THE α-ANALOGUE r-WHITNEY NUMBERS VIA NORMAL ORDERING

ABSTRACT. The normal ordering of an integral power of the number operator $a^\dagger a$ in terms of boson annihilation a and creation operators $a^\dagger$ operators is expressed with the help of the Stirling numbers of the second kind. The normal ordering problems directly links the problems to combinatorics. In this paper, we note that the normal ordering of a certain quantity involving the number operator is expressed in terms of the analogues of r-Whitney numbers of the first and second kind, respectively. Moreover, we derived some interesting properties, recurrence relations and several identities on those numbers from such normal ordering.

1. INTRODUCTION

Dowling constructed Dowling lattice $Q_n(G)$ [1] for a finite group of order m using the Möbious function. He also introduced the Whitney numbers of the first and second kind, $V_m(n,k)$ and $W_m(n,k)$ $\quad (n \geq k \geq 0, m \geq 1)$ respectively which are independent if the group G itself, but depend only on its order m. For the trivial group, $V_1(n,k) = S_1(n+1,k+1)$ and $W_1(n,k) = S_2(n+1,k+1)$. Benoumhani [2, 3] studied some properties of the Whitney numbers of both kind. Mezö [4] introduced the r-Whitey numbers of the first and second kind, respectively. Since then, many scholars have derived interesting results about r-Whitney numbers [5-9]. Moreover, the translated Whitney numbers of the second kind which count the number of partitions of a set with n elements into k subsets such that the elements of each subset can mutate in α ways $(\alpha \in \mathbb{N})$, except the dominant one, were studied by Belbachir et al [10] and Mangontarum et al [11] as follows:

$$\widetilde{W}_{(\alpha)}(n,k) = \widetilde{W}_{(\alpha)}(n-1,k-1) + \alpha k \widetilde{W}_{(\alpha)}(n-1,k) \quad \text{and} \quad x^n = \sum_{k=0}^{n} \widetilde{W}_{(\alpha)}(n,k)(x)_{n,-\alpha}.$$

On the other hand, the normal ordering problems directly links the problems to combinatorics. Let a and $a^\dagger$ be the boson annihilation and creation operators satisfying the commutation relation $[a, a^\dagger] = aa^\dagger - a^\dagger a = 1$. Katriel [12] showed that the normal ordering of integral power of the number operator $a^\dagger a$ in terms of boson operators was expressed with the help of the Stirling numbers of the second kind. Recently, some physicists and mathematicians have studied the normal ordering problems (see [8, 12-22]). In addition, Kim-Kim-Kim studied some identities on λ-analogues of r-Stirling numbers of the second kind [23], some identities involving degenerate Stirling numbers arising from normal ordering [20] and degenerate r-Bell polynomials arising from degenerate normal ordering [21]. Kim et al [24] also studied the analogue of r-Stirling numbers via Boson operators.

In this paper, we consider the α-analogue r-Whitney numbers of the first kind and those of the second kind, respectively, different from the translated r-Whitney numbers. We derived some properties, recurrence relations and several identities on those numbers from such normal ordering.

We first introduce the definitions and properties needed this manuscript.

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For $\alpha \in \mathbb{R}$, the α-analogue of falling factorial sequence is defined by

(1)
$$(x)_{n,\alpha} = x(x-\alpha)(x-2\alpha)\cdots(x-(n-1)\alpha) \quad \text{and} \quad (x)_{0,\alpha} = 1, \qquad (n \geq 1), \qquad \text{(see [14, 25, 26])}.$$

For $\alpha \in \mathbb{R}$, the α-analogue of rising factorial sequence is defined by

(2)
$$\langle x \rangle_{n,\alpha} = x(x+\alpha)\cdots(x+(n-1)\alpha) \quad \text{and} \quad \langle x \rangle_{0,\alpha} = 1, \qquad (n \geq 1).$$

The Stirling numbers of the first kind are given by

(3)
$$(x)_n = \sum_{l=0}^{n} S_1(n,l)x^l \quad \text{and} \quad \frac{1}{k!}\big(\log(1+t)\big)^k = \sum_{n=k}^{\infty} S_1(n,k)\frac{t^n}{n!}, \qquad \text{(see [14, 17, 25-27])}.$$

For $r \in \mathbb{N} \cup \{0\}$, the α-analogues of r-Stirling numbers of the first kind are defined by

(4)
$$(x+r)_{n,\alpha} = \sum_{k=0}^{n} S_{1,\alpha}^{(r)}(n+r,k+r)x^k, \qquad (n \geq 0), \qquad \text{(see [17, 24])}.$$

When $\alpha \to 1$, we have the r-Stirling numbers of the first kind given by

$$(x+r)_n = \sum_{k=0}^{n} S_1^{(r)}(n+r,k+r)x^k, \qquad (n \geq 0), \qquad \text{(see [17, 21, 24, 26, 27])}.$$

The generating function of the α-analogues of r-Stirling numbers of the first kind is

(5)
$$(1+\alpha t)^{\frac{r}{\alpha}} \frac{1}{k!}\left(\frac{\log(1+\alpha t)}{\alpha}\right)^k = \sum_{n=k}^{\infty} S_{1,\alpha}^{(r)}(n+r,k+r)\frac{t^n}{n!}, \qquad \text{(see [17, 24])}.$$

When $r = 0$, we have the α-analogues of the Stirling numbers of the first kind as

(6)
$$(x)_{n,\alpha} = \sum_{k=0}^{n} S_{1,\alpha}(n,k)x^k \quad \text{and} \quad \frac{1}{k!}\left(\frac{\log(1+\alpha t)}{\alpha}\right)^k = \sum_{n=k}^{\infty} S_{1,\alpha}(n,k)\frac{t^n}{n!}, \qquad \text{(see [17, 24])}.$$

The Stirling numbers of the second kind $S_2(n,k)$ are given by

(7)
$$x^n = \sum_{l=0}^{n} S_2(n,l)(x)_l \quad \text{and} \quad \frac{1}{k!}(e^t-1)^k = \sum_{n=k}^{\infty} S_2(n,k)\frac{t^n}{n!}, \qquad \text{(see [14, 23, 26-28])}.$$

The ordinary Bell polynomials is given by

(8)
$$bel_n(x) = \sum_{k=0}^{n} S_2(n,k)x^k \quad \text{and} \quad \sum_{n=0}^{\infty} bel_n(x)\frac{t^n}{n!} = e^{x(e^t-1)} \qquad \text{(see [14, 26, 28])}.$$

The r-Stirling numbers of the second kind $S_2^{(r)}(n,k)$ are given by

(9)
$$e^{rt}\frac{1}{k!}(e^t-1)^k = \sum_{n=k}^{\infty} S_2^{(r)}(n,k)\frac{t^n}{n!}, \qquad \text{(see [14, 21, 23, 26-28])}.$$

The r-Bell polynomials is given by

(10)
$$\sum_{n=0}^{\infty} bel_n^{(r)}(x)\frac{t^n}{n!} = e^{x(e^t-1)+rt} \qquad \text{(see [14, 21, 26, 28])}.$$

As the inversion formula of (4), the α-analogues of r-Stirling numbers of the second kind are given by

(11)
$$(x+r)^n = \sum_{k=0}^{n} S_{2,\alpha}^{(r)}(n+r,k+r)(x)_{k,\alpha}, \qquad (n \geq 0), \qquad \text{(see [23])}$$

and the generating function as

$$\frac{1}{k!}\left(\frac{e^{\alpha t}-1}{\alpha}\right)^{k}e^{rt}=\sum_{n=k}^{\infty}S_{2,\alpha}^{(r)}(n+r,k+r)\frac{t^{n}}{n!},\qquad \text{(see [23])}. \tag{12}$$

When $r=0$, as the inversion formula of (6), the α-analogues of Stirling numbers of the second kind are given by

$$x^{n}=\sum_{k=0}^{n}S_{2,\alpha}(n,k)(x)_{k,\alpha}\quad\text{and}\quad \frac{1}{k!}\left(\frac{e^{\alpha t}-1}{\alpha}\right)^{k}=\sum_{n=k}^{\infty}S_{2,\alpha}(n,k)\frac{t^{n}}{n!},\quad (n\geq 0),\quad \text{(see [23])}. \tag{13}$$

When $\alpha\to 1$, we have the r-Stirling numbers of the second kind given by

$$(x+r)^{n}=\sum_{k=0}^{n}S_{2}^{(r)}(n+r,k+r)(x)_{k},\qquad (n\geq 0),\qquad \text{(see [21, 23, 27])}.$$

Kim-Kim [29] introduced the α-analogue differential operator given by

$$\left(x\frac{d}{dx}\right)_{k,\alpha}=\left(x\frac{x}{dx}\right)\left(x\frac{d}{dx}-\alpha\right)\cdots\left(x\frac{d}{dx}-(k-1)\alpha\right). \tag{14}$$

From (14), we have

$$\left(x\frac{d}{dx}\right)_{k,\alpha}x^{n}=(n)_{k,\alpha}x^{n},\quad \text{(see [29])}. \tag{15}$$

Let a and $a^{\dagger}$ be the boson annihilation and creation operators satisfying the commutation relation

$$[a,a^{\dagger}]=aa^{\dagger}-a^{\dagger}a=1,\qquad \text{(see [12, 13, 15, 16])}. \tag{16}$$

The number states $|l\rangle$, $l=1,2,\ldots$, are defined as

$$a|l\rangle=\sqrt{l}|l-1\rangle,\quad a^{\dagger}|l\rangle=\sqrt{l+1}|l+1\rangle,\qquad \text{(see [2, 5, 12, 24])}. \tag{17}$$

The normal ordering of an integral power of the number operator $a^{\dagger}a$ in terms of boson operators a and $a^{\dagger}$ can be written in the form

$$(a^{\dagger}a)^{k}=\sum_{l=0}^{k}S_{2}(k,l)(a^{\dagger})^{l}a^{l},\qquad \text{(see [12, 13, 15, 16, 24])}. \tag{18}$$

The coherent states $|z\rangle$, where z is a complex number, satisfy $a|z\rangle=z|z\rangle$, $z|z\rangle=1$. To show a connection to coherent states, we recall that the harmonic oscillator has Hamiltonian $\hat{n}=a^{\dagger}a$ (neglecting the zero point energy) and the usual eigenstates $|n\rangle$ (for $n\in\mathbb{N}$) satisfying $\hat{n}|n\rangle=n|n\rangle$ and $\langle l|n\rangle=\delta_{l,n}$, where $\delta_{l,n}$ is the Kronecker's symbol.

The evaluation of the expectation values of both sides of (17) with respect to the coherent state $|z\rangle$ yields

$$\langle z|(a^{\dagger}a)^{k}|z\rangle=\sum_{l=0}^{k}S_{2}(k,l)|z|^{2l},\qquad \text{(see [12, 15, 16, 24])}. \tag{19}$$

Since $[a,\,a^{\dagger}]=[\frac{d}{dx},\,x]=1$, we use both forms by identifying formally $a=\frac{d}{dx}$ and $a^{\dagger}=x$. When we consider the action of $(x\frac{d}{dx})^{n}$ on a polynomials $f(x)$, we have

$$\left(x\frac{d}{dx}\right)^{n}f(x)=\sum_{k=0}^{n}S_{2}(n,k)x^{k}\left(\frac{d}{dx}\right)^{k}f(x),\qquad \text{(see [12-15, 19, 20, 24])} \tag{20}$$

or, alternatively

$$(a^\dagger a)^n = \sum_{k=0}^{n} S_2(n,k)(a^\dagger)^k a^k, \qquad \text{(see [12-15, 19, 20, 24])}. \tag{21}$$

Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$. In view of (20), by (4) and (15), we observe that

$$(a^\dagger a + r)_{k,\alpha} = \sum_{j=0}^{k} S_{1,\alpha}^{(r)}(k+r, j+r)(a^\dagger a)^j, \quad \text{(see [24])}. \tag{22}$$

When $r = 0$, we have

$$(a^\dagger a)_{k,\alpha} = \sum_{j=0}^{k} S_{1,\alpha}(k,j)(a^\dagger a)^j, \quad \text{(see [24])}. \tag{23}$$

To obtain the inverse formula of (11), from (15), we observe that

$$(a^\dagger a + r)^k = \sum_{j=0}^{k} S_{2,\alpha}^{(r)}(k+r, j+r)(a^\dagger a)_{j,\alpha}, \quad \text{(see [24])}. \tag{24}$$

Form (23) and (24), we get

$$(a^\dagger a + r)^k = \sum_{j=0}^{k}\sum_{l=0}^{j} S_{2,\alpha}^{(r)}(k+r, j+r) S_{1,\alpha}(j,l)(a^\dagger a)^l, \quad \text{(see [24])}. \tag{25}$$

When $r = 0$, from (11) and (15), we have the inverse formula of (23) as follows:

$$(a^\dagger a)^k = \sum_{j=0}^{k} S_{2,\alpha}(k,j)(a^\dagger a)_{j,\alpha}, \quad \text{(see [24])}. \tag{26}$$

When $r = 0$, we note that

$$(a^\dagger a)^k = \sum_{j=0}^{k}\sum_{l=0}^{j} S_{2,\alpha}(k,j) S_{1,\alpha}(j,l)(a^\dagger a)^l.$$

For the Dowling lattice $Q_n(G)$ of rank n over a finite group G of order m, Dowling [11] defined the Whitney numbers $\omega_m(n,k)$ and $W_m(n,k)$ of the first and second kind, respectively. Dowling showed the following Stirling relations:

$$m^n \left(\frac{x-1}{m}\right)_n = \sum_{k=0}^{n} V_m(n,k) x^k \quad \text{and} \quad x^n = \sum_{k=0}^{n} W_m(n,k) m^k \left(\frac{x-1}{m}\right)_k. \tag{27}$$

Equivalently, we have

$$m^n (x)_n = \sum_{k=0}^{n} V_m(n,k)(mx+1)^k$$
$$\text{and} \quad (mx+1)^n = \sum_{k=0}^{n} W_m(n,k) m^k (x)_k, \quad \text{(see [1, 3-5])}. \tag{28}$$

For $r \in \mathbb{N}$, the r-Whitney numbers of the first kind $\omega_m^{(r)}(n,k)$ and those of the second kind $W_m^{(r)}(n,k)$, respectively defined by

$$m^n (x)_n = \sum_{k=0}^{n} V_m^{(r)}(n,k)(mx+r)^k, \qquad \text{(see [1, 3-5])} \tag{29}$$

and

(30) $$(mx+r)^n = \sum_{k=0}^{n} m^k W_m^{(r)}(n,k)(x)_k, \qquad (n \geq 0), \qquad \text{(see [1, 3-5])}.$$

2. λ-ANALOGUES OF r-WHITNEY NUMBERS OF THE FIRST AND SECOND KIND VIA NORMAL ORDERING

First, we define by the α-analogues of r-Whitney numbers of the first kind $V_{m,\alpha}^{(r)}(n,k)$ and those $W_{m,\alpha}^{(r)}(n,k)$, respectively.

(31) $$m^n (x)_{n,\alpha} = \sum_{k=0}^{n} V_m^{(r)}(n,k)(mx+r)^k$$

and

(32) $$(mx+r)^n = \sum_{k=0}^{n} W_m^{(r)}(n,k) m^k (x)_{k,\alpha}, \qquad (n \geq 0).$$

Note that

$$\lim_{\alpha \to 1} V_{m,\alpha}^{(r)}(n,k) = V_m^{(r)}(n,k) \quad \text{and} \quad \lim_{\alpha \to 1} W_{m,\alpha}^{(r)}(n,k) = W_m^{(r)}(n,k).$$

Proposition 1. [17] *For $n,k \in \mathbb{N} \cup \{0\}$ with $n \geq k$, we have the generating function of the λ-analogue r-Whitney of the second kind as*

$$\sum_{n=k}^{\infty} W_{m,\alpha}^{(r)}(n,k) \frac{t^n}{n!} = e^{rt} \frac{1}{k!} \frac{1}{\alpha^k} \left(\frac{e^{\alpha mt} - 1}{m} \right)^k.$$

Proposition 2. *For $n,k \in \mathbb{N} \cup \{0\}$ with $n \geq k$, we have the generating function of the λ-analogue r-Whitney of the first kind as*

$$e_{\alpha m}^{-r}(t) \frac{1}{k!} \frac{1}{\alpha^k} \left(\frac{\log(1+\alpha mt)}{m} \right)^k = \sum_{n=k}^{\infty} (-1)^{n-k} V_{m,\alpha}^{(r)}(n,k) \frac{t^n}{n!},$$

where $e_{\alpha m}^{-r}(t) = (1+\alpha mt)^{-\frac{r}{\alpha m}}$.

Proof. From (31), we observe that

$$\begin{aligned}
\sum_{k=0}^{\infty}\left\{\sum_{n=k}^{\infty}V_{m,\alpha}^{(r)}(n,k)\frac{t^n}{n!}\right\}(mx+r)^k &= \sum_{n=0}^{\infty}\left\{\sum_{k=0}^{n}V_{m,\alpha}^{(r)}(n,k)(mx+r)^k\right\}\frac{t^n}{n!}\\
&= \sum_{n=0}^{\infty}m^n(x)_{n,\alpha}\frac{t^n}{n!} = (1+\alpha mt)^{\frac{x}{\alpha}}\\
&= (1+\alpha mt)^{\frac{mx+r}{m\alpha}-\frac{r}{m\alpha}}\\
&= (1+\alpha mt)^{-\frac{r}{\alpha m}}\exp\left(\frac{mx+r}{m\alpha}\log(1+\alpha mt)\right)\\
&= e_{\alpha m}^{-r}(t)\sum_{k=0}^{\infty}\left(\frac{mx+r}{m\alpha}\right)^k\big(\log(1+\alpha mt)\big)^k\frac{1}{k!}\\
&= e_{\alpha m}^{-r}(t)\sum_{k=0}^{\infty}\frac{1}{k!}\frac{1}{\alpha^k}\left(\frac{\log(1+\alpha mt)}{m}\right)^k(mx+r)^k.
\end{aligned} \tag{33}$$

By comparing of the coefficients of both sides of (33), we have the desire the generating function. □

When $m=1$, for $r\geq 1$, by (6), (13), Proposition 1 and Proposition 2, we note that

$$V_{1,\alpha}^{(r)}(n,k) = S_{1,\alpha}^{(-r)}(n+r,k+r) \quad \text{and} \quad W_{1,\alpha}^{(r)}(n,k) = S_{2,\alpha}^{(r)}(n+r,k+r).$$

Theorem 3. *For $n,j\in\mathbb{N}\cup\{0\}$ with $n\geq j$, we have*

$$m^n(a^\dagger a)_{n,\alpha} = \sum_{j=0}^{n}V_{m,\alpha}^{(r)}(n,j)(ma^\dagger a+r)^j$$

and

$$(ma^\dagger a+r)^n = \sum_{j=0}^{n}W_m^{(r)}(n,j)m^j(a^\dagger a)_{j,\alpha}.$$

Proof. From (31) and (15), we note that

$$\begin{aligned}
m^k\left(x\frac{d}{dx}\right)_{k,\alpha}f(x) &= \sum_{n=0}^{\infty}a_n m^k\left(x\frac{d}{dx}\right)_{k,\alpha}x^n = \sum_{n=0}^{\infty}a_n m^k(n)_{k,\alpha}x^n\\
&= \sum_{n=0}^{\infty}a_n\sum_{j=0}^{k}V_{m,\alpha}^{(r)}(k,j)(mn+r)^j x^n\\
&= \sum_{j=0}^{k}V_{m,\alpha}^{(r)}(k,j)\left(mx\frac{d}{dx}+r\right)^j f(x).
\end{aligned} \tag{34}$$

Thus, by (34), we have

$$m^k(a^\dagger a)_{k,\alpha} = \sum_{j=0}^{k}V_{m,\alpha}^{(r)}(k,j)(ma^\dagger a+r)^j. \tag{35}$$

To obtain the inverse formula of (35), from (32) and (15), we observe that

$$
\begin{aligned}
\left(mx\frac{d}{dx}+r\right)^k f(x) &= \left(mx\frac{d}{dx}+r\right)^k \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n (mn+r)^k x^n \\
&= \sum_{n=0}^{\infty} a_n \sum_{j=0}^{k} W_m^{(r)}(k,j) m^j (n)_{j,\alpha} x^n \\
&= \sum_{j=0}^{k} W_m^{(r)}(k,j) m^j \left(x\frac{d}{dx}\right)_{j,\alpha} f(x).
\end{aligned} \tag{36}
$$

By (36), we get

$$
(ma^\dagger a + r)^k = \sum_{j=0}^{k} W_m^{(r)}(k,j) m^j (a^\dagger a)_{j,\alpha}. \tag{37}
$$

□

Theorem 4. *For $n, j \in \mathbb{N} \cup \{0\}$ with $n \geq j$, we have*

$$
V_{m,\alpha}^{(0)}(n,j) = m^{n-j} S_{1,\alpha}(n,j) \quad \textit{and} \quad W_{m,\alpha}^{(0)}(n,j) = m^{n-j} S_{2,\alpha}(n,j).
$$

Proof. From (23) and (35), we get

$$
\begin{aligned}
\sum_{j=0}^{n} V_{m,\alpha}^{(0)}(n,j)(ma^\dagger a)^j &= m^n (a^\dagger a)_{n,\alpha} \\
&= m^n \sum_{j=0}^{n} S_{1,\alpha}(n,j)(a^\dagger a)^j = \sum_{j=0}^{n} S_{1,\alpha}(n,j) m^{n-j} (ma^\dagger a)^j.
\end{aligned} \tag{38}
$$

By comparing the coefficients of both side of (38), we have

$$
V_{m,\alpha}^{(0)}(n,j) = m^{n-j} S_{1,\alpha}(n,j). \tag{39}
$$

In addition, from (26) and (37), we get

$$
\begin{aligned}
\sum_{j=0}^{n} W_{m,\alpha}^{(0)}(n,j) m^j (a^\dagger a)_{j,\alpha} &= (ma^\dagger a)^n = m^n (a^\dagger a)^n \\
&= m^n \sum_{j=0}^{n} S_{2,\alpha}(n,j)(a^\dagger a)_{j,\alpha}.
\end{aligned} \tag{40}
$$

By comparing the coefficients of both side of (40), we have

$$
W_{m,\alpha}^{(0)}(n,j) = m^{n-j} S_{2,\alpha}(n,j). \tag{41}
$$

□

For $r \in \mathbb{N} \cup \{0\}$, the unsigned r-Stirling numbers of the first kind are given by

$$
\langle x+r \rangle_n = \sum_{k=0}^{n} \begin{bmatrix} n+r \\ k+r \end{bmatrix}_r x^k, \quad (n \geq 0), \quad (\text{see } [9, 25]). \tag{42}
$$

From (42), we consider the λ-analogue unsigned r-Stirling numbers defined by

$$
\langle x+r \rangle_{n,\alpha} = \sum_{k=0}^{n} \begin{bmatrix} n+r \\ k+r \end{bmatrix}_{r,\alpha} x^k, \quad (n \geq 0). \tag{43}
$$

From (43), we observe that

$$\begin{aligned}(x-r)_{n,\alpha} &= (-1)^n \langle -x+r\rangle_{n,\alpha} = (-1)^n \sum_{k=0}^{n} \begin{bmatrix} n+r \\ k+r \end{bmatrix}_{r,\alpha} (-x)^k \\ &= \sum_{k=0}^{n} (-1)^{n-k} \begin{bmatrix} n+r \\ k+r \end{bmatrix}_{r,\alpha} x^k.\end{aligned} \tag{44}$$

By replacing x by $x+r$ in (44), we have

$$(x)_{n,\alpha} = \sum_{k=0}^{n} (-1)^{n-k} \begin{bmatrix} n+r \\ k+r \end{bmatrix}_{r,\alpha} (x+r)^k. \tag{45}$$

For any polynomial $f(x) = \sum_{n=0}^{\infty} a_n x_n$ by (15) and (45), we get

$$\left(x\frac{d}{dx}\right)_{k,\alpha} f(x) = \sum_{j=0}^{k} (-1)^{k-j} \begin{bmatrix} k+r \\ j+r \end{bmatrix}_{r,\alpha} \left(x\frac{d}{dx}+r\right)^k \tag{46}$$

or equivalently,

$$(a^\dagger a)_{k,\alpha} = \sum_{j=0}^{k} (-1)^{k-j} \begin{bmatrix} k+r \\ j+r \end{bmatrix}_{r,\alpha} (a^\dagger a+r)^k. \tag{47}$$

Theorem 5. *For $n,\ j \geq 0$ with $n \geq j$, we have*

$$V_{1,\alpha}^{(r)}(n,j) = \begin{bmatrix} n+r \\ j+r \end{bmatrix}_{r,\alpha} \quad \textit{and} \quad W_{1,\alpha}^{(r)}(n,j) = S_{2,\alpha}^{(r)}(n+r,j+r).$$

Proof. By Theorem 5, (24) and (47), we have

$$\sum_{j=0}^{n} V_{1,\alpha}^{(r)}(n,j)(a^\dagger a+r)^j = (a^\dagger a)_{n,\alpha} = \sum_{j=0}^{n} (-1)^{n-j} \begin{bmatrix} n+r \\ j+r \end{bmatrix}_{r,\alpha} (a^\dagger a+r)^n \tag{48}$$

and

$$\sum_{j=0}^{n} W_1^{(r)}(n,j)(a^\dagger a)_{j,\alpha} = (a^\dagger a+r)^n = \sum_{j=0}^{n} S_{2,\alpha}^{(r)}(n+r,j+r)(a^\dagger a)_{j,\alpha}. \tag{49}$$

By comparing the coefficients of both sides of (48) and (49), respectively, we have the desired result.

□

Theorem 6. *For $n,\ j \geq 0$ with $n \geq j$, we have*

$$V_{m,\alpha}^{(r)}(n,j) = \sum_{l=j}^{n} \binom{l}{j} m^{n-l} (-1)^{n+l} r^{l-j} S_{1,\alpha}(n,l)$$

and

$$W_{m,\alpha}^{(r)}(n,j) = \sum_{l=j}^{n} \binom{n}{l} m^{l-j} r^{n-l} S_{2,\alpha}(l,j).$$

Proof. From (22) and (35), we have

$$\begin{aligned}
\sum_{j=0}^{n} V_{m,\alpha}^{(r)}(n,j)(ma^\dagger a+r)^j &= m^n (a^\dagger a)_{n,\alpha} \\
&= m^n \sum_{l=0}^{n} S_{1,\alpha}(n,l)(a^\dagger a)^l \\
&= m^n \sum_{l=0}^{n} S_{1,\alpha}(n,l)\frac{1}{m^l}(ma^\dagger a+r-r)^l \\
&= m^n \sum_{l=0}^{n} S_{1,\alpha}(n,l)\frac{1}{m^l}\sum_{j=0}^{l}\binom{l}{j}(ma^\dagger a+r)^j(-r)^{l-j} \\
&= \sum_{j=0}^{n}\left(\sum_{l=j}^{n}\binom{l}{j}m^{n-l}(-r)^{l-j}S_{1,\alpha}(n,l)\right)(ma^\dagger a+r)^j.
\end{aligned} \tag{50}$$

By comparing the coefficients of both sides of (50), we have

$$V_{m,\alpha}^{(r)}(n,j) = \sum_{l=j}^{n}\binom{l}{j}m^{n-l}(-r)^{l-j}S_{1,\alpha}(n,l). \tag{51}$$

From (24) and (37), we observe that

$$\begin{aligned}
\sum_{j=0}^{n} W_{m,\alpha}^{(r)}(n,j)m^j(a^\dagger a)_{j,\alpha} = (ma^\dagger a+r)^n &= \sum_{l=0}^{n}\binom{n}{l}m^l(a^\dagger a)^l r^{n-l} \\
&= \sum_{l=0}^{n}\binom{n}{l}m^l r^{n-l}\sum_{j=0}^{l}S_{2,\alpha}(l,j)(a^\dagger a)_{j,\alpha} \\
&= \sum_{j=0}^{n}\left(\sum_{l=j}^{n}\binom{n}{l}m^l r^{n-l}S_{2,\alpha}(l,j)\right)(a^\dagger a)_{j,\alpha}.
\end{aligned} \tag{52}$$

By comparing the coefficients of both sides of (52), we have

$$W_{m,\alpha}^{(r)}(n,j) = \sum_{l=j}^{n}\binom{n}{l}m^{l-j}r^{n-l}S_{2,\alpha}(l,j). \tag{53}$$

□

By (6) and (42), we note that

$$\begin{bmatrix} n \\ j \end{bmatrix}_{\alpha} = (-1)^{n-j}S_{1,\alpha}(n,j), \qquad (n \geq j \geq 0). \tag{54}$$

When $m=1$, by Theorem , we get the following corollary.

Corollary 7. *For $n,\ j \geq 0$ with $n \geq j$, we have*

$$V_{1,\alpha}^{(r)}(n,j) = \begin{bmatrix} k+r \\ j+r \end{bmatrix}_{r,\alpha} = \sum_{l=j}^{n}\binom{l}{j}(-1)^{n-l}r^{l-j}S_{1,\alpha}(n,l)$$

and

$$W_{1,\alpha}^{(r)}(n,j) = S_{2,\alpha}^{(r)}(n+r,j+r) = \sum_{l=j}^{n}\binom{n}{l}r^{n-l}S_{2,\alpha}(l,j).$$

Theorem 8. *For $n,\ j \geq 0$ with $n \geq j$, we have*

$$V_{m,\alpha}^{(r)}(n+1,j) = V_{m,\alpha}^{(r)}(n,j-1) - (mn\alpha + r)V_{m,\alpha}^{(r)}(n,j)$$

and

$$W_{m,\alpha}^{(r)}(n+1,j) = W_{m,\alpha}^{(r)}(n,j-1) + (mj\alpha + r)W_{m,\alpha}^{(r)}(n,j).$$

Proof. From (35), we have

$$\begin{aligned}
\sum_{j=0}^{n+1} V_{m,\alpha}^{(r)}(n+1,j)(ma^\dagger a+r)^j &= m^{n+1}(a^\dagger a)_{n+1,\alpha} \\
&= m^{n+1}(a^\dagger a)_{n,\alpha}\left(\frac{ma^\dagger a - mn\alpha + r - r}{m}\right) \\
&= \sum_{j=0}^{n} V_{m,\alpha}^{(r)}(n,j)(ma^\dagger a+r)^{j+1} - (mn\alpha+r)\sum_{j=0}^{n} V_{m,\alpha}^{(r)}(n,j)(ma^\dagger a+r)^j \\
&= \sum_{j=0}^{n+1} V_{m,\alpha}^{(r)}(n,j-1)(ma^\dagger a+r)^j - (mn\alpha+r)\sum_{j=0}^{n+1} V_{m,\alpha}^{(r)}(n,j)(ma^\dagger a+r)^j \\
&= \sum_{j=0}^{n+1}\left\{V_{m,\alpha}^{(r)}(n,j-1) - (mn\alpha+r)\sum_{j=0}^{n+1} V_{m,\alpha}^{(r)}(n,j)\right\}(ma^\dagger a+r)^j.
\end{aligned} \tag{55}$$

By comparing the coefficients of both side of (55), we have

$$V_{m,\alpha}^{(r)}(n+1,j) = V_{m,\alpha}^{(r)}(n,j-1) - (mn\alpha+r)V_{m,\alpha}^{(r)}(n,j). \tag{56}$$

By (37), we get

$$\begin{aligned}
\sum_{j=0}^{n+1} W_{m,\alpha}^{(r)}(n+1,j)m^j(a^\dagger a)_{j,\alpha} &= (ma^\dagger a+r)^{n+1} = (ma^\dagger a+r)^n(ma^\dagger a+r) \\
&= (ma^\dagger a+r)^n(ma^\dagger a - mj\alpha + mj\alpha + r) \\
&= \sum_{j=0}^{n} W_{m,\alpha}^{(r)}(n,j)m^{j+1}(a^\dagger a)_{j+1,\alpha} + (mj\alpha+r)\sum_{j=0}^{n} W_{m,\alpha}^{(r)}(n,j)m^j(a^\dagger a)_{j,\alpha} \\
&= \sum_{j=1}^{n+1} W_{m,\alpha}^{(r)}(n,j-1)m^j(a^\dagger a)_{j,\alpha} + (mj\alpha+r)\sum_{j=0}^{n} W_{m,\alpha}^{(r)}(n,j)m^j(a^\dagger a)_{j,\alpha} \\
&= \sum_{j=0}^{n+1}\left\{W_{m,\alpha}^{(r)}(n,j-1) + (mj\alpha+r)W_{m,\alpha}^{(r)}(n,j)\right\}m^j(a^\dagger a)_{j,\alpha}.
\end{aligned} \tag{57}$$

By comparing the coefficients of both side of (57), we have

$$W_{m,\alpha}^{(r)}(n+1,j) = W_{m,\alpha}^{(r)}(n,j-1) + (mj\alpha+r)W_{m,\alpha}^{(r)}(n,j). \tag{58}$$

□

Corollary 9. *For $n \geq 0$, we have*

$$V_{m,\alpha}^{(r)}(n,0) = (-1)^n\prod_{l=0}^{n-1}(ml\alpha+r) \quad \textit{and} \quad W_{m,\alpha}^{(r)}(n,0) = r^n.$$

Proof. From Theorem 10, we note that

$$
\begin{aligned}
V_{m,\alpha}^{(r)}(n+1,0) &= -(mn\alpha+r)V_{m,\alpha}^{(r)}(n,0) \\
&= (-1)^2(mn\alpha+r)(m(n-1)\alpha+r)V_{m,\alpha}^{(r)}(n-1,0) \\
&= \cdots \\
&= (-1)^n(mn\alpha+r)(m(n-1)\alpha+r)\cdots rV_{m,\alpha}^{(r)}(0,0).
\end{aligned} \tag{59}
$$

By (59), we have

$$
V_{m,\alpha}^{(r)}(n,0) = (-1)^n \prod_{l=0}^{n-1}(ml\alpha+r). \tag{60}
$$

From Theorem10, we observe that

$$
W_{m,\alpha}^{(r)}(n+1,0) = rW_{m,\alpha}^{(r)}(n,0) = r^2 W_{m,\alpha}^{(r)}(n-1,0) = \cdots = r^{n+1}W_{m,\alpha}^{(r)}(0,0). \tag{61}
$$

By (61), we have

$$
W_{m,\alpha}^{(r)}(n,0) = r^n. \tag{62}
$$

□

Theorem 10. *For $n,\ l,\ j \in \mathbb{N}\cup\{0\}$, we have*

$$
V_{m,\alpha}^{(r+1)}(n,l) = \sum_{j=l}^{n}\binom{j}{l}V_{m,\alpha}^{(r)}(n,j) \quad \text{and} \quad W_{m,\alpha}^{(r+1)}(n,j) = \sum_{l=j}^{n}\binom{n}{l}W_{m,\alpha}^{(r)}(l,j).
$$

Proof. From (35), we have

$$
\begin{aligned}
m^n(a^+a)_{n,\alpha} &= \sum_{j=0}^{n} V_{m,\alpha}^{(r)}(n,j)(ma^\dagger a+r)^j \\
&= \sum_{j=0}^{n} V_{m,\alpha}^{(r)}(n,j)(ma^\dagger a+r+1-1)^j \\
&= \sum_{j=0}^{n} V_{m,\alpha}^{(r)}(n,j)\sum_{l=0}^{j}\binom{j}{l}(ma^\dagger a+r+1)^l \\
&= \sum_{l=0}^{n}\left\{\sum_{j=l}^{n}\binom{j}{l}V_{m,\alpha}^{(r)}(n,j)\right\}(ma^\dagger a+r+1)^l.
\end{aligned} \tag{63}
$$

On the other hand, by (35), we have

$$
m^n(a^\dagger a)_{n,a} = \sum_{l=0}^{n} V_{m,\alpha}^{(r+1)}(n,l)(ma^\dagger a+r+1)^l. \tag{64}
$$

By (63) and (64), we have

$$
V_{m,\alpha}^{(r+1)}(n,l) = \sum_{j=l}^{n}\binom{j}{l}V_{m,\alpha}^{(r)}(n,j). \tag{65}
$$

From (37), we have

$$\begin{aligned}\sum_{j=0}^{n} W_{m,\alpha}^{(r+1)}(n,j)m^j(a^\dagger a)_{j,\alpha} &= (ma^\dagger a+r+1)^n \\ &= \sum_{l=0}^{n}\binom{n}{l}(ma^\dagger a+r)^l \\ &= \sum_{l=0}^{n}\binom{n}{l}\sum_{j=0}^{l} W_{m,\alpha}^{(r)}(l,j)m^j(a^\dagger a)_{j,\alpha} \\ &= \sum_{j=0}^{n}\left\{\sum_{l=j}^{n}\binom{n}{l}W_{m,\alpha}^{(r)}(l,j)\right\}m^j(a^\dagger a)_{j,\alpha}.\end{aligned} \tag{66}$$

Comparing the coefficients of both sides of (66), we have

$$W_{m,\alpha}^{(r+1)}(n,j) = \sum_{l=j}^{n}\binom{n}{l}W_{m,\alpha}^{(r)}(l,j). \tag{67}$$

□

We recall that the coherent state

$$|z\rangle = e^{-\frac{|z|^2}{z}}\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!}}|n\rangle, \quad \text{(see [15, 16, 19])}, \tag{68}$$

where z is an arbitrary complex constant $a|z\rangle = z|z\rangle = 1$ and $\langle z|z\rangle = 1$.

For $x,\ y\in\mathbb{C}$, we note that

$$\langle x|y\rangle = e^{-\frac{1}{2}(|x|^2+|y|^y)}+\bar{x}y = e^{-\frac{|x|^2}{2}-\frac{|y|^2}{2}}\sum_{n=0}^{\infty}\frac{(\bar{x}y)^n}{n!}, \quad \text{(see [15, 16, 19])}. \tag{69}$$

By using the properties of coherent state, from (8) and (22),

$$\langle z|(a^\dagger a)^n|z\rangle = bel_n(|z|^2). \tag{70}$$

It is easy to show that

$$\begin{aligned} a^\dagger a e^{a^\dagger a-\alpha+r}(t) &= e^{a^\dagger a+r-\alpha}(t)a^\dagger a \\ &= a^\dagger e^{aa^\dagger-\alpha+r}(t)a = a^\dagger e^{a^\dagger a+1+r-\alpha}(t)a, \quad \text{(see [8, 19])}\end{aligned} \tag{71}$$

Theorem 11. *For $k,\ n\geq 0$, we have*

$$\sum_{q=0}^{[\frac{k+n}{2}]}\sum_{j=l}^{n}\sum_{l=k}^{j}\binom{k}{q}\binom{j}{l}q!m^l r^{j-l}V_{m,\alpha}^{(r)}(n,j)S_2(l,q) = m^n(k)_{n,\alpha}.$$

Proof. By (35), we have

$$
\begin{aligned}
\langle z|m^n(a^\dagger a)_{n,\alpha}|z\rangle &= \sum_{j=0}^{n} V_{m,\alpha}^{(r)}(n,j)\langle z|(ma^\dagger a+r)^j|z\rangle \\
&= \sum_{j=0}^{n} V_{m,\alpha}^{(r)}(n,j)\sum_{l=0}^{j}\binom{j}{l} m^l\langle z|(a^\dagger a)^l|z\rangle r^{j-l} \\
&= \sum_{j=0}^{n} V_{m,\alpha}^{(r)}(n,j)\sum_{l=0}^{j}\binom{j}{l} m^l r^{j-l}\sum_{q=0}^{l} S_2(l,q)|z|^{2q} \\
&= \sum_{q=0}^{n}\sum_{j=l}^{n}\sum_{l=q}^{j}\binom{j}{l} m^l r^{j-l} V_{m,\alpha}^{(r)}(n,j) S_2(l,q)|z|^{2q}.
\end{aligned} \tag{72}
$$

On the other hand, we observe that

$$
\begin{aligned}
\langle z|m^n(a^\dagger a)_{n,\alpha}|z\rangle &= m^n\langle z|(a^\dagger a)_{n,\alpha}|z\rangle \\
&= m^n\sum_{k,l=0}^{\infty}\frac{z^k\bar{z}^l}{\sqrt{k!}\sqrt{l!}}\langle l|k\rangle (k)_{n,\alpha} e^{-\frac{|z|^2}{z}} e^{-\frac{|\bar{z}|^2}{z}} \\
&= m^n e^{-|z|^2}\sum_{k=0}^{\infty}\frac{|z|^{2k}}{k!}(k)_{n,\alpha}.
\end{aligned} \tag{73}
$$

By (72) and (73), we have

$$
e^{|z|^2}\sum_{q=0}^{n}\sum_{j=l}^{n}\sum_{l=q}^{j}\binom{j}{l} m^l r^{j-l} V_{m,\alpha}^{(r)}(n,j) S_2(l,q)|z|^{2q} = m^n\sum_{k=0}^{\infty}\frac{|z|^{2k}}{k!}(k)_{n,\alpha}. \tag{74}
$$

The left-hand sides of (74) is

$$
\begin{aligned}
& e^{|z|^2}\sum_{q=0}^{n}\sum_{j=l}^{n}\sum_{l=q}^{j}\binom{j}{l} m^l r^{j-l} V_{m,\alpha}^{(r)}(n,j) S_2(l,q)|z|^{2q} \\
&= \sum_{k=0}^{\infty}\sum_{q=0}^{[\frac{k+n}{2}]}\sum_{j=l}^{n}\sum_{l=q}^{j}\binom{k}{q}\binom{j}{l} m^l r^{j-l} V_{m,\alpha}^{(r)}(n,j) S_2(l,q)|z|^{2k}.
\end{aligned} \tag{75}
$$

From (74) and (75), we have the desired identity.

□

Theorem 12. *For $n \geq 0$, we have*

$$
\sum_{j=0}^{n}\sum_{l=0}^{j} W_{m,\alpha}^{(r)}(n,j) m^j S_{1,\alpha}(j,l) bel_l(|z|^2) = bel_n^{(r)}(|z|^2).
$$

Proof. Let $f(t) = \langle\, z \mid e^{t(ma^\dagger a+r)} \mid z\,\rangle$. Then, by (71), we observe that

$$
\begin{aligned}
\frac{\partial f(t)}{\partial t} &= \langle\, z \mid (ma^\dagger a+r)e^{t(ma^\dagger a+r)} \mid z\,\rangle \\
&= \langle\, z \mid ma^\dagger e^{t(ma^\dagger a+1+r)} a \mid z\,\rangle + r\langle\, z \mid e^{t(ma^\dagger a+r)} \mid z\,\rangle \\
&= e^t\bar{z}z\langle\, z \mid e^{t(ma^\dagger a+r)} \mid z\,\rangle + r\langle\, z \mid e^{t(ma^\dagger a+1+r)} \mid z\,\rangle \\
&= \left(e^t|z|^2 + r\right) f(t).
\end{aligned} \tag{76}
$$

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Note that $f(0)=1$. From (76), we have

$$\log f(t)=\int_0^t \frac{f'(t)}{f(t)}dt=\int_0^t (|z|^2e^t+r)dt=|z|^2(e^t-1)+rt. \tag{77}$$

Thus, (10) and (77), we get

$$f(t)=\exp\left(|z|^2(e^t-1)+rt\right)=e^{rt}e^{|z|^2(e^t-1)}=\sum_{n=0}^{\infty} bel_n^{(r)}(|z|^2), \tag{78}$$

where $\exp(t)=e^t$.

On the other hand, from (23), (37) and (70), we note that

$$\begin{aligned} f(t)=\langle\, z \mid e^{t(ma^\dagger a+r)} \mid z\,\rangle &= \sum_{n=0}^{\infty}\langle z|(ma^\dagger a+r)^n|z\rangle \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\sum_{j=0}^{n} W_{m,\alpha}^{(r)}(n,j)m^j\langle z|(a^\dagger a)_{j,\alpha}|z\rangle\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\sum_{j=0}^{n} W_{m,\alpha}^{(r)}(n,j)m^j\sum_{l=0}^{j}S_{1,\alpha}(j,l)\langle z|(a^\dagger a)^l|z\rangle\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\sum_{j=0}^{n}\sum_{l=0}^{j} W_{m,\alpha}^{(r)}(n,j)m^jS_{1,\alpha}(j,l)bel_l(|z|^2)\frac{t^n}{n!}. \end{aligned} \tag{79}$$

By comparing the coefficients of (78) and (79), we have the desired identity. □

3. Further Remark

In view of (8), we consider the α-analogue r-Dowling polynomials of the first kind

$$D_{m,\alpha}^{(r)}(n,x)=\sum_{k=0}^{n} V_{m,\alpha}^{(r)}(n,k)x^k, \qquad (n\geq 0). \tag{80}$$

When $x=1$, $D_{m,\alpha}^{(r)}(n,1)=D_{m,\alpha}^{(r)}(n)$ are called the α-analogue r-Dowling numbers of the first kind.

By Proposition 2 and (80), we get the generating function of the α-analogue r-Dowling numbers of the first kind as follows.

Proposition 13. *For $n\geq 0$, we have*

$$\sum_{n=0}^{\infty} D_{m,\alpha}^{(r)}(n,x)\frac{t^n}{n!}=e_{\alpha m}^{-r}(t)\exp\left(\frac{\log(1+\alpha mt)}{\alpha m}\right),$$

where $e_{\alpha m}^{-r}(t)=(1+\alpha mt)^{-\frac{r}{\alpha m}}$ and $\exp(t)=e^t$.

In view of (8), we define the α-analogue r-Dowling polynomials of the second kind as

$$d_{m,\alpha}^{(r)}(n,x)=\sum_{k=0}^{n} W_{m,\alpha}^{(r)}(n,k)x^k, \qquad (n\ \geq\ 0). \tag{81}$$

when $x=1, d_{m,\alpha}^{(r)}(n,1)=d_{m,\alpha}^{(r)}(n)$ are called the α-analogue r-Dowling numbers of the second kind.

By Proposition 1 and (81), we have the generating function of r-Dowling polynomials of the second kind as follows.

Proposition 14. *For $n \geq 0$, we have*

$$e^{rt} \exp\left(\frac{e^{\alpha mt}-1}{\alpha m}\right) = \sum_{n=0}^{\infty} d_{m,\alpha}^{(r)}(n,x)\frac{t^n}{n!},$$

where $\exp(t) = e^t$.

4. Conclusion

We studied various combinatorial properties of the the α-analogue r-Whitney of the first and second kind from the algebraic properties of the boson operators of these numbers. For future projects, we would like to conduct research into some potential applications of the numbers and polynomials (for example, α-analogue r-Dowling polynomials of the first and second kind in Proposition 13 and 14 derived in this paper).

Availability of data and material
Not applicable.

Ethics approval and consent to participate
The author declare that there is no ethical problem in the production of this paper.

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