

On the Riemann Hypothesis

Abstract

This paper provides an exact characterisation of the zeroes of the Riemann zeta function.

1 Riemann Hypothesis

We study the Riemann zeta function (Riemann [1859]), by defining it explicitly as a function on a subset $S \subseteq \mathbb{R}^2$, mapping into \mathbb{R}^2 . Namely, we define the set S as

$$S = \{(\sigma, t) \in \mathbb{R}^2 : \sigma \in (0, 1); t \neq 0\} \quad (1)$$

corresponding to the space of all complex numbers, whose real part is in $(0, 1)$. The space of all complex numbers is itself, of course, identified with the space \mathbb{R}^2 . For any given $s = (\sigma, t) \in \mathbb{R}^2$, we interpret σ as the real part and t as the imaginary part.

We now recall some definitions about complex numbers regarding addition, multiplication and exponentiation of complex numbers [2-4]. Addition is defined simply as vector addition in \mathbb{R}^2 i.e. if we have points $s = (\sigma, t)$ and $s' = (\sigma', t')$ in \mathbb{R}^2 , then $s + s' = (\sigma + \sigma', t + t')$. Multiplication, as defined, incorporates the assimilation of the arithmetic of real numbers with i , using the fact that $i^2 = -1$. Hence, $s \times s' := (\sigma\sigma' - tt', \sigma t' + t\sigma')$. Exponentiation, by applying Euler's formula, gives the following expression

$$e^s := e^\sigma (\cos(t), \sin(t)), \quad (2)$$

in which $e^s \in \mathbb{R}^2$ is the vector defined by multiplying the scalar e^σ with the vector $(\cos(t), \sin(t)) \in \mathbb{R}^2$. Further, $(\cos(t), \sin(t))$ also belongs to the unit circle in \mathbb{R}^2 . By now applying this expression for exponentiation, we obtain that for any positive integer $n \geq 1$, and $s = (\sigma, t)$, we have that

$$\frac{1}{n^s} = e^{-\sigma \ln(n)} (\cos(-t \ln(n)), \sin(-t \ln(n))). \quad (3)$$

1.1 The Riemann Hypothesis and the expectation of a random vector

The Riemann zeta function is defined as the function $\zeta : S \rightarrow \mathbb{R}^2$,

$$\zeta(s) := \sum_{n=1}^{\infty} \left(1 - \frac{1}{2^{1-s}}\right) \times \frac{(-1)^{n+1}}{n^s}. \quad (4)$$

Now, we think of the set of all positive integers \mathbb{Z}^+ as a measurable space with the power set sigma-field i.e. $(\mathbb{Z}^+, 2^{\mathbb{Z}^+})$. For any point $s \in \mathbb{R}^2$, we define the measurable function $X_s : \mathbb{Z}^+ \rightarrow \mathbb{R}^2$ and measure $\mu_s : 2^{\mathbb{Z}^+} \rightarrow [0, \infty]$ as

$$X_s(n) := (-1)^{n+1} \left[\left(1 - \frac{1}{2^{1-s}}\right) \times (\cos(-t \ln(n)), \sin(-t \ln(n))) \right], \quad (5)$$

and

$$\mu_s(E) := \sum_{n \in E} e^{-\sigma \ln(n)} \quad (6)$$

respectively.

Hence, it is apparent from the definition of X_s and μ_s above, that $\zeta(s) = \mathbb{E}_{\mu_s}[X_s]$ and that hence $\zeta(s) \neq 0$ if and only if $\mathbb{E}_{\mu_s}[X_s] \neq 0$, denoting \mathbb{E}_{μ_s} as the expectation operator corresponding to the measure μ_s , defined in this context as follows : for any measure μ , the operator \mathbb{E}_μ is defined as any linear function $\mathbb{E}_\mu : l^\infty \times l^\infty \rightarrow \mathbb{R}^2$ such that

$$\mathbb{E}_\mu(X) = \sum_{n \in \mathbb{Z}^+} \mu(\{n\})X(n), \quad (7)$$

if it is true that the partial sums $\sum_{n=1}^m \mu(\{n\})X(n)$ converge for the given $X \in l^\infty \times l^\infty$. For a subset $E \subseteq \mathbb{Z}^+$, the expectation over the set E is given by $\mathbb{E}_\mu[\mathbb{I}_E X]$.

We first prove a proposition that would lead to the characterisation. The proposition is about convergent infinite series in \mathbb{R}^d . For points $s, s' \in \mathbb{R}^d$, we define the line joining the points as $\langle s, s' \rangle := \{\theta s + (1-\theta)s' : \theta \in \mathbb{R}\}$ and define $\langle s \rangle := \langle s, 0 \rangle$.

Proposition 1.1. *Suppose that $\{x_n\}_{n=1} \subseteq \mathbb{R}^d$ such that there exist $n, m \in \mathbb{Z}^+$ with*

$$x_m \notin \langle x_n \rangle. \quad (8)$$

Further, suppose that $\sum_{n \in \mathbb{Z}^+} x_n$ exists. Then, we have that $\sum_{n \in \mathbb{Z}^+} x_n \neq 0$ if and only if there exists $m \in \mathbb{Z}^+$ such that $0 \notin \langle \sum_{n=1}^m x_n, \sum_{n=m+1}^\infty x_n \rangle$.

Proof. We first prove the "if" part of the proposition. Suppose that it is true that there exists $m \in \mathbb{N}$ such that $0 \notin \langle \sum_{n=1}^m x_n, \sum_{n=m+1}^\infty x_n \rangle$. Suppose, for contradiction $\sum_{n \in \mathbb{Z}^+} x_n = 0$. Then, we have

$$\frac{1}{2} \sum_{n=1}^m x_n + \frac{1}{2} \sum_{n=m+1}^\infty x_n = 0. \quad (9)$$

Hence, $0 \in \langle \sum_{n=1}^m x_n, \sum_{n=m+1}^\infty x_n \rangle$, which is a contradiction.

We next prove the only if part of the proposition. Suppose that $\sum_{n \in \mathbb{Z}^+} x_n \neq 0$. Define,

$$m := \min\{m' : x_{m'} \notin \langle \sum_{n \in \mathbb{Z}^+} x_n \rangle\}. \quad (10)$$

The condition that is given by 8 shows that m is well-defined. By definition, we have for m that

$$\sum_{n=1}^m x_n \notin \langle \sum_{n \in \mathbb{Z}_+} x_n \rangle. \quad (11)$$

This also means that $\sum_{n=1}^m x_n \neq 0$ and hence from the above conclusion, we have that $\sum_{n=m+1}^{\infty} x_n \neq 0$ and that

$$\sum_{n \in \mathbb{Z}_+} x_n \notin \langle \sum_{n=1}^m x_n \rangle. \quad (12)$$

Suppose, for contradiction that $0 \in \langle \sum_{n=1}^m x_n, \sum_{n=m+1}^{\infty} x_n \rangle$. Then, we have that $\sum_{n=m+1}^{\infty} x_n \in \langle \sum_{n=1}^m x_n \rangle$. This would then mean that

$$\sum_{n=1}^m x_n + \sum_{n=m+1}^{\infty} x_n \in \langle \sum_{n=1}^m x_n \rangle, \quad (13)$$

which contradicts 12. Hence, $0 \notin \langle \sum_{n=1}^m x_n, \sum_{n=m+1}^{\infty} x_n \rangle$. \square

We prove the following main proposition about the function ζ .

Proposition 1.2. *Suppose $s \in S$. Then, $\zeta(s) = 0$ if and only if for each subset $E \subseteq \mathbb{Z}^+$, we have that $0 \in \langle \mathbb{E}_{\mu_s}[\mathbb{I}_E X_s], \mathbb{E}_{\mu_s}[\mathbb{I}_{\mathbb{Z}^+ \setminus E} X_s] \rangle$.*

Proof. The proof follows from Proposition 1.1. \square

We state an application of the above proposition. For each $m \in \mathbb{Z}_+$, define the functions $\zeta_m : S \rightarrow \mathbb{R}^2$ and $\zeta_{-m} : S \rightarrow \mathbb{R}^2$ as

$$\zeta_m(s) := \sum_{n=1}^m \left(1 - \frac{1}{2^{1-s}}\right) \times \frac{(-1)^{n+1}}{n^s}, \quad (14)$$

$$\zeta_{-m}(s) := \sum_{n=m+1}^{\infty} \left(1 - \frac{1}{2^{1-s}}\right) \times \frac{(-1)^{n+1}}{n^s}. \quad (15)$$

Hence, for each $s \in S$, we have the vectors $\zeta_m(s) = (\zeta_{m,1}(s), \zeta_{m,2}(s)) \in \mathbb{R}^2$ and $\zeta_{-m}(s) = (\zeta_{-m,1}(s), \zeta_{-m,2}(s)) \in \mathbb{R}^2$. By applying Heron's formula, one may define the function $F_m : S \rightarrow \mathbb{R}$ as

$$F_m(s) := 4(\zeta_{m,1}^2(s) + \zeta_{m,2}^2(s))(\zeta_{-m,1}^2(s) + \zeta_{-m,2}^2(s)) - (\zeta_{m,1}^2(s) + \zeta_{m,2}^2(s) \quad (16)$$

$$+ \zeta_{-m,1}^2(s) + \zeta_{-m,2}^2(s) - (\zeta_{m,1}(s) - \zeta_{-m,1}(s))^2 - (\zeta_{m,2}(s) - \zeta_{-m,2}(s))^2)^2. \quad (17)$$

From Proposition 1.2, we have that $\zeta(s) = 0$, if and only if $F_m(s) = 0$, for each $m \in \mathbb{Z}_+$. Hence, since $F_m(s) \geq 0$, we have that if $\zeta(s) = 0$, then $s \in \arg \min_{s' \in S} F_m(s')$, for each $m \in \mathbb{Z}_+$. Hence, if $\zeta(s) = 0$, one obtains that $\nabla_{s'} F_m(s) = 0$, for each $m \in \mathbb{Z}_+$.

The expectation operator defined previously also leads to another characterisation of the zeroes of the Riemann zeta function. Let $\mathfrak{Z} = \{s \in S : \zeta(s) = 0\}$ and $\mathfrak{M} = \{(X, \mu) \in l^\infty \times l^\infty : \mathbb{E}_\mu(X) = 0\}$. Then, \mathfrak{M} is nowhere dense in $l^\infty \times l^\infty$, with the topology of pointwise convergence. Further, the map $f : S \rightarrow l^\infty \times l^\infty$ defined by $f(s) = (X_s, \mu_s)$ is continuous. The next proposition characterises the behaviour of the zeroes of the Riemann zeta function on compact subsets of S .

Proposition 1.3. *Let $S' \subseteq S$ be a compact subset of S . Suppose $f(S') \cap \mathfrak{M}$ is nowhere dense in the subspace topology generated by $f(S')$. Then, $S' \cap \mathfrak{Z}$ is nowhere dense in S' . Hence, if ψ is a category measure on S' , it follows that $\psi(S' \cap \mathfrak{Z}) = 0$.*

Proof. By definition of f , we have that $f^{-1}(f(S') \cap \mathfrak{M}) = S' \cap \mathfrak{Z}$. Suppose for contradiction, $S' \cap \mathfrak{Z}$ is not nowhere dense. Then, this means that $\text{int}(\text{cl}(S' \cap \mathfrak{Z})) \neq \emptyset$. Hence, $\text{int}(\text{cl}(f^{-1}(f(S') \cap \mathfrak{M}))) \neq \emptyset$. Since S' is compact, the map $f|_{S'}$ is an open map from S' to $f(S')$, with the subspace topology on $f(S')$. Hence, the set $f(\text{int}(\text{cl}(f^{-1}(f(S') \cap \mathfrak{M}))))$ is an open set in $f(S')$. However, observe that $f(\text{int}(\text{cl}(f^{-1}(f(S') \cap \mathfrak{M})))) \subseteq f(\text{cl}(f^{-1}(f(S') \cap \mathfrak{M}))) \subseteq f(f^{-1}(\text{cl}(f(S') \cap \mathfrak{M}))) \subseteq \text{cl}(f(S') \cap \mathfrak{M})$ (the second inclusion follows since f is continuous). This is a contradiction, since we have that $\text{int}(\text{cl}(f(S') \cap \mathfrak{M})) = \emptyset$. \square

The Riemann zeta function, for values $\sigma > 1$, is given by the infinite sum

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s}. \tag{18}$$

In the next proposition, applying the perspectives presented above involving the expectations operator, we may show a proposition previously known, for the Riemann zeta function on this domain.

Proposition 1.4. *For $s = (\sigma, t)$ such that $\sigma \geq 2$, we have that $\zeta(s) \neq 0$.*

Proof. Define, on the set of positive integers, the probability measure $\mu_s(\{n\}) := \frac{(1/n)^\sigma}{\sum_{m \geq 1} (1/m)^\sigma}$ and the random variable $X_s(n) = (\cos(-t \ln(n)), \sin(-t \ln(n)))$. Note that

$$\mu_s(\{1\}) = \frac{1}{1 + \sum_{m \geq 2} (1/m)^\sigma} \tag{19}$$

$$\geq \frac{1}{1 + \sum_{m \geq 2} (1/m)^2} \tag{20}$$

$$> \frac{1}{1 + \sum_{m \geq 1} (1/2)^m} \tag{21}$$

$$= 1/2 \tag{22}$$

for the value $n = 1$, and we have that $X(\{1\}) = (1, 0)$ as $\ln(1) = 0$. Suppose, for contradiction that $\zeta(s) = 0$. Then, for the set $E = \{1\}$, we have that $\mathbb{E}_{\mu_s}[X_s] = \mu_s(E)\mathbb{E}_{\mu_s}[X_s|E] + (1 - \mu_s(E))\mathbb{E}_{\mu_s}[X_s|\mathbb{Z}^+ \setminus E] = 0$. Let $\alpha = \mu_s(E)$, $x = \mathbb{E}_{\mu_s}[X_s|E]$ and $z = \mathbb{E}_{\mu_s}[X_s|\mathbb{Z}^+ \setminus E]$. Since $\alpha > 1/2$, $x = (1, 0)$ and $z = (z, 0)$ such that $z \in [-1, 1]$, we have that $\alpha x + (1 - \alpha)z \neq 0$, which is a contradiction. \square

References

Bernhard Riemann. On the number of prime numbers below a given size. Ges. Math. Werke und Wissenschaftlicher Nachlass, 2:145-155, 1859