Original Research Article

Region Polynomial of a Knot Diagram

1 Abstract

We introduce a new polynomial for knot diagrams. The coefficients of this polynomial are dependent on the number of the arcs on the regions of the knot diagrams. A relation between the sum of the coefficients and the number of crossing of the knot diagram is established. Some other properties of the developed region based polynomial are discussed. The behaviour of the polynomial against Reidemeister moves is also studied.

Keywords: Knot diagram; crossing number; Reidemeister moves; Warping polynomial.

2 Introduction

In the field of knot theory, there are many knot polynomial, for example: Alexander polynomial [1], Jones polynomial [2], HOMFLY polynomial [3], Warping polynomial [4] etc. These polynomial can be obtained recursively using skein relations. Warping polynomial is crossing based polynomial.
The Alexander polynomial is first knot polynomial discovered by James Waddell Alexander in 1923. The procedure for computing the Alexander polynomial was as: Take an n crossing oriented diagram of the knot then the knot diagram contains $n+2$ regions. To compute the Alexander polynomial, first he generated a $(n, n+2)$ size incidence matrix [5] having values of $0, 1, -1,$

t or $-t$. The n rows are correspond to the n crossings, whereas the $n+2$ columns relate to the regions. If the region is not connected to the crossing, that entry is 0 and if region is connected to crossing then the entry is determined by its position. The entries is decided by the position of the region at the crossing from the viewpoint of the approaching under crossing line accordingly as: left side before the under crossing: $-t$, on right side before the under crossing: 1, on left side after the under crossing: t, on right side after the under crossing: -1. The determinants of the sub matrix obtained after deleting any two adjacent region's columns yields Alexander polynomial.
The Seifert matrix can also be used to calculate the Alexander polynomial.

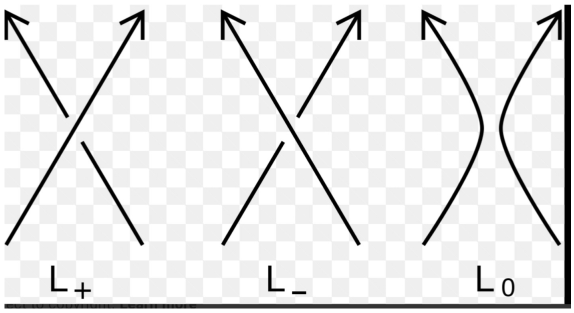


Figure 1
Alexander polynomial structure

John Conway, in 1969 discovered a new version of Alexander polynomial, now known as Alexander - Conway polynomial [6], could be determined by a skein relationship. Assume we have an orientated diagram of link, where L_+,L_-, and L_0 are produced by crossings changes and smoothing on a particular region of a link diagram, as indicated in Figure 1. Then Conway's skein relations are:
$(i)\ \bigtriangledown(Unknot) = 1$
$(ii)\ \bigtriangledown(L_+) - \bigtriangledown(L_-) = \mathrm{Z} \bigtriangledown(L_0)$.
The Alexander polynomial and Alexander Conway's polynomial are related by $\triangle_L(t^2) = \bigtriangledown_L(t - t^{-1})$. where $\triangle$ and $\bigtriangledown$ denotes the Alexander polynomial and Alexander Conway's polynomial of a knot diagram respectively.
In 1984, a new polynomial named Jones polynomial was discovered by Vaughan Jones. Jones polynomial assigns a Laurent Polynomial in the variable $x^{1/2}$ to each oriented knot or link. The skein relations for this polynomial are

$$\left(x^{1/2} - x^{-1/2}\right) V(L_0) = x^{-1} V(L_+) - \ xV(L_-)$$

where L_+, L_- and L_0 are diagrams as shown in Figure 1.
In 1985, a new polynomial, named HOMFLY polynomial, was discovered. Jim Hoste, Adrian Ocneanu, Kenneth Millett, Peter J. Freyd, W. B. R. Lickorish, and David N. Yetter are the six co-discoverers of the term HOMFLY.

This polynomial is in two variables s and l. Alexander polynomial and Jones polynomial are special cases of HOMFLY polynomial. These polynomial can be obtained from HOMFLY polynomial by providing a particular substitution. The skein relations of this polynomial are:
(i) $P(Unknot) = 1$
(ii) $\ell\ P(L_+) + \ell^{-1}\ P(L_-) + s\ P(L_0) = 0$,
where L_+, L_- and L_0 are diagram as indicated in Figure 1.
In 1989, Louis Kauffman discovered the 2-variable knot polynomial, named Kauffman polynomial [7]. It is described as
$F(K)(b, z) = b^{-w(K)}\ L(K)$,
where $w(K)$ is the Link diagram's writhe [8] and $L(K)$ is a link diagram-defined polynomial in 'z' and 'b'.
In 2012, Warping polynomial of a knot diagram was introduced by Ayaka Shimizu [4]. She established a relationship of warping polynomial with the crossing number, dealternating number [9], the warping degree [10], span of knot [4] and monotone knot diagram [11]. The warping polynomial was dependent on both crossing and orientation of the knot diagram. She also introduced about warping matrix and its rank [12, 13]. Now in this paper, we introduced a new polynomial defined using region in diagram. This region polynomial $f_D(x)$ of a knot diagram D has non-negative integer coefficients. This polynomial is not affected by the orientation of the knot diagram.

3 Main Results

In this section, we define the region numbering for a region in a knot diagram. An edge/arc of diagram D is a path which has no crossings between two consecutive crossing points.
Region Numbering of a Region: Let D be a knot diagram and R is any region of D, then the region numbering of R is the number of arcs that R has in common with other regions of D. It is denoted by $\gamma(R)$
For Example: The shaded portion in Figure 2 has region numbering 7 as this shaded region has 7 arcs common with other regions of this knot diagram.

The region numbering of all the regions of the knot diagram in Figure 3 are as:

Region Polynomial of a Knot Diagram: Let $\Re D$ denotes the set of all regions of the diagram D. Then region polynomial is defined as $f_D(x) =$

$$\sum_{R\in\Re D} x^{\gamma(R)}$$

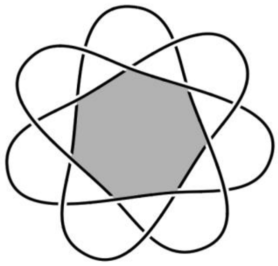


Figure 2 Knot diagram

(a) (b)

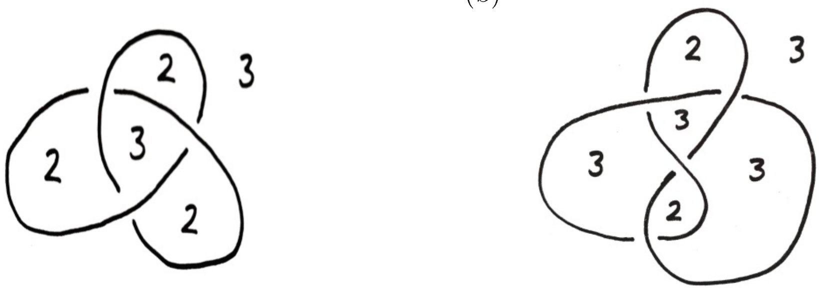


Figure 3: Region numbering for all regions of Figure 3a and Figure 3b

So degree of the polynomial is the largest region numbering of any region in D. Number of terms in this polynomial is same as the number of regions. The above get reduced in polynomial form as

$$f_D(x) = a_1 x^1 + a_2 x^2 + \cdots + a_k x^k$$

where in x^r, r is the region numbering of the region and a_r means total number of regions with region numbering r.
For example:
(i) Region polynomial of Figure $(3a)$ and $(3b)$ is $f(x) = 3x^2 + 2x^3$ and $2x^2 + 4x^3$ respectively.
(ii) Region polynomial of diagram of trivial knot of zero crossing is $f(x) = 2x$

Since the number of terms in region polynomial is same as the number of regions, being one term for one region. So number of regions in the diagram D is equal to sum of coefficients of the region polynomial i.e., number of regions $= f(1)$ or $\sum_{i=0}^{k} a_i$

Number of crossings in the diagram $D = f(1) - 2$.
Proof. For a knot diagram,

number of regions = number of crossing + 2
$\implies f(1) =$ number of crossing + 2
$\implies$ number of crossing $= f(1) - 2$.

This polynomial is independent of orientation or reflection.

3.1 Connected Sum of Two Knot Diagrams

Two oriented knots can be added by arranging them side by side and connecting them with straight bars, preserving the orientation in the sum. The knot sum is also called composition or connected sum. The knot sum of knot diagrams P and Q is denoted as $P\#Q$. If P has m regions and Q has n regions then $P\#Q$ will have $m + n - 2$ regions.

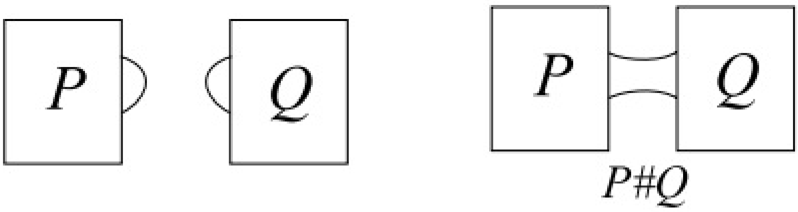


Figure 4: Connected sum of two knot diagram P and Q

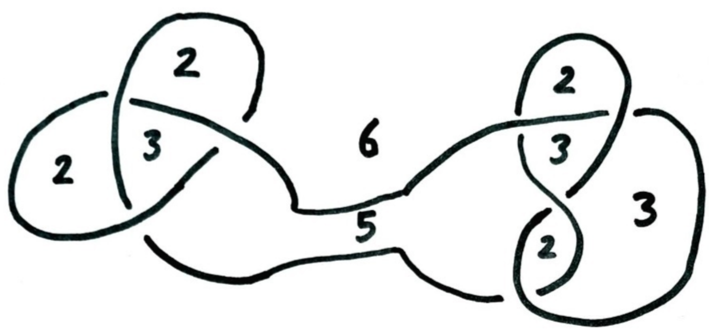


Figure 5: Connected sum of Figure 3a and Figure 3b

Let D be a knot diagram which is connected sum of P and Q. Let e_1 and e_2 be the connecting regions of P and Q and, e_3 and e_4 be the outer regions of diagram P and Q respectively. Let $e_{1\#2}$ and $e_{3\#4}$ are the regions in D, obtained after sum of P and Q as in Figure 5 . Then region polynomial of D is

$$f(D) = f(P) + f(Q) - x^{\gamma(e_1)} - x^{\gamma(e_2)} + x^{\gamma(e_{1\#2})} - x^{\gamma(e_3)} - x^{\gamma(e_4)} + x^{\gamma(e_{3\#4})}$$

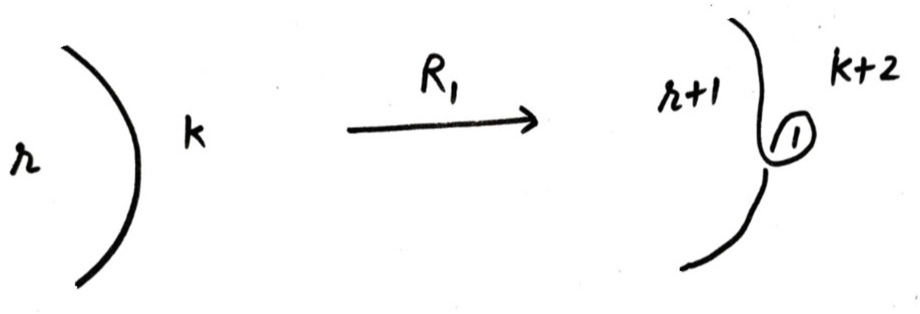
Note that in the connected sum $P\#Q$, the regions e_1 and e_2 get eliminated and new region $e_{1\#2}$ get generated. Two outer regions e_3 and e_4 are also get combined as new region $e_{3\#4}$. Hence $x^{\gamma(e_1)} + x^{\gamma(e_2)} + x^{\gamma(e_3)} + x^{\gamma(e_4)}$ get eliminated and $x^{\gamma(e_{1\#2})} + x^{\gamma(e_{3\#4})}$ is included in region polynomial of $P\#Q$. Also $\gamma(e_{1\#2}) = \gamma(e_1) + \gamma(e_2)$. Since the number of arcs on $e_{1\#2}$ is sum of number of arcs on e_1 and e_2. Similar is true for $\gamma(e_{3\#4})$ i.e. $\gamma(e_{3\#4}) = \gamma(e_3) + \gamma(e_4)$.

(i) Region polynomial of knot diagram of Figure 5 ($3a$ # $3b$) is $3x^2 + 2x^3 + 2x^2 + 4x^3 - x^2 - x^3 + x^5 - x^3 - x^3 + x^6$
$= 4x^2 + 3x^3 + x^5 + x^6$.
It is easy to note that addition of any knot diagram with zero crossing knot diagram does not affect the region polynomial of the knot diagram i.e., $f(P\#O) = f(P)$ where O represents zero crossing diagram.

3.2 Effect of Reidemeister Moves on Region Polynomial

Now, we will discuss about the effects of Reidemeister moves on the region polynomial of a knot diagram.

3.2.1 Effects of 1^{st} Reidemeister move (R_1) on region polynomial



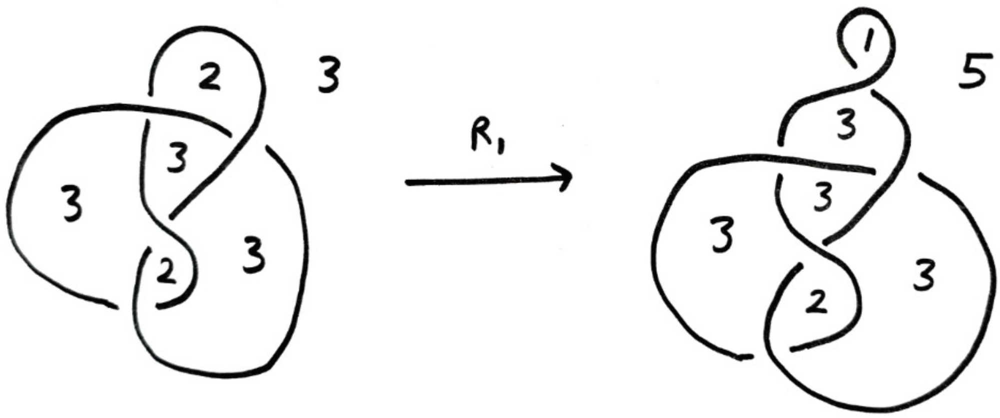
(a) (b)

Figure 6: R_1 move

Let r and k be the region numberings around an arc where we are willing to perform R_1. The change in region numbering under R_1 move is shown in

Figure 6. After R_1 move, the number of regions increase by one. The new region will have region numbering one whereas older region's numbering will increase by one and two as shown in Figure 6. Overall, R_1 changes region numbering of two old regions in above said fashion.
If the region polynomial of Figure 6(a) is $f(x)$ then the region polynomial of Figure 6(b) will be $f(x) - x^r - x^k + x + x^{r+1} + x^{k+2}$.

Region polynomial of diagram in Figure 7(a) is $2x^2 + 4x^3$ and region polynomial of Figure 7(b) is $2x^2 + 4x^3 - x^2 - x^3 + x + x^3 + x^5 = x + x^2 + 4x^3 + x^5$.



(a) (b)

Figure 7: Example of R_1 move

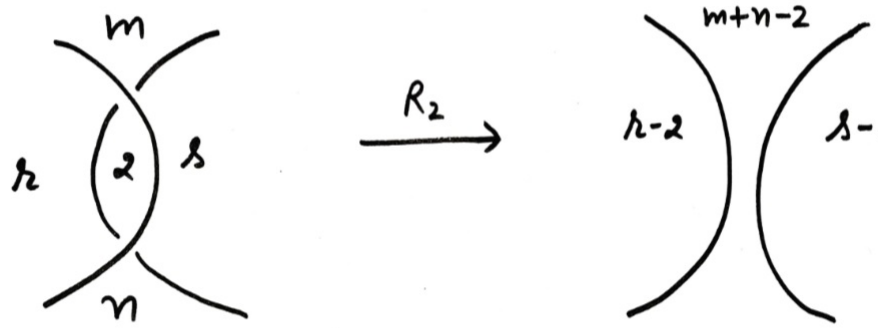
(a) (b)

Figure 8: Example of reverse case of R_1 move

Similar effects can be observed in reverse process of R_1 move where one re-

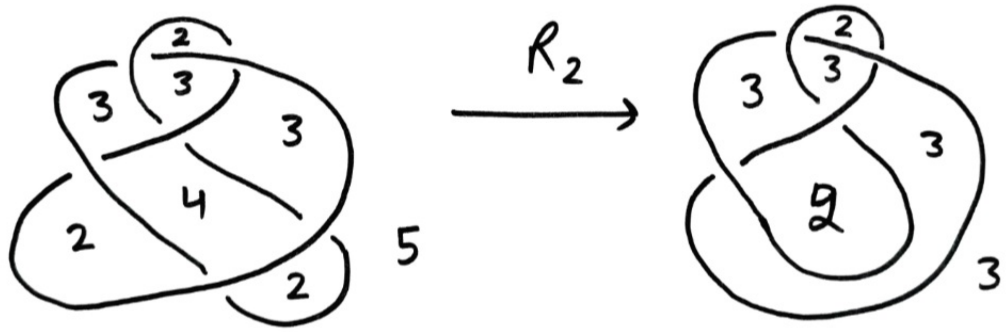
gion get deleted and region numbering of two regions get decreased by one and two.
Region polynomial of diagram in Figure 8(a) is $x + 2x^2 + 2x^3 + x^5$. After R_1, a region with region numbering one get deleted and decrement in region numbering of two other regions can be seen. So region polynomial of Figure 8(b) is $x + 2x^2 + 2x^3 + x^5 - x^3 - x^5 - x + x^2 + x^3 = 3x^2 + 2x^3$.

3.2.2 Effects of 2^{nd} Reidemeister move (R_2) on region polynomial



(a) (b)

Figure 9: R_2 move



(a) (b)

Figure 10: Example of R_2 move

It can be seen from Figure 9 that after R_2, number of regions decrease by two. One region of region numbering two get deleted. In this case, two

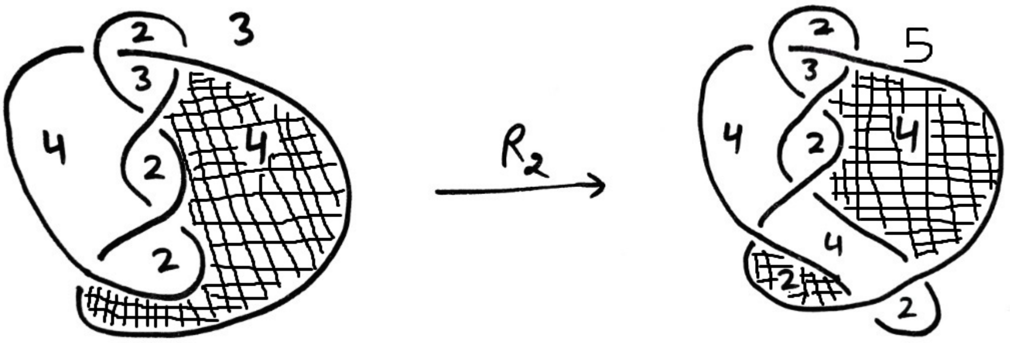


Figure 11: Example of reverse case of R_2 move

regions with region numbering m and n (Figure 9) become one region with region numbering $m+n-2$.
If the region polynomial of Figure 9(a) is $f(x)$ then the region polynomial of Figure 9(b) will be $f(x) - x^r - x^s - x^m - x^n - x^2 + x^{r-2} + x^{s-2} + x^{m+n-2}$.

Region polynomial of diagram in Figure 10(a) is $3x^2 + 3x^3 + x^4 + x^5$ and region polynomial of diagram in Figure 10(b) is $3x^2 + 3x^3 + x^4 + x^5 - x^5 - x^4 - x^3 - x^2 - x^2 + x^3 + x^2 + x^3 = 2x^2 + 4x^3$.

In the reverse case, region with region numbering $m+n-2$ get divided in two parts. But from here, values of m and n can not be identified uniquely as shown in Figure 11 and Figure 12.

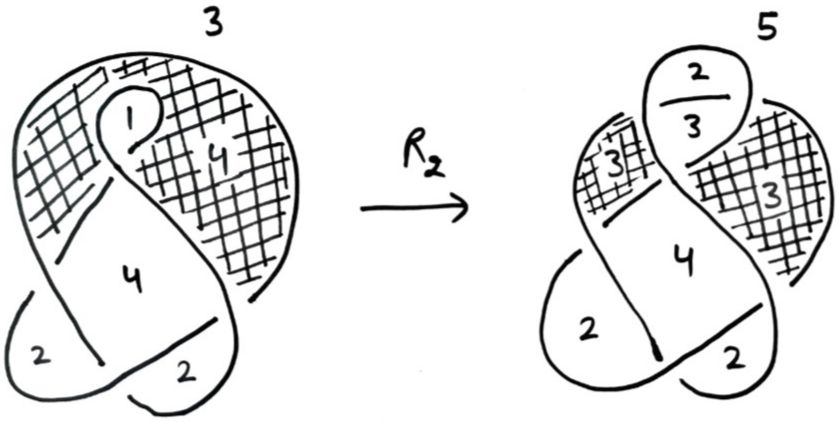


Figure 12: Example of reverse case of R_2 move

It can be seen from Figure 11 that after R_2, shaded region having region numbering $m+n-2=4$ get divided into two regions having region numbering four and two. Similarly from Figure 12, after R_2, shaded region having region numbering $m+n-2=4$ get divided into two regions having region numbering three and three. So these values of m and n can not be identified uniquely.

3.2.3 Effects of 3^{rd} Reidemeister move (R_3) on region polynomial

It can be seen from Figure 13 that there is no change in the number of regions after R_3 move. There will be three regions whose region numbering increases by one and three regions whose region numbering decreases by one.

Region polynomial of diagram of Figure 14(a) is $2x^2 + 4x^3$ and region

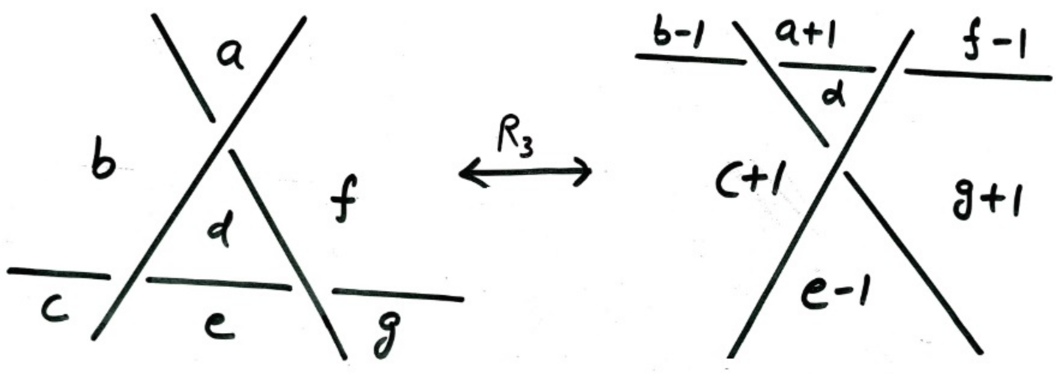


Figure 13: R_3 move

polynomial of diagram of Figure 14(b) is $2x^2+4x^3-x^3-x^3-x^3-x^2-x^2+x^2+x^2+x^3+x+x^5 = x+2x^2+2x^3+x^5$

4 Conclusion

A region polynomial, using number of arcs on the boundary of a region of a knot diagram, is introduced. This polynomial is independent of the orientation of the knot. A relation between the sum of the coefficients and the number of crossing of the knot diagram is established. The effect of Reidemeister moves on polynomial is studied with examples. The region polynomial of connected sum of two knot diagrams is also derived.

(a) (b)

Figure 14: Example of R_3 move

References

[1] J. W. Alexander, Topological invariants of knots and links, Transactions of the American Mathematical Society 30 (2) (1928) 275–306.

[2] V. F. Jones, A polynomial invariant for knots via von neumann algebras, in: Fields Medallists' Lectures, World Scientific, 1997, pp. 448–458.

[3] P. Freyd, D. Yetter, J. Hoste, W. R. Lickorish, K. Millett, A. Ocneanu, A new polynomial invariant of knots and links, Bulletin (new series) of the American mathematical society 12 (2) (1985) 239–246.

[4] A. Shimizu, The warping polynomial of a knot diagram, Journal of Knot Theory and Its Ramifications 21 (14) (2012) 1250124.

[5] J. L. Gross, J. Yellen, M. Anderson, Graph theory and its applications, Chapman and Hall/CRC, 2018.

[6] J. Sawollek, On alexander-conway polynomials for virtual knots and links, arXiv preprint math/9912173 (1999).

[7] L. H. Kauffman, An invariant of regular isotopy, Transactions of the American Mathematical Society 318 (2) (1990) 417–471.

[8] D. Cimasoni, Computing the writhe of a knot, Journal of Knot Theory and Its Ramifications 10 (03) (2001) 387–395.

[9] C. Adams, Why knot?: an introduction to the mathematical theory of knots, Springer Science & Business Media, 2004.

[10] A. Shimizu, The warping degree of a knot diagram, Journal of Knot Theory and Its Ramifications 19 (07) (2010) 849–857.

[11] A. Kawauchi, A. Shimizu, On the orientations of monotone knot diagrams, Journal of Knot Theory and Its Ramifications 26 (10) (2017) 1750053.

[12] A. Shimizu, The warping matrix of a knot diagram, Contemp. Math 670 (2016) 337–344.

[13] T. Akiyama, A. Shimizu, R. Watanabe, The rank of a warping matrix, Topology and its Applications 206 (2016) 228–240.