha_2 2}, \ldots, a_{\alpha_n n}$ appear in different rows of the matrix (5.10), one from each row, then the numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$ are all different and represent some permutation of the set $[1,n]$. By an inversion in the sequence $\{\alpha_i\}_{i=1}^{n}$, we mean an arrangement of two indices such that the larger index comes before the smaller index. The total number of inversions will be denoted by

$$\pi(\alpha_1, \alpha_2, \ldots, \alpha_n). \qquad (5.12)$$

If the number of inversions in the sequence $\{\alpha_i\}_{i=1}^{n}$ is even, we put a plus sign before the product (5.11); if the number is odd, we put a minus sign before the product.

In other words, we agree to write in front of each product of the form (5.11) the sign determined by the expression

$$(-1)^{\pi(\alpha_1, \alpha_2, \ldots, \alpha_n)}. \qquad (5.13)$$

The total number of products of the form (5.11) which can be formed from the elements of a given matrix of order n is equal to the total number of permutations of the set $[1,n]$. As is well known, this number is equal to $n!$.

Definition 5.3. By the determinant $\mathbf{D}$ of the matrix (4.5.1) is meant the external sum of the $n!$ products of the form (5.11), each preceded by the sign determined by the rule just given, i.e.,

$$\mathbf{D} = Ext\text{-}\sum (-1)^{\pi(\alpha_1, \alpha_2, \ldots, \alpha_n)} \left(Ext\text{-}\prod_{m=1}^{n} a_{\alpha_m m} \right) = \\ Ext\text{-}\sum (-1)^{\pi(\alpha_1, \alpha_2, \ldots, \alpha_n)} (Ext\text{-}(a_{\alpha_1 1} \times a_{\alpha_2 2} \times \ldots \times a_{\alpha_n n})). \qquad (5.14)$$

Henceforth, the products of the form (5.11) will be called the terms of the determinant $\mathbf{D}$. The elements a_{ij} of the matrix (5.10) will be called the elements of $\mathbf{D}$ and the order of (5.10) will be called the order of $\mathbf{D}$. We denote the determinant $\mathbf{D}$ corresponding to the matrix (5.10) by one of the following symbols:

$$\mathbf{D} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{m2} & \cdots & a_{nn} \end{vmatrix} = \det\|a_{ij}\| = |\mathbf{A}|. \qquad (5.15)$$

5.6.Determinant and Cofactors.

General procedure for evaluating the determinants of any order is by expanding determinant in terms of a row or column, which is called Laplaces' expansion. If such an expansion is made along the i-th row of an array, it has the following form

$$|\mathbf{A}| = Ext\text{-}\sum a_{ik}A_{ik}, \qquad (5.16)$$

where all a_{ik} are the elements of $\mathbf{A}$ and all A_{ik} are cofactors. These cofactors are formed by deleting the i-th row and k-th column of the array (so that the remaining elements form a determinant, called minor, $\mathbf{M}$, which is of order one less than $|\mathbf{A}|$) and prefixing the result by the multiplier $(-1)^{i+k}$, which predetermines the sign of the minor.

5.7.The transposition of the external hyperfinite matrix

Let $\mathbf{A}^t$ be a hyperfinite matrix

$$\mathbf{A}^t = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \cdot & \cdot & \cdot & \cdot \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{bmatrix} \qquad (5.17)$$

is obtained from a hyperfinite matrix (5.10) by interchanging rows an columns

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \qquad (5.18)$$

The determinant $|\mathbf{A}^t|$ obtained from the determinant $|\mathbf{A}|$ by interchanging rows and columns with the same indices is said to be the transpose of the determinant $|\mathbf{A}|$. We now show that the transpose of a determinant has the same value as the original determinant. In fact, the determinants $|\mathbf{A}|$ and $|\mathbf{A}^t|$ obviously consist of the same terms; therefore it is enough for us to show that identical terms in the determinants $|\mathbf{A}|$ and $|\mathbf{A}^t|$ have identical signs. Transposition of the matrix of a determinant is clearly the result of rotating it (in space) through 180° about the principal diagonal an, $a_{22},\ldots,a_{nn}$. As a result of this rotation, every segment with negative slope (e.g., making an angle $\alpha < 90^\circ$ with the rows of the matrix) again becomes a segment with negative slope (i.e., making the angle 90° - a with the rows of the matrix). Therefore the number of segments with negative slope joining the elements of a given term does not change after transposition.

Consequently the sign of the term does not change either. Thus the signs of all the terms are preserved, which means that the value of the determinant remains unchanged.

The property just proved establishes the equivalence of the rows and columns of a determinant. Therefore further properties of determinants will be stated and proved only for columns.

5.8. The antisymmetry property.

By the property of being antisymmetric with respect to columns, we mean the fact that a determinant changes sign when two of its columns are interchanged. We consider first the case where two adjacent columns are interchanged, for example columns j and $j+1$. The determinant which is obtained after these columns are interchanged obviously still consists of the same terms as the original determinant. Consider any of the terms of the original determinant. Such a term contains an element of the j-th column and an element of the $(j+1)$-th column. If the segment joining these two elements originally had negative slope, then after the interchange of columns, its slope becomes positive, and conversely. As for the other segments joining pairs of elements of the term in question, each of these segments does not change the character of its slope after the column interchange. Consequently the number of segments with negative slope joining the elements of the given term changes by one when the two columns are interchanged; therefore each term of the determinant, and hence the determinant itself, changes sign when the columns are interchanged.

Suppose now that two nonadjacent columns are interchanged, e.g., column j and column k with $j<k$, where there are hyper finitely many $m \in \mathbb{N}^{\#}\backslash\mathbb{N}$ other columns between. This interchange can be accomplished inductively by successive interchanges of adjacent columns as follows:

First column j is interchanged with column $j+1$, then with columns $j+2, j+3, \ldots, k$. Then the column $k-1$ so obtained (which was formerly column k) is interchanged with columns $k-2, k-3, \ldots, j$. In all, $m+1+m=2m+1$ interchanges of adjacent columns are required, each of which, according to what has just been proved, changes the sign of the determinant. Therefore, at the end of the process, the determinant will have a sign opposite to its original sign (since for any hyperinteger $m \in \mathbb{N}^{\#}\backslash\mathbb{N}$, the number $2m+1$ is odd).

Remark 5.1. Note that the process mention above is well defined by hyperfinite induction axiom [2]-[4].

Corollary 5.1. A hyperfinite determinant with two identical columns vanishes.

Proof. Interchanging the columns does not change the determinant $\mathbf{D}$. On the other hand, as just proved, the determinant must change its sign. Thus $\mathbf{D}=-\mathbf{D}$, which implies that $\mathbf{D}=0$.

5.9. The linear properties of determinant

This property can be formulated as follows:

Theorem 5.1.If all the elements of the j-th column of a determinant $\mathbf{D}$ are linear combinations of two columns of numbers, i.e., if

$$a_{ij} = b_i + c_i, 1 \leq i \leq n, \tag{5.19}$$

where $\lambda, \mu \in \mathbb{R}_c^{\#}$ or $\lambda, \mu \in \mathbb{C}_c^{\#}$ are fixed numbers, then D is equal to a linear combination of two determinants

$$\mathbf{D} = \lambda\mathbf{D}_1 + \mu\mathbf{D}_2 \tag{5.20}$$

Here both determinants $\mathbf{D}_1$ and $\mathbf{D}_2$ have the same columns as the determinant $\mathbf{D}$ except for the j-th column', the j-th column of $\mathbf{D}_1$ consists of the numbers b_i, while the j-th column of $\mathbf{D}_2$ consists of the numbers c_i.

Proof. Every term of the determinant D can be represented in the form

$$\begin{gathered}\mathbf{D} = \left(Ext\text{-}\prod_{i=1}^{j-1} a_{\alpha_i i}\right) a_{\alpha_j j} \left(Ext\text{-}\prod_{i=j+1}^{n} a_{\alpha_i i}\right) = \\ \left(Ext\text{-}\prod_{i=1}^{j-1} a_{\alpha_i i}\right)(\lambda b_{\alpha_j j} + \mu c_{\alpha_j j})\left(Ext\text{-}\prod_{i=j+1}^{n} a_{\alpha_i i}\right) = \\ \lambda\left(Ext\text{-}\prod_{i=1}^{j-1} a_{\alpha_i i}\right) b_{\alpha_j j}\left(Ext\text{-}\prod_{i=j+1}^{n} a_{\alpha_i i}\right) + \mu\left(Ext\text{-}\prod_{i=1}^{j-1} a_{\alpha_i i}\right) c_{\alpha_j j}\left(Ext\text{-}\prod_{i=j+1}^{n} a_{\alpha_i i}\right).\end{gathered} \tag{5.21}$$

Adding up all the first terms (with the signs which the corresponding terms have in the original determinant), we clearly obtain the determinant $\mathbf{D}_1$, multiplied by the number λ.

Similarly, adding up all the second terms, we obtain the determinant $\mathbf{D}_2$, multiplied by the number μ.

Remark 5.2.It is convenient to write this formula in a somewhat different form. Let $\mathbf{D}$ be an arbitrary fixed determinant. Denote by $\mathbf{D}_j(p_i)$ the determinant which is obtained by replacing the elements of the j-th column of $\mathbf{D}$ by the numbers $p_i, 1 \leq i \leq n \in \mathbb{N}^{\#}\backslash\mathbb{N}$. Then Eq.(5.21) takes the form

$$\mathbf{D}_j(\lambda b_i + \mu c_i) = \lambda\mathbf{D}_j(b_i) + \mu\mathbf{D}_j(c_i) \tag{5.22}$$

The linear property of determinants can be extended to the case where every element of the j-th column is a linear combination not of two terms but of any other number of terms, i.e.

$$a_{ij} = Ext\text{-}\sum_{k=1}^{r} \lambda_k b_i^k. \tag{5.23}$$

In this case

$$\mathbf{D}_j(a_{ij}) = \mathbf{D}_j\left(Ext\text{-}\sum_{k=1}^{r} \lambda_k b_i^k\right) = Ext\text{-}\sum_{k=1}^{r} \lambda_k \mathbf{D}_j(b_i^k). \tag{5.24}$$

Corollary 5.2. Any common factor of a column of a determinant can be factored out of the determinant.

Proof. If $a_{ij} = \lambda b_i$, then by (5.22) we have

$\mathbf{D}_j(a_{ij}) = \mathbf{D}_j(\lambda b_i) = \lambda \mathbf{D}_j(b_i)$.

Corollary 5.3. If a column of a determinant consists entirely of zeros, then the determinant vanishes.

Proof. Since 0 is a common factor of the elements of one of the columns, we can factor it out of the determinant, obtaining $D_j(0) = D_j(0 \cdot 1) = 0 \cdot D_j(1)$.

5.10.Addition of an arbitrary multiple of one column to another column.

Theorem 5.2. The value of a determinant is not changed by adding the elements of one column multiplied by an arbitrary number to the corresponding elements of another column.

Proof. Suppose we add the /cth column multiplied by the number λ to the j-th column $(\kappa \neq j)$. The j-th column of the resulting determinant consists of elements of the form $a_{ij} + \lambda a_{ik}, 1 \leq i \leq n$. By (5.22) we have $\mathbf{D}_j(a_{ij} + \lambda a_{ik}) = \mathbf{D}_j(a_{ij}) + \lambda \mathbf{D}_j(a_{ik})$.

The j-th column of the second determinant consists of the elements aik, and hence is identical with the /cth column. It follows from Corollary 3.8.1 that $\mathbf{D}_j(a_{ik}) = 0$, so that $\mathbf{D}_j(a_{ij} + \lambda a_{ik}) = \mathbf{D}_j(a_{ij})$.

Remark 5.3.Theorem 5.2 can be formulated in the following more general form: The value of a determinant is not changed by adding to the elements of its j-th column first the corresponding elements of the k-th column multiplied by λ, next the elements of the l-th column multiplied by μ, etc., and finally the elements of the p-th column multiplied by τ $(\kappa \neq j, l \neq j, \ldots, p \neq j)$.

Remark 5.4.Because of the invariance of determinants under transposition, all the properties of determinants proved above for columns remain valid for rows as well.

5.11.Cofactors and minors

Consider any column, the j-th say, of the determinant $\mathbf{D}$. Let a$_{ij}$ be any element of this column. Add up all the terms containing the element a_{ij} appearing in the right-hand side of equation (5.26),i.e.,

$$\mathbf{D} = Ext\text{-}\sum(-1)^{\pi(\alpha_1,\alpha_2,\ldots,\alpha_n)}\left(Ext\text{-}\prod_{m=1}^{n} a_{\alpha_m m}\right) = Ext\text{-}\sum(-1)^{\pi(\alpha_1,\alpha_2,\ldots,\alpha_n)}(Ext\text{-}(a_{\alpha_1 1} \times a_{\alpha_2 2} \times \ldots \times a_{\alpha_n n})). \quad (5.25)$$

and then factor out the element ais. The quantity which remains, denoted by A_{ij}, is called the cofactor of the element a_{ij} of the determinant $\mathbf{D}$.Since every term of the determinant $\mathbf{D}$ contains an element from the j-th column, (5.25) can be written in the form

$$Ext\text{-}\sum_{i=1}^{n} a_{ik}A_{ij} = Ext\text{-}\left(a_{1k}A_{1j} + a_{2k}A_{2j} + \ldots + a_{nk}A_{nj}\right) \quad (5.26)$$

called the expansion of the determinant $\mathbf{D}$ with respect to the (elements of the) j-th column. Naturally, we can write a similar formula for any row of the determinant $\mathbf{D}$. For example, for the ith row we have the formula

$$Ext\text{-}\sum_{j=1}^{n} a_{ij}A_{ij} = Ext\text{-}\left(a_{i1}A_{i1} + a_{i2}A_{i2} + \ldots + a_{in}A_{in}\right). \quad (5.27)$$

Thus one obtains.

Theorem 5.3. The sum of all the products of the elements of any column (or row) of the determinant $\mathbf{D}$ with the corresponding cofactors is equal to the determinant $\mathbf{D}$ itself.

Remark 5.5.Equations (5.26) and (5.27) can be used to calculate determinants, but first we must know how to calculate cofactors.

Remark 5.6.Next we note a consequence of (5.26) and (5.327 which will be useful later. Equation (5.26) is an identity in the quantities $a_{1j}, a_{2j}, \ldots, a_{nj}$. Therefore it remains valid if we replace a_{ij} $(1 \leq i \leq n)$ by any other quantities. The quantities $A_{1j}, A_{2j}, \ldots, A_{nj}$ remain unchanged when such a replacement is made, since they do not depend on the elements ais. Suppose that in the right and left-hand sides of the equality (5.26) we replace the elements $a_{1j}, a_{2j}, \ldots, a_{nj}$ by the corresponding elements of any other column, say the k-th. Then the determinant in the left-hand side of (5.26) will have two identical columns and will therefore vanish, according to Corollary 5.3. Thus one obtains the relation

$$Ext\text{-}\sum_{i=1}^{n} a_{ik}A_{ij} = Ext\text{-}\left(a_{1k}A_{1j} + a_{2k}A_{2j} + \ldots + a_{nk}A_{nj}\right) = 0 \tag{5.28}$$

for $k \neq j$. Similarly from Eq.(5.27) one obtains the relation

$$Ext\text{-}\sum_{j=1}^{n} a_{lj}A_{ij} = Ext\text{-}(a_{i1}A_{i1} + a_{i2}A_{i2} + \ldots + a_{ln}A_{in}) = 0 \tag{5.29}$$

for $l \neq i$. Thus one obtains the following.

Theorem 5.4. The sum of all the products of the elements of a column (or row) of the determinant $\mathbf{D}$ with the cofactors of the corresponding elements of another column (or row) is equal to zero.

Remark 5.7.If we delete a row and a column from a matrix of hyperfinite order n, then, of course, the remaining elements form a hyperfinite matrix of order $n-1$. The determinant of this matrix is called a minor of the original n-th-order matrix (and also a minor of its determinant $\mathbf{D}$).

If we delete the j-th row and the j-th column of $\mathbf{D}$, then the minor so obtained is denoted by $\mathbf{M}_{ij}$ or $\mathbf{M}_{ij}(\mathbf{D})$.

We now show that the relation

$$A_{ij} = (-1)^{i+j}\mathbf{M}_{ij} \tag{5.30}$$

holds, so that the calculation of cofactors reduces to the calculation of the corresponding minors. First we prove (4.11.6) for the case $i = 1, j = 1$. We add up all the terms in the right-hand side of (4.11.1) which contain the element a_{11}, and consider one of these terms. It is clear that the product of all the elements of this term except a_{11} gives a term c of the minor $\mathbf{M}_{11}$. Since in the matrix of the determinant $\mathbf{D}$, there are no segments of negative slope joining the element an with the other elements of the term selected, the sign ascribed to the term $a_{11}c$ of the determinant $\mathbf{D}$ is the same as the sign ascribed to the term c in the minor $\mathbf{M}_{11}$. Moreover, by suitably choosing a term of the determinant $\mathbf{D}$ containing a_{11} and then deleting a_{11}, we can obtain any term of the minor $\mathbf{M}_{11}$. Thus the algebraic hyperfinite external sum of all the terms of the determinant $\mathbf{D}$ containing a_{11}, with a_{11} deleted, equals the product $\mathbf{M}_{11}$.

But according to results obtained above, this sum is equal to the product A_{11}. Therefore, $A_{11} = \mathbf{M}_{11}$ as required.

Now we prove (4.11.6) by hyper infinite induction for arbitrary i and j, making essential use of the fact that the formula is valid for $i = j = 1$. Consider the element $a_{ii} = a$.

appearing in the i-th row and the j-th column of the determinant $\mathbf{D}$. By successively interchanging adjacent rows and columns, we can move the element a over to the upper left-hand corner of the matrix; to do this, we need $i-1+j-1=i+j-2$ hyper interchanges. As a result, we obtain the determinant $\mathbf{D}_1$ with the same terms as those of the original determinant $\mathbf{D}$ multiplied by $(-1)^{i+j-2}=(-1)^{i+j}$.
The minor $\mathbf{M}_{11}(\mathbf{D}_1)$ of the determinant $\mathbf{D}_1$ is clearly identical with the minor $\mathbf{M}_{ij}(\mathbf{D})$ of the determinant $\mathbf{D}$. By what has been proved already, the sum of the terms of the determinant $\mathbf{D}_1$ which contain the element a, with a deleted, is equal to $\mathbf{M}_{11}(\mathbf{D}_1)$. Therefore the sum of the terms of the original determinant $\mathbf{D}$ which contain the element $a_{ij}=a$, with a deleted, is equal to

$$(-1)^{i+j}\mathbf{M}_{11}(\mathbf{D}_1)=(-1)^{i+j}\mathbf{M}_{ij}(\mathbf{D}). \tag{4.11.7}$$

According to results obtained above, this sum is equal to A_{ij}. Consequently $A_{ij}=(-1)^{i+j}\mathbf{M}_{ij}$, which completes the proof of (5.30).

Theorem 5.5. Formulas (4.11.2) and (4.11.3) can now be written in the following form

$$\begin{gathered}\mathbf{D}=Ext\text{-}\sum_{k=1}^{n}Ext\text{-}(-1)^{k+j}a_{kj}\mathbf{M}_{kj}=\\ Ext\text{-}\left((-1)^{1+j}a_{1j}\mathbf{M}_{1j}+(-1)^{2+j}a_{2j}\mathbf{M}_{2j}+\ldots+(-1)^{n+j}a_{nj}\mathbf{M}_{nj}\right)\end{gathered} \tag{4.11.8}$$

and

$$\begin{gathered}\mathbf{D}=Ext\text{-}\sum_{k=1}^{n}(-1)^{i+k}a_{ik}\mathbf{M}_{ik}=\\ Ext\text{-}\left((-1)^{i+1}a_{i1}\mathbf{M}_{i1}+(-1)^{i+2}a_{i2}\mathbf{M}_{i2}+\ldots+(-1)^{i+n}a_{in}\mathbf{M}_{in}\right).\end{gathered} \tag{4.11.9}$$

Example 4.10.1. An hyperfinite n-th-order determinant

$$\mathbf{D}_n=\begin{vmatrix}a_{11} & 0 & 0 & \cdots & 0\\ a_{21} & a_{22} & 0 & \cdots & 0\\ a_{31} & a_{32} & a_{33} & \cdots & 0\\ \cdot & \cdot & \cdot & \cdots & \cdot\\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn}\end{vmatrix} \tag{4.11.10}$$

is called triangular. Expanding $\mathbf{D}_n$ with respect to the first row, we find that $\mathbf{D}_n$ equals the product of the element a_{11} with the triangular determinant

$$\mathbf{D}_{n-1}=\begin{vmatrix}a_{22} & 0 & 0 & \cdots & 0\\ a_{32} & a_{33} & 0 & \cdots & 0\\ \cdot & \cdot & \cdots & \cdots & \cdot\\ \cdot & \cdot & \cdots & \cdot & 0\\ a_{n2} & a_{n3} & \cdots & \cdots & a_{nn}\end{vmatrix} \tag{4.11.11}$$

of the order $n-1$. Again expanding $\mathbf{D}_{n-1}$with respect to the first row, we find that

$$\mathbf{D}_{n-1}=a_{22}\mathbf{D}_{n-2}, \tag{4.11.12}$$

where $\mathbf{D}_{n-2}$ is triangular determinant of the order $n-2$. By hyper infinite induction finally we obtain

$$\mathbf{D}_n = Ext\text{-}\prod_{i=1}^{n} a_{ii}. \qquad (4.11.13)$$

5.12.Generalized Cramer's Rule for hyperfinite system.

We are now can to solve external hyperfinite systems of linear equations.
First we consider hyperfinite system of the special form

$$\begin{aligned} &Ext\text{-}\sum_{i=1}^{n} a_{1i}x_i = b_1,\\ &Ext\text{-}\sum_{i=1}^{n} a_{2i}x_i = b_2,\\ &\ldots\ldots\ldots\ldots\ldots\\ &Ext\text{-}\sum_{i=1}^{n} a_{ni}x_i = b_n. \end{aligned} \qquad (4.12.1)$$

i.e., a system which has the same number of unknowns and equations. The coefficients a_{ij} $(i,j = 1,2,\ldots,n)$ form the coefficient matrix of the system; we assume that the determinant of this matrix is different from zero. We now show that such a system is always compatible and determinate, and we obtain a formula which gives the unique solution of the system.
We begin by assuming that $c_1, c_2, .\quad .., c_n$ is a solution of (4.12.1),so that

$$\begin{aligned} &Ext\text{-}\sum_{i=1}^{n} a_{1i}c_i = b_1,\\ &Ext\text{-}\sum_{i=1}^{n} a_{2i}c_i = b_2,\\ &\ldots\ldots\ldots\ldots\ldots\\ &Ext\text{-}\sum_{i=1}^{n} a_{ni}c_i = b_n. \end{aligned} \qquad (4.12.2)$$

We multiply the first of the equations (4.12.2) by the cofactor A_{11} of the element a_{11} in the coefficient matrix, then we multiply the second equation by A_{21}, the third by A_{31}, and so on, and finally the last equation by A_{n1}. Then we add all the equations so obtained. The result is

$$\begin{aligned} &Ext\text{-}(a_{11}A_{11} + a_{21}A_{21} + \ldots + a_{n1}A_{n1})c_1 + \\ &+Ext\text{-}(a_{12}A_{11} + a_{22}A_{21} + \ldots + a_{n2}A_{n1})c_2 + \ldots + \\ &+Ext\text{-}(a_{1n}A_{11} + a_{2n}A_{21} + \ldots + a_{nn}A_{n1})c_n = b_1A_{11} + b_2A_{21} + \ldots + b_nA_{n1}. \end{aligned} \qquad (4.12.3)$$

By Theorem 4.11.1, the coefficient of c_1 in (4.12.3) equals the determinant $\mathbf{D}$ itself.
By Theorem 4.11.2, the coefficients of all the other $c_j (j \neq 1)$ vanish. The expression in the right-hand side of (4.12.3) is the expansion of the determinant

$$\mathbf{D}_1 = \begin{vmatrix} b_1 & a_{12} & \cdots & a_{1n} \\ b_2 & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ b_n & a_{n2} & \cdots & a_{nn} \end{vmatrix} \qquad (4.12.4)$$

with respect to its first column. Therefore (19) can now be written in the form $\mathbf{D}c_1 = \mathbf{D}_1$, so that

$$c_1 = \frac{\mathbf{D}_1}{\mathbf{D}}. \qquad (3.12.5)$$

In a completely analogous way, we can obtain the expression

$$c_j = \frac{\mathbf{D}_j}{\mathbf{D}}, \qquad (4.12.6)$$

$1 \leq j \leq n$, where

$$\mathbf{D}_j = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1j-1} & b_1 & a_{1j+1} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j-1} & b_2 & a_{2j+1} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nj-1} & b_n & a_{nj+1} & \cdots & a_{nn} \end{vmatrix} \qquad (4.12.7)$$

is the determinant obtained from the determinant $\mathbf{D}$ by replacing its j-th column by the numbers $b_1 b_2, \ldots, b_n$. Thus we obtain the following result.

Theorem 4.12.1. If a solution of the system (4.12.1) exists, then (4.12.6) expresses the solution in terms of the coefficients of the system and the numbers in the right-hand side of (4.12.1). In particular, we find that if a solution of the system (4.12.3) exists, it is unique.

Remark 4.12.1.We must still show that a solution of the system (4.12.1) always exists. Consider the quantities $c_j = \mathbf{D}_j/\mathbf{D}, 1 \leq j \leq n$ and substitute them into the system (4.12.1) in place of the unknowns $x_1 x_2, \ldots, x_n$. Then this reduces all the equations of the system (4.12.1) to identities. In fact, for the i-th equation we obtain

$$\begin{aligned} &Ext\text{-}(a_{i1}c_1 + a_{i2}c_2 + \ldots + a_{in}c_n) = a_{i1}\frac{\mathbf{D}_1}{\mathbf{D}} + a_{i2}\frac{\mathbf{D}_2}{\mathbf{D}} + \ldots + a_{in}\frac{\mathbf{D}_n}{\mathbf{D}} = \\ &\mathbf{D}^{-1}[a_{i1}(Ext\text{-}(b_1A_{11} + b_2A_{21} + \ldots + b_nA_{n1})) + \\ &+ a_{i2}(Ext\text{-}(b_1A_{12} + b_2A_{22} + \ldots + b_nA_{n2})) + \ldots + \\ &+ a_{in}(Ext\text{-}(b_1A_{1n} + b_2A_{2n} + \ldots + b_nA_{nn}))] = \\ &= \mathbf{D}^{-1}[b_1(Ext\text{-}(a_{i1}A_{11} + a_{i2}A_{12} + \ldots + a_{in}A_{1n})) + \ldots + \\ &+ b_2(Ext\text{-}(a_{i1}A_{21} + a_{i2}A_{22} + \ldots + a_{in}A_{2n})) + \ldots + \\ &+ b_n(Ext\text{-}(a_{i1}A_{n1} + a_{i2}A_{n2} + \ldots + a_{in}A_{nn}))]. \end{aligned} \qquad (4.12.8)$$

By Theorems 4.11.1 and 4.11.2, only one of the coefficients of the quantities $b_1, b_2, \ldots, b_n$ is different from zero, namely the coefficient of b_i, which is equal to the determinant D itself. Consequently, the above expression reduces to

$$\mathbf{D}^{-1} b_i \mathbf{D} = b_i, \qquad (4.12.9)$$

i.e., is identical with the right-hand side of the i-th equation of the system.

Thus the quantities c_j $(1 \leq j \leq n)$ actually constitute a solution of the system (4.12.1), and we have found the following prescription (Generalized Cramer's rule) for obtaining solutions of hyperfinite system (4.12.1).

Theorem 4.12.2. If the determinant of the system (4.12.1) is different from zero, then (4.12.1) has a unique solution, namely, for the value of the unknown x_j $(1 \leq j \leq n)$ we take the fraction whose denominator is the determinant $\mathbf{D}$ of (4.12.1) and whose numerator is the determinant obtained by replacing the j-th column of $\mathbf{D}$ by the column consisting of the constant terms of (4.12.1),i.e., the numbers in the right-hand sides of the system.

Remark 4.12.2. One sometimes encounters systems of linear equations whose constant terms are not numbers but vectors, e.g., in analytic geometry or in mechanics.

Cramer's rule and its proof remain valid in this case as well; one must only bear in mind that the values of the unknowns $x_1, x_2, \ldots, x_n$ will then be vectors rather than numbers.

5.13.Minors of arbitrary hyperfinite order. Generalized Laplace's Theorem.

Theorem 4.11.3 on the expansion of a determinant with respect to a row or a column is a special case of a more general theorem on the expansion of a determinant with respect to a whole set of rows or columns. Before formulating this general theorem (Generalized Laplace's theorem), we introduce some new notation.

Suppose that in a square external matrix of hyperfinite order $n \in \mathbb{N}^{\#}/\mathbb{N}$ we specify any $k \leq n$ different rows and the same number of different columns. The elements appearing at the intersections of these rows and columns form a square matrix of hyperfinite order k. The determinant of this matrix is called a minor of order k of the original matrix of order n (also a minor of order k of the determinant $\mathbf{D}$); it is denoted by

$$\mathbf{M} = \mathbf{M}^{i_1 . i_2 .... i_k}_{j_1 . j_2 ... j_k}, \tag{4.13.1}$$

where $i_1, i_2, \ldots, i_k$, are the numbers of the deleted rows, and $j_1, j_2, \ldots, j_k$ are the numbers of the deleted columns.

If in the original matrix we delete the rows and columns which make up the minor $\mathbf{M}$, then the remaining elements again form a square matrix, this time of order $n - k$. The determinant of this matrix is called the complementary minor of the minor M, and is denoted by the symbol

$$\overline{\mathbf{M}} = \overline{\mathbf{M}}^{i_1 . i_2 .... i_k}_{j_1 . j_2 ... j_k}. \tag{4.13.2}$$

In particular, if the original minor is of order 1, i.e., is just some element a_{ij} of the determinant $\mathbf{D}$, then the complementary minor is the same as the minor $\mathbf{M}_{ij}$ discussed in Sec. .Consider now the minor

$$\mathbf{M}_1 = \mathbf{M}^{1,2,\ldots,k}_{1,2,\ldots,k} \tag{4.13.3}$$

formed from the first k rows and the first k columns of the determinant $\mathbf{D}$; its complementary minor is

$$\mathbf{M}_2 = \overline{\mathbf{M}}^{1,2,\ldots,k}_{11,2,\ldots,k}. \tag{4.13.4}$$

In the right-hand side of equation (4.5.5), put group together all the terms of the determinant whose first k elements belong to the minor $\mathbf{M}_1$ (and thus whose remaining $n-k$ elements belong to the minor $\mathbf{M}_2$). Let one of these terms be denoted by $c;$ we now wish to determine the sign which must be ascribed to c. The first k elements of c belong to a term $c_1,$ of the minor $\mathbf{M}_1$. If we denote by N_1 the number of segments of negative slope corresponding to these elements, then the sign which must be put in front of the term c_1 in the minor $\mathbf{M}_1$ is $(-1)^{N_1}$.The remaining $n-k$ elements of c belong to a term c_2 of the minor $\mathbf{M}_2$; the sign which must be put in front of this term in the minor $\mathbf{M}_2$ is $(-1)^{N_2}$, where N_2 is the number of segments of negative slope corresponding to the $n-k$ elements of c_2. Since in the matrix of the determinant $\mathbf{D}$ there is not a single segment with negative slope joining an element of the minor $\mathbf{M}_1$ with an element of the minor $\mathbf{M}_2$, the total number of segments of negative slope joining elements of the term c equals the sum $N_1+N_2.$ Therefore the sign which must be put in front of the term c is given by the expression $(-1)^{N_1+N_2}$, and hence is equal to the product of the signs of the terms c_1 and c_2 in the minors $\mathbf{M}_1$ and $\mathbf{M}_2$. Moreover, we note that the product of any term of the minor $\mathbf{M}_1$ and any term of the minor $\mathbf{M}$, gives us one of the terms of the determinant $\mathbf{D}$ that have been grouped together. It follows that the sum of all the terms that we have grouped together from the expression for the determinant $\mathbf{D}$ given by (4.5.5) is equal to the product of the minors $\mathbf{M}_1$ and $\mathbf{M}_2$.Next we solve the analogous problem for an arbitrary minor

$$\mathbf{M}_1=\mathbf{M}^{i_1,i_2,\ldots,i_k}_{j_1,j_2,\ldots,j_k} \tag{4.13.5}$$

with complementary minor $\mathbf{M}_2$. By successively interchanging adjacent rows and columns, we can move the minor $\mathbf{M}_1$ over to the upper left-hand comer of the determinant $\mathbf{D}$; to do so, we need a total of $Ext\text{-}\sum_{\alpha=1}^{k}(i_\alpha-\alpha)+Ext\text{-}\sum_{\alpha=1}^{k}(j_\alpha-\alpha)$ interchanges. As a result, we obtain a determinant $\mathbf{D}_1$ with the same terms as in the original determinant but multiplied by $(-1)^{i+j},$ where $i=Ext\text{-}\sum_{\alpha=1}^{k}(i_\alpha-\alpha),$ $j=Ext\text{-}\sum_{\alpha=1}^{k}(j_\alpha-\alpha)$ by what has just been proved, the sum of all the terms in the determinant $\mathbf{D}_1$ whose first k elemeflts appear in the minor $\mathbf{M}_1$ is equal to the product $\mathbf{M}_1\mathbf{M}_2$. It follows from this that the sum of the corresponding terms of the determinant $\mathbf{D}$ is equal to the product $(-1)^{i+j}\mathbf{M}_1\mathbf{M}_2=\mathbf{M}_1A_2,$where the quantity $A_2=(-1)^{i+j}\mathbf{M}_2$ is called the cofactor of the minor $\mathbf{M}_1$ in the determinant $\mathbf{D}$. Sometimes one uses the notation $A_2=\overline{A}^{i_1,i_2,\ldots,i_k}_{j_1,j_2,\ldots,j_k},$ where the indices indicate the numbers of the deleted rows and columns.Finally, let the rows of the determinant $\mathbf{D}$ with indices $i_1,i_2,\ldots,i_k$ be fixed; some elements from these rows appear in every term of $\mathbf{D}$. We group together all the terms of $\mathbf{D}$ such that the elements from the fixed rows $i_1,i_2,\ldots,i_k$ belong to the columns with indices $j_1,j_2,\ldots,j_k.$ Then, by what has just been proved, the sum of all these terms equals the product of the minor with the corresponding cofactor. In this way, all the terms of $\mathbf{D}$ can be divided into groups, each of which is characterized by specifying k columns. The sum of the terms in each group is equal to the product of the corresponding minor and its cofactor. Therefore the entire determinant can be represented as the sum

$$\mathbf{D}=Ext\text{-}\sum\mathbf{M}^{i_1,i_2,\ldots,i_k}_{j_1,j_2,\ldots,j_k}\overline{A}^{i_1,i_2,\ldots,i_k}_{j_1,j_2,\ldots,j_k}, \tag{4.13.6}$$

where the indices $i_1,i_2,\ldots,i_k$ (the indices selected above) are fixed, and the sum is over all possible values of the column indices $j_1,j_2,\ldots,j_k$ $(1<j_1<j_2<\ldots<j_k<n)$.

The expansion of $\mathbf{D}$ given by (4.13.6) is called Laplace's theorem. Clearly, Laplace's theorem constitutes a generalization of the formula for expanding a determinant with respect to one of its rows. There is an analogous formula for expanding the determinant

$\mathbf{D}$ with respect to a fixed set of columns.

4.14.Linear dependence between hyperfinite columns.

Suppose we are given m columns of hyperreal numbers with n numbers in each:

$$A_1 = \begin{Vmatrix} a_{11} \\ a_{21} \\ \cdot \\ \cdot \\ \cdot \\ a_{n1} \end{Vmatrix}, A_2 = \begin{Vmatrix} a_{12} \\ a_{22} \\ \cdot \\ \cdot \\ \cdot \\ a_{n2} \end{Vmatrix}, \ldots, A_m = \begin{Vmatrix} a_{1m} \\ a_{2m} \\ \cdot \\ \cdot \\ \cdot \\ a_{nm} \end{Vmatrix}. \qquad (4.14.1)$$

We multiply every element of the first column by some number λ_1, every element of the second column by λ_2, etc., and finally every element of the last (mth) column by λ_m; we then add corresponding elements of the columns.
As a result, we get a new column of numbers, whose elements we denote by $c_1, c_2, \ldots, c_n$. We can represent all these operations schematically as follows:

$$Ext\text{-}\left(\lambda_1 \begin{Vmatrix} a_{11} \\ a_{21} \\ \cdot \\ \cdot \\ \cdot \\ a_{n1} \end{Vmatrix} + \lambda_2 \begin{Vmatrix} a_{12} \\ a_{22} \\ \cdot \\ \cdot \\ \cdot \\ a_{n2} \end{Vmatrix} + \ldots + \lambda_m \begin{Vmatrix} a_{1m} \\ a_{2m} \\ \cdot \\ \cdot \\ \cdot \\ a_{nm} \end{Vmatrix} \right) = \begin{Vmatrix} c_1 \\ c_2 \\ \cdot \\ \cdot \\ \cdot \\ c_n \end{Vmatrix}, \qquad (4.14.2)$$

or more briefly as

$$Ext\text{-}\sum_{i=1}^{m} \lambda_i A_i = C, \qquad (4.14.3)$$

where C denotes the column whose elements are $c_1, c_2, \ldots, c_n \in {}^*\mathbb{R}_c^\#$. The column

C is called a linear combination of the columns $A_1, A_2, \ldots, A_m$, and the hyperreal numbers $\lambda_1, \lambda_2, \ldots, \lambda_m \in {}^*\mathbb{R}_c^\#$ are called the coefficients of the linear combination.
As special cases of the linear combination C, we have the sum of the columns if $\lambda_1 = \lambda_2 = \ldots = \lambda_m = 1$ and the product of a column by a number if $m = 1$.
Suppose now that our columns are not chosen independently, but rather make up a determinant $\mathbf{D}$ of order $n \in \mathbb{N}^\# / \mathbb{N}$. Then we have the following
Theorem 4.14.1. If one of the columns of the determinant $\mathbf{D}$ is a linear combination of the other columns, then $\mathbf{D} = 0$.
Proof. Suppose, for example, that the q-th column of the determinant $\mathbf{D}$ is a linear combination of the j-th, k-th, . . . , p-th columns of $\mathbf{D}$, with coefficients $\lambda_j, \lambda_k, \ldots, \lambda_p$, respectively. Then by subtracting from the q-th column first the j-th column multiplied by λ_j, then the k-th column multiplied by λ_k, etc., and finally the p-th column multiplied

by λ_p, we do not change the value of the determinant $\mathbf{D}$.
However, as a result, the q-th column consists of zeros only, from which it follows that $\mathbf{D} = 0$.

Remark 4.14.1.It is remarkable that the converse is also true, i.e., if a given determinant $\mathbf{D}$ is equal to zero, then (at least) one of its columns is a linear combination of the other columns. The proof of this theorem requires some preliminary considerations, to which we now turn.

Again suppose we have $m \in \mathbb{N}^{\#}\backslash\mathbb{N}$ columns of numbers with $n \in \mathbb{N}^{\#}\backslash\mathbb{N}$ elements in each.
We can write them in the form of a matrix

$$\mathbf{A} = \left\| \begin{matrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{matrix} \right\| \qquad (4.14.4)$$

with n rows and m columns. If k columns and k rows of this matrix are held fixed, then the elements appearing at the intersections of these columns and rows form a square matrix of order κ, whose determinant is a minor of order κ of the original matrix $\mathbf{A}$; this determinant may either be vanishing or nonvanishing. If, as we shall always assume, not all of the a_{ik} are zero, then we can always find an integer r which has the following two properties:

1.The matrix $\mathbf{A}$ has a minor of order r which does not vanish;
2. Every minor of the matrix $\mathbf{A}$ of order $r+1$ and higher (if such actually exist) vanishes.

Definition 4.14.1.The number r which has these properties is called the rank of the matrix A. If all the a_{ik} vanish, then the rank of the matrix $\mathbf{A}$ is considered to be zero ($r = 0$). Henceforth we shall assume that $r > 0$. The minor of order r which is different from zero is called the basis minor of the matrix $\mathbf{A}$. (Of course, $\mathbf{A}$ can have several basis minors, but they all have the same order r.) The columns which contain the basis minor are called the basis columns.

Concerning the basis columns, we have the following important

Theorem 4.14.2. (**Basis minor theorem**). Any column of the matrix A is a linear combination of its basis columns.

Proof. To be explicit, we assume that the basis minor of the matrix is located in the first r rows and first r columns of A. Let s be any integer from 1 to m, let κ be any integer from 1 to n, and consider the determinant

$$\mathbf{D} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1r} & a_{1s} \\ a_{21} & a_{22} & \cdots & a_{2r} & a_{2s} \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ a_{r1} & a_{r2} & \cdots & a_{rr} & a_{rs} \\ a_{k1} & a_{k2} & \cdots & a_{kr} & a_{ks} \end{vmatrix} \qquad (4.14.5)$$

of order $r+1$. If $k \leq n$, the determinant $\mathbf{D}$ is obviously zero, since it then has two identical rows. Similarly, $\mathbf{D} = 0$ for $s < r$. If $k > r$ and $s > r$, then the determinant $\mathbf{D}$ is also equal to zero, since it is then a minor of order $r + 1$ of a matrix of rank r. Consequently $\mathbf{D} = 0$ for any values of k and s. We now expand $\mathbf{D}$ with respect to its last row, obtaining the relation

$$Ext\text{-}(a_{k1}A_{k1} + a_{k2}A_{k2} + a_{kr}A_{kr}) + a_{ks}A_{ks} = 0, \qquad (4.14.6)$$

where the numbers $A_{k1}, A_{k2}, \ldots, A_{kr}, A_{ks}$ denote the cofactors of the elements $a_{k1}, a_{k2}, \ldots, a_{kr}, a_{ks}$ appearing in the last row of $\mathbf{D}$. These cofactors do not depend on the number k, since they are formed by using elements a_{ij} with $i < r$. Therefore we can introduce the notation

$$A_{k1} = c_1, A_{k2} = c_2, \ldots, A_{kr} = c_r, A_{ks} = c_s. \qquad (4.14.7)$$

Substituting the values $\kappa = 1, 2, \ldots, n$ in turn into (4.14.6), we obtain hyperfinite system of equations

$$\begin{aligned} &Ext\text{-}\sum_{j=1}^{r} c_j a_{1j} + c_s a_{1s} = Ext\text{-}(c_1 a_{11} + c_2 a_{12} + \ldots + c_r a_{1r}) + c_s a_{1s} = 0, \\ &Ext\text{-}\sum_{j=1}^{r} c_j a_{2j} + c_s a_{2s} = Ext\text{-}(c_1 a_{21} + c_2 a_{22} + \ldots + c_r a_{2r}) + c_s a_{2s} = 0, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &Ext\text{-}\sum_{i=1}^{r} c_j a_{nj} + c_s a_{ns} = Ext\text{-}(c_1 a_{n1} + c_2 a_{n2} + \ldots + c_r a_{nr}) + c_s a_{ns} = 0. \end{aligned} \qquad (4.14.8)$$

The number $c_s = A_{ks}$ is different from zero, since A_{ks} is a basis minor of the matrix A. Dividing each of the equations (4.14.8) by cs, transposing all the terms except the last to the right-hand side, and denoting $-c_j/c_s$ by λ_j $(1 \leq j \leq r)$, we obtain

$$\begin{aligned} &Ext\text{-}\sum_{j=1}^{r} \lambda_j a_{1j} = Ext\text{-}(\lambda_1 a_{11} + \lambda_2 a_{12} + \ldots + \lambda_r a_{1r}) = a_{1s}, \\ &Ext\text{-}\sum_{j=1}^{r} \lambda_j a_{2j} + c_s a_{2s} = Ext\text{-}(\lambda_1 a_{21} + \lambda_2 a_{22} + \ldots + \lambda_r a_{2r}) = a_{2s}, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &Ext\text{-}\sum_{j=1}^{r} \lambda_j a_{nj} + \lambda_s a_{ns} = Ext\text{-}(\lambda_1 a_{n1} + \lambda_2 a_{n2} + \ldots + \lambda_r a_{nr}) = a_{ns}. \end{aligned} \qquad (4.14.9)$$

These equations show that the s-th column of the matrix A is a linear combination of the first r columns of the matrix (with coefficients $\lambda_1, \lambda_2, \ldots, \lambda_r$). The proof of the theorem is now complete, since s can be any number such that $1 \leq s \leq m$.

Theorem 4.14.2. If the determinant $\mathbf{D}$ vanishes, then it has at least one column which is a linear combination of the other columns.

Proof. Consider the matrix of the determinant $\mathbf{D}$. Since $\mathbf{D} = 0$, the basis minor of this matrix is of order $r < n$. Therefore, after specifying the r basis columns, we can still find at least one column which is not one of the basis columns. By the basis minor theorem, this column is a linear combination of the basis columns. Thus we have found a column of the determinant $\mathbf{D}$ which is a linear combination of the other columns. Note that we can include all the remaining columns of the determinant $\mathbf{D}$ in

this linear combination by assigning them zero coefficients.
Remark 4.14.2.The results obtained above can be formulated in a somewhat more symmetric way. If the coefficients $\lambda_1,\lambda_2,\ldots,\lambda_m$ of a linear combination of m columns $A_1,A_2,\ldots,A_m$ are equal to zero, then obviously the linear combination is just the zero column, i.e., the column consisting entirely of zeros. But it may also be possible to obtain the zero column from the given columns by using coefficients $\lambda_1,\lambda_2,\ldots,\lambda_m$ which are not all equal to zero. In this case, the given columns $A_1,A_2,\ldots,A_m$ are called linearly dependent.
A more detailed statement of the definition of linear dependence is the following: The hyperfinite columns

$$A_1 = \left\| \begin{array}{c} a_{11} \\ a_{21} \\ \cdot \\ \cdot \\ \cdot \\ a_{n1} \end{array} \right\|, A_2 = \left\| \begin{array}{c} a_{12} \\ a_{22} \\ \cdot \\ \cdot \\ \cdot \\ a_{n2} \end{array} \right\|, \ldots, A_m = \left\| \begin{array}{c} a_{1m} \\ a_{2m} \\ \cdot \\ \cdot \\ \cdot \\ a_{nm} \end{array} \right\|. \qquad (4.14.10)$$

are called linearly dependent if there exist numbers $\lambda_1,\lambda_2,\ldots,\lambda_m$, not all equal to zero, such that the system of equation

$$\begin{gathered} Ext\text{-}\sum_{j=1}^{m} \lambda_j a_{1j} = Ext\text{-}(\lambda_1 a_{11} + \lambda_2 a_{12} + \ldots + \lambda_m a_{1m}) = 0, \\ Ext\text{-}\sum_{j=1}^{m} \lambda_j a_{2j} = Ext\text{-}(\lambda_1 a_{21} + \lambda_2 a_{22} + \ldots + \lambda_m a_{2m}) = 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ Ext\text{-}\sum_{j=1}^{m} \lambda_j a_{nj} = Ext\text{-}(\lambda_1 a_{n1} + \lambda_2 a_{n2} + \ldots + \lambda_r a_{nr}) = 0. \end{gathered} \qquad (4.14.11)$$

is satisfied, or equivalently such that

$$Ext\text{-}\sum_{i=1}^{m} \lambda_i A_i = \mathbf{0}, \qquad (4.14.12)$$

where the symbol $\mathbf{0}$ on the right-hand side denotes the zero column. If one of the columns $A_1,A_2,\ldots,A_m$, (e.g., the last column) is a linear combination of the others, i.e.,

$$A_m = Ext\text{-}\sum_{i=1}^{m-1} \lambda_i A_i. \qquad (4.14.13)$$

then the columns $A_1,A_2,\ldots,A_m$ are linearly dependent. In fact, (4.14.13) is equivalent to the relation

$$A_m - Ext\text{-}\sum_{i=1}^{m-1} \lambda_i A_i = 0 \qquad (4.14.14)$$

Consequently, there exists a linear combination of the columns $A_1,A_2,\ldots,A_m$, whose coefficients are not equal to zero (e.g., with the last coefficient equal to -1 whose sum is the zero column; this just means that the columns $A_1,A_2,\ldots,A_m$ are linearly

dependent.
Conversely, if the columns $A_1,A_2,\ldots,A_m$ are linearly dependent, then {at least) one of the columns is a linear combination of the other columns. In fact, suppose that in the relation

$$\lambda_m A_m + Ext\text{-}\sum_{i=1}^{m-1} \lambda_i A_i = 0 \tag{4.14.15}$$

expressing the linear dependence of the columns $A_1,A_2,\ldots,A_m$, the coefficient λ_m, say, is nonzero. Then (4.14.15) is equivalent to the relation

$$A_m = -\left(Ext\text{-}\sum_{i=1}^{m-1} \frac{\lambda_i}{\lambda_m} A_i\right). \tag{4.14.16}$$

Remark 4.14.3.Theorems 4.14.1 and 4.14.2 show that the determinant $\mathbf{D}$ vanishes if and only if one of its columns is a linear combination of the other columns. Using the results obtained above, we have the following.

Theorem 4.14.3. The determinant $\mathbf{D}$ vanishes if and only if there is linear dependence between its columns.

Remark 4.14.4.Since the value of a determinant does not change when it is transposed and since transposition changes columns to rows, we can change columns to rows in all the statements made above. In particular, the determinant $\mathbf{D}$ vanishes if and only if there is linear dependence between its rows.

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Conclusion

Though the history of infinitesimals and infinity is long and tortuous,nonstandard analysis, as a canonical formulation of the method of infinitesimals, is only about 60 years old. Hence, definitive answers for many of its methodological issues are yet to be found. In 1960, Abraham Robinson, exploiting the power of the theory of formal language reinvented the method of infinitesimals, which he called nonstandard analysis because it used nonstandard models of analysis. K. Hrbacek argue for acceptance of $BNST^+$ (Basic Nonstandard Set Theory plus additional Idealization axioms) [20]. $BNST^+$ has nontrivial consequences for standard set theory; for example, it implies existence of inner models with measurable cardinals.It has been proved in [21]-[22] that any set theory wich implies existence of inner models with measurable cardinals is inconsistent. However hyper Infinitary first-order logic ${}^2L^{\#}_{\infty^\#}$ with restricted rules of conclusions obviously can save $BNST^+$ from a triviality.

References

[1] A,Tarski, A ber Unerreichbare Kardinalzahlen. Fundamenta Mathematicae 30,68–89 (1938)

[2] C. E. Brown,K. Pąk, A Tale of Two Set Theories,In: Kaliszyk C., Brady E., Kohlhase A., Sacerdoti Coen C. (eds) Intelligent Computer Mathematics. CICM 2019. Lecture Notes in Computer Science, vol 11617. Springer, Cham. https://doi.org/10.1007/978-3-030-23250-4_4

[3] Trybulec, A.: Tarski Grothendieck set theory. J. Formalized Math. Axiomatics. Number 1, January 1990 Universit ´e Catholique de Louvain https://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.304.7824&rep=rep1&type=pdf#page=9

[4] M. Kaufmann,Set Theory With a Filter Quantifier, The Journal of Symbolic Logic Vol. 48, No. 2 (Jun., 1983), pp. 263-287, 25 pp.

[5] Foukzon, J. (2021). Set Theory $INC^{\#}$ Based on Intuitionistic Logic with Restricted Modus Ponens Rule (Part. I). Journal of Advances in Mathematics and Computer Science, 36(2), 73-88. https://doi.org/10.9734/jamcs/2021/v36i230339

[6] Foukzon, J. (2021). Set Theory $INC^{\#}_{\infty^{\#}}$ Based on Infinitary Intuitionistic Logic with Restricted Modus Ponens Rule (Part.II) Hyper Inductive Definitions. Journal of Advances in Mathematics and Computer Science, 36(4), 90-112. https://doi.org/10.9734/jamcs/2021/v36i430359

[7] Foukzon, J. (2021). Set Theory $INC^{\#}_{\infty^{\#}}$ Based on Innitary Intuitionistic Logic with Restricted Modus Ponens Rule. Hyper Inductive Denitions. Application in Transcendental Number Theory. Generalized Lindemann-Weierstrass Theorem. Journal of Advances in Mathematics and Computer Science, 36(8), 70-119. https://doi.org/10.9734/jamcs/2021/v36i830394

[8] Foukzon, J. (2022). Internal Set Theory $IST^{\#}$ Based on Hyper Infinitary Logic with Restricted Modus Ponens Rule: Nonconservative Extension of the Model Theoretical NSA. Journal of Advances in Mathematics and Computer Science, 37(7), 16-43. https://doi.org/10.9734/jamcs/2022/v37i730463

[9] Aczel, Peter, 1941. Non-well-founded sets http://www.irafs.org/courses/materials/aczel_set_theory.pdf https://papers.ssrn.com/sol3/papers.cfm?abstract_id=3989960

[10] S. Albeverio, J. E. Fenstad, R. Høegh-Krohn,Nonstandard Methods in Stochastic Analysis and Mathematical Physics (Dover Books on Mathematics) , February 26, 2009 Paperback : 526 pages ISBN-10 : 0486468992,ISBN-13 : 978-0486468990

[11] A.H.Lightstone, A.Robinson, Nonarchimedean Fields and Asymptotic Expansions,eBook ISBN: 9781483257440,1st Edition - January 1, 1975

[12] Foukzon, J, The Solution of the Invariant Subspace Problem.

Part I. Complex Hilbert Space (February 20, 2022). Available at SSRN: https://ssrn.com/abstract=4039068 or http://dx.doi.org/10.2139/ssrn.4039068