Original Research Article

Useful Extensions of the Juchez Probability Distribution: Properties and Application

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ABSTRACT

Inverse and Power Juchez Distribution were derived and studied in this paper as post-development of Juchez distribution. The two extensions serve to make up for the limitations of their baseline distribution. Some relevant properties: shape of the PDF, moments, survival function, hazard function, quantile function, stress-strength reliability, order statistics and parameter estimations were studied. The hazard shapes of the distributions were found to be - inverted bathtub shape for the IJD; and three different shapes namely: increasing function, decreasing function and bathtub shape, for PJD. This implies that the distributions can altogether model many varieties of datasets emanating from different life phenomena. This statement "if $S(\infty) = 0$, then $H(\infty) = \infty$" was examined and was discovered to apply for exponential distribution but not any of the extended distributions and the baseline distribution too. A generator for other generalized distributions termed Juchez-G was developed to suffice for the relevance of robust model development. Finally, IJD and PJD showed to be a better fit over both the baseline distribution and their respective counterpart distributions, with respect to the datasets used. For the PJD, the superiority was accountably hinged on the extra parameters; since the dataset has an outlier.

Keyword: Inverse transformation, power transformation, Juchez-G, hazard shapes, AIC, baseline distribution.

1. INTRODUCTION

Echebiri and Mbegbu (2022) developed Juchez distribution as a three component mixture of exponential and gamma distributions $g_i(x, \theta, \alpha)$ with suitable mixing proportions d_i using mixture model; where the gamma distribution is characterized by a constant scale parameter θ and two different shape parameters: $\alpha = 2$ and 4:

$$j(x, \theta) = d_1 g_1(x, \theta, 1) + d_2\, g_2(x, \theta, 2) + d_3\, g_3(x, \theta, 4) \tag{1}$$

$$= \frac{\theta^3}{\theta^3+\theta^2+6}\left\{\theta e^{-\theta x}\right\} + \frac{\theta^2}{\theta^3+\theta^2+6}\left\{\theta^\alpha x^{\alpha-1}\, e^{-\theta x}\right\}\Big|_{\alpha=2} + \frac{6}{\theta^3+\theta^2+6}\left\{\frac{\theta^\alpha x^{\alpha-1}\, e^{-\theta x}}{6}\right\}\Big|_{\alpha=4} \tag{2}$$

Consequently, the probability density function (Pdf) and cumulative distribution function (Cdf) of the Juchez distribution (JD) are, respectively, given as

$$j(x, \theta) = \frac{\theta^4}{\theta^3+\theta^2+6}\,(1 + x + x^3)e^{-\theta x}, \quad x > 0, \ \theta > 0 \tag{3}$$

$$J(x,\theta) = 1 - \left(1 + \frac{\theta x\,[\,\theta^2 + \theta^2 x^2 + 3\theta x + 6\,]}{\theta^3 + \theta^2 + 6}\right) e^{-\theta x} \quad (4)$$

The very many statistical features of JD were investigated, but remarkably, the comparative study on coefficient of variation showed that it is a flexible distribution. The distribution was applied in the modeling of online live-streaming and cancer data and it was found to be a better fit than some univariate one parameter distributions. JD exhibits a shape as increasing failure rate; hence, limited in a sense to modeling phenomenon emanating from other hazard rate types, say, decreasing failure rate, bathtub and or inverted bathtub shapes. However, some probability distributions with bathtub shape hazard function have been proposed for the analysis of lifetime data; notwithstanding, exhaustive work has not been done in the analysis of models with inverted bathtub shape hazard rate function. Very recently, Sharma (2015) discussed the use of inverted bathtub shape hazard rate function in real applications of the Inverse Lindley distribution. Keller and Kamath (1982) proposed the inverse exponential distribution, which can also model data sets with inverted bathtub shaped failure rate.

Furthermore, the limitations of inverse exponential distribution to properly model data sets that are highly skewed or that exhibit heavy tails was detailed in the development of Generalized Inverse Exponential distribution Abouammoh and Alshingiti (2009). Extra parameter which was added in the model, made it feasible to model for heavily tailed and highly skewed datasets. It is usually a conventional deduction that any inclusion of an extra parameter(s) into an existing distribution improves its ability to model data with wild observations. In like manner, Power Lindley distribution Ghitany (2013) was seen to model data sets that are also highly skewed and that exhibit heavy tails. As studied, the hazard rate function of Power Akash distribution has different shapes which include monotonically increasing and decreasing failure rate. Consequently, ample study is owed to be done as regarding other hazard shapes of distribution extensions like power distributions aside the increasing and decreasing failure rate.

Howbeit, generators in distributions are employed in the development of generalizations; which serve as models that meet the rising need of wide applications on data emanating from different real-life events. Some examples of these distribution family generators are the beta generalized family (Beta-G) Eugene (2002), Gamma-G (type 1) Zografos and Balakrishnan (2009), the Kumaraswamy-G Cordeiro and de Castro (2011), Gamma-G (type 2) Ristic (2012), Gamma-G (type 3) Torabi and Montazari (2012), Exponentiated-G (EG) Cordeiro (2013), Weibull-G Bourguignon (2014), Logistic-G Torabi and Montazari (2014), Lomax-G family by Cordeiro (2014), new Weibull-G family Tahir (2016), Lindley-G family Cakmakyapan and Ozel (2016), Gompertz-G family Alizadeh (2017) etc. It commonly can be deduced that the extra parameters these generated generalized distributions come with, enhances for greater flexibility, and hence could model data with wild observations. The development of generators from mixture model distributions is still a very rare adoption or development; which would improve the flexibility of distributions in this category.

The paper aims at introducing few extensions of Juchez probability distribution, which involve Inverse Juchez distribution (IJD) and Power Juchez Distribution (PJD), without leaving out their properties. They primarily would suffice for the limitation of the baseline distribution to model datasets from systems with other forms of failure rate apart from the increasing failure rate, as shown in the hazard function of the baseline distribution. More so, the paper proposes a distribution generator termed Juchez-G family of distribution, which could serve as a robust version of the baseline distribution in the modeling of datasets with wild observations.

2. FORMULATION OF JUCHEZ EXTENSIONS

Down through this study, we represent the pdf and cdf of Juchez distribution as $g(y)$ and $G(y)$ respectively, which form the *baseline distribution* in this study.

UNDER PEER REVIEW

$$g(y) = \frac{\theta^4}{\theta^3+\theta^2+6} \; (1 + y + y^3)e^{-\theta x}, y > 0, \; \theta > 0 \qquad (5)$$

$$G(y) = 1 - \left(1 + \frac{\theta y\,[\,\theta^2 + \theta^2 y^2 + 3\theta y + 6\,]}{\theta^3 + \theta^2 + 6}\right) e^{-\theta y} \qquad (6)$$

2.1 Inverse Distribution

Proposition 1: Let Y denote a non-negative continuous random variable such that; $Y \sim Juchez(x, \theta)$, then the cdf and pdf of the Inverse Juchez distribution (IJD) are derived thus:
Using the transformation $Y = \frac{1}{X}$ of a random variable, the technique for deriving its cdf and pdf are respectively given by:

$$F(x) = 1 - G(\tfrac{1}{x}) \qquad (7)$$

$$f(x) = \frac{1}{x^2}\, g\left(\frac{1}{x}\right) \qquad (8)$$

Now, a new inverted form of Juchez distribution is obtained by making substitutions accordingly in equations (1) and (2) following the cdf and pdf of Juchez Distribution.

$$I_{jd}(x,\theta) = 1 - \left[1 - \left(1 + \frac{\frac{\theta^3}{x} + \frac{\theta^3}{x^3} + \frac{3\theta^2}{x^2} + \frac{6\theta}{x}}{\theta^3 + \theta^2 + 6}\right) e^{-\frac{\theta}{x}}\right]$$

$$= \left[1 + \frac{\frac{\theta^3}{x} + \frac{\theta^3}{x^3} + \frac{3\theta^2}{x^2} + \frac{6\theta}{x}}{\theta^3 + \theta^2 + 6}\right] e^{-\frac{\theta}{x}} \quad \text{and} \qquad (9)$$

$$i_{jd}(x,\theta) = \frac{1}{x^2}\left[\frac{\theta^4}{\theta^3+\theta^2+6}\left(1 + \frac{1}{x} + \frac{1}{x^3}\right) e^{-\frac{\theta}{x}}\right]$$

$$i_{jd}(x,\theta) = \frac{\theta^4}{\theta^3+\theta^2+6}\left(\frac{1}{x^2} + \frac{1}{x^3} + \frac{1}{x^5}\right) e^{-\frac{\theta}{x}}, \quad x > 0, \; \theta > 0 \qquad (10)$$

The Inverse Juchez Distribution is a valid probability density function, that is: $\int f(x)dx = 1$
Proof.

$$\int f(x)dx = \int_0^\infty i_{jd}(x,\theta)dx = \int_0^\infty \frac{\theta^4}{\theta^3+\theta^2+6}\left(\frac{1}{x^2} + \frac{1}{x^3} + \frac{1}{x^5}\right) e^{-\frac{\theta}{x}}\, dx \qquad (11)$$

let $m = \frac{1}{x} \to x = \frac{1}{m}, \frac{\partial x}{\partial m} = \frac{1}{m^2} \to \partial x = \frac{\partial m}{m^2}$

$$= \int_0^\infty \frac{\theta^4}{\theta^3+\theta^2+6}\,(m^2 + m^3 + m^5)e^{-\theta m}\,\frac{dm}{m^2}$$

$$= \frac{\theta^4}{\theta^3+\theta^2+6}\int_0^\infty (1 + m + m^3)e^{-\theta m}\, dm$$

but $\int_0^\infty x^C e^{-\theta x}\,\text{dx} = \frac{\Gamma(C+1)}{\theta^{C+1}}$

Hence,

$$\int_0^\infty \frac{\theta^4}{\theta^3+\theta^2+6}\,(1 + m + m^3)e^{-\theta m}\,\text{dm}$$

$$= \frac{\theta^4}{\theta^3+\theta^2+6} * \frac{\theta^3+\theta^2+6}{\theta^4} = 1 \qquad (12)$$

UNDER PEER REVIEW

2.2 Properties of Inverse Juchez Distribution

2.2.1 Shape of Inverse Juchez Distribution

The first derivative of IJD-pdf at equation (10) gives the mode of the distribution:

$$\frac{d[i_{jd}(x,\theta)]}{dx} = \frac{d\left[\frac{\theta^4}{\theta^3+\theta^2+6}\left(\frac{1}{x^2}+\frac{1}{x^3}+\frac{1}{x^5}\right)e^{-\frac{\theta}{x}}\right]}{dx}$$

$$\theta^5[x^{-5}+x^{-3}+x^{-2}] - \theta^4[5x^{-4}+3x^{-2}+2x^{-1}] = 0 \qquad (13)$$

$$\left.\frac{d[i_{jd}(x,\theta)]}{dx}\right|_{x=M_0} = 0$$

2.2.2 Moments of Inverse Juchez Distribution

The r^{th} moment of the Inverse Juchez distribution is obtained

$$E(X^r) = \int_0^{\infty} x^r\, f(x)\, dx \qquad (14)$$

$$= \int_0^{\infty} x^r\, \frac{\theta^4}{\theta^3+\theta^2+6}\left(\frac{1}{x^2}+\frac{1}{x^3}+\frac{1}{x^5}\right)e^{-\frac{\theta}{x}}\, dx$$

$$E(X^r) = \frac{\theta^r\left[\theta^3-\theta^2(r-1)-(r-3)(r-2)(r-1)\right]\Gamma(1-r)}{(\theta^3+\theta^2+6)} \qquad (15)$$

$E(X^r)$ is conditional, valid at $r < 1$. Hence, $E\left(X^r|_{r=1,2,3\ldots\infty}\right)$ does not converge. This implies that mean and variance and other measures of dispersion are indeterminate.

2.2.3 Reliability Analysis of Inverse Juchez Distribution

The Survival function gives the description of the probability that a component will sustain after a given time; whereas, hazard function is the likelihood that a system will fail after a given period of time or cycle. The survival $S_{ijd}(x,\theta)$ and hazard rate models $H_{ijd}(x,\theta)$ of Inverse Juchez distribution are given as:

$$S(x,\theta) = P(X \geq x) = \int_x^{\infty} f(t)\, dt \qquad (16)$$

$$S_{ijd}(x,\theta) = 1 - I_{jd}(x,\theta)$$

$$= 1 - \left[1 + \frac{\frac{\theta^3}{x}+\frac{\theta^3}{x^3}+\frac{3\theta^2}{x^2}+\frac{6\theta}{x}}{\theta^3+\theta^2+6}\right]e^{-\frac{\theta}{x}} \qquad (17)$$

$$H_{ijd}(x,\theta) = \frac{i_{jd}(x,\theta)}{S_{ijd}(x,\theta)} = \frac{\theta^4[\theta^3+\theta^2+1]e^{-\frac{\theta}{x}}}{x^5\left[(\theta^3+\theta^2+6)-\left\{(\theta^3+\theta^2+6)+\frac{\theta^3}{x}+\frac{\theta^3}{x^3}+\frac{3\theta^2}{x^2}+\frac{6\theta}{x}\right\}e^{-\frac{\theta}{x}}\right]} \qquad (18)$$

In addition, Lawless (1982) derived the relationship between survival and hazard function as given by:

$$S_{ijd}(x) = e^{-[H_{ijd}(x)]} \qquad (19)$$

UNDER PEER REVIEW

He also stated that: "if $S(\infty) = 0$, then $H(\infty) = \infty$". These would be verified in the empirical section alongside the behavior of $S_{ijd}(x)$ *and* $H_{ijd}(x,\theta)\Big]_{\substack{x=0 \\ x=\infty}}$.

2.2.4 Parameter Estimation for Inverse Juchez Distribution

Let $X_i, i = 1,2,3,\ldots,n,$ be a random variable from Inverse Juchez Distribution, the log-likelihood function $ln\mathrm{L}f(x,\theta)$, is obtained as:

$$L(x,\theta) = \prod f(x,\theta) \tag{20}$$

$$L(x,\theta) = \left[\frac{\theta^4}{\theta^3+\theta^2+6}\right]^n \sum_i^n\left(\frac{1}{x^2} + \frac{1}{x^3} + \frac{1}{x^5}\right) e^{-\theta\sum_{i=1}^n\frac{1}{x_i}}$$

$$ln\mathrm{L}i_{jd}(x,\theta) = 4\mathrm{n}ln\theta - \mathrm{n}ln(\theta^3+\theta^2+6) + \sum_{i=1}^n \ln\left(\frac{1}{x^2} + \frac{1}{x^3} + \frac{1}{x^5}\right) - \theta\sum_{i=1}^n\frac{1}{x_i} \tag{21}$$

In estimation of MLE, the estimator is maximized at $\frac{\partial\, ln\mathrm{L}}{\partial\theta} = 0$

Therefore, $\frac{\partial\, ln\mathrm{L}i_{jd}(x,\theta)}{\partial\theta} = \frac{4n}{\theta} - \frac{n(3\theta^2+2\theta)}{\theta^3+\theta^2+6} - \sum_{i=1}^n\frac{1}{x_i} = 0$

$$\rightarrow \quad (\theta^3+2\theta^2+24) - (\theta^4+\theta^3+6\theta)\left(\frac{1}{\mathrm{n}}\sum_{i=1}^n\frac{1}{x_i}\right) = 0 \tag{22}$$

2.2.5 Quantile Function and Median of Inverse Juchez Distribution

The mathematical expression for quantile function is given as

$$F(x) = u \rightarrow x = F^{-1}(u) \tag{23}$$

Hence, engaging the cdf of IJD, we obtain thus:

$$u = I_{jd}(x,\theta) = \left[1 + \frac{\frac{\theta^3}{x} + \frac{\theta^3}{x^3} + \frac{3\theta^2}{x^2} + \frac{6\theta}{x}}{\theta^3+\theta^2+6}\right] e^{-\frac{\theta}{x}} \tag{24}$$

Taking the natural logarithm of both sides

$$\ln(u) = ln\left[1 + \frac{\frac{\theta^3}{x} + \frac{\theta^3}{x^3} + \frac{3\theta^2}{x^2} + \frac{6\theta}{x}}{\theta^3+\theta^2+6}\right] - \frac{\theta}{x}$$

The quantile of inverse Juchez Distribution is not an explicit expression and is given by:

$$ln\left[1 + \frac{\frac{\theta^3}{x} + \frac{\theta^3}{x^3} + \frac{3\theta^2}{x^2} + \frac{6\theta}{x}}{\theta^3+\theta^2+6}\right] - \ln(u) - \frac{\theta}{x} = 0 \tag{25}$$

Where $u \sim uniform(0,1)$

The median of inverse Juchez distribution can be obtained by substituting for $u = 0.5$

2.2.6 Stress-Strength Reliability of Inverse Juchez Distribution

The stress-strength reliability projects the description of a component's life, which has random strength X that is subjected to a random stress Y. When the stress applied to it overrides the strength, the system collapses at once, and it will function satisfactorily till $Y < X$. Therefore, $R = P(X > Y)$ is a measure of system reliability and in statistical literature it is termed stress-strength parameter. It is widely applied in various fields of life especially in structuring engineering, deterioration of rocket motors, aging of concrete pressure vessels etc.

UNDER PEER REVIEW

Let X and Y be independent strength and stress random variables having Inverse Juchez distribution with parameter $i_{jd}(x, \theta_1) and\, i_{jd}(x, \theta_2)$ respectively; then the stress-strength reliability R of IJD can be obtained as

$$R = P(X > Y) = \int_0^\infty P(X > Y|X = x) f(x) dx \tag{26}$$

$$= \int_0^\infty I_{jd}(x, \theta_2)\, i_{jd}(x, \theta_1)\, dx$$

$$= \int_0^\infty \left\{ \left[\left(1 + \frac{\frac{\theta_2{}^3}{x} + \frac{\theta_2{}^3}{x^3} + \frac{3\theta_2{}^2}{x^2} + \frac{6\theta_2}{x}}{\theta_2{}^3 + \theta_2{}^2 + 6} \right) e^{-\frac{\theta_2}{x}} \right] \left[\frac{\theta_1{}^4}{\theta_1{}^3 + \theta_1{}^2 + 6} \left(\frac{1}{x^2} + \frac{1}{x^3} + \frac{1}{x^5} \right) e^{-\frac{\theta_1}{x}} \right] \right\} dx$$

$$= \left\{ \frac{\theta_1{}^4}{(6 + \theta_1{}^2 + \theta_1{}^3)(\theta_1 + \theta_2)^7 (6 + \theta_2{}^2 + \theta_2{}^3)} \left[\begin{matrix} (6(\theta_1 + \theta_2)^3 (6 + (\theta_1 + \theta_2)^2 + (\theta_1 + \theta_2)^3) + \\ 6b(\theta_1 + \theta_2)^2 (24 + 2(\theta_1 + \theta_2)^2 + (\theta_1 + \theta_2)^3) + \\ \vartheta^2 (\theta_1 + \theta_2)(360 + 24(\theta_1 + \theta_2)^2 + 6(\theta_1 + \theta_2)^3 + \\ (\theta_1 + \theta_2)^4 + (\theta_1 + \theta_2)^5) + \theta_2{}^3 (720 + 48(\theta_1 + \theta_2)^2 + \\ 12(\theta_1 + \theta_2)^3 + 2(\theta_1 + \theta_2)^4 + 2(\theta_1 + \theta_2)^5 + (\theta_1 + \theta_2)^6)) \end{matrix} \right] \right\}, \ [\theta_1 + \theta_2] > 0 \tag{27}$$

2.2.7 Order Statistics (Inverse Juchez Distribution)

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from Inverse Juchez Distribution. Let $X_1 < X_2 <, \ldots, < X_n$ denote the corresponding order statistics. The pdf of the kth order statistics say $Y = X_k$ is given by:

$$f_{i:n} = \frac{n!}{(k-1)!(n-k)!} \sum_{l=0}^{n-k} (-1)^l \binom{n-k}{l} f(y) F^{k+l-1}(y) \tag{28}$$

Thus the pdf of kth order statistics can be expressed from equation (28) given by

$$f_{i:n} = \frac{n!\, \theta^4 (\frac{1}{x^2} + \frac{1}{x^3} + \frac{1}{x^5})\, e^{-\frac{\theta}{x}}}{(\theta^3 + \theta^2 + 6)(k-1)!(n-k)!} \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^l \left[1 - \left(1 + \frac{\theta x [\theta^2 + \theta^2 x^2 + 3\theta x + 6]}{\theta^3 + \theta^2 + 6} \right) e^{-\frac{\theta}{x(i+k+1)}} \right] \tag{29}$$

That implies that the pdf of minimum order statistics is obtained by substituting $j = k = 1$ in equation (29), to have:

$$f_{1:n} = \frac{n[\theta^4 (\frac{1}{x^2} + \frac{1}{x^3} + \frac{1}{x^5})\, e^{-\frac{\theta}{x}}]}{(\theta^3 + \theta^2 + 6)} \sum_{l=0}^{n-1} \binom{n-1}{l} (-1)^l \left[1 - \left(1 + \frac{\theta x [\theta^2 + \theta^2 x^2 + 3\theta x + 6]}{\theta^3 + \theta^2 + 6} \right) e^{-\frac{\theta}{x(i+2)}} \right] \tag{30}$$

While the corresponding pdf of maximum order statistics is obtained by making the substitution of $j = k = n$ in equation (29)

$$f_{n:n} = \frac{n[\theta^4 (\frac{1}{x^2} + \frac{1}{x^3} + \frac{1}{x^5})\, e^{-\frac{\theta}{x}}]}{(\theta^3 + \theta^2 + 6)} \left[1 - \left(1 + \frac{\theta x [\theta^2 + \theta^2 x^2 + 3\theta x + 6]}{\theta^3 + \theta^2 + 6} \right) e^{-\frac{\theta}{x(i+n+1)}} \right] \tag{31}$$

2.3 Power Juchez Distribution (PJD)

Proposition 2: Let Y denote a non-negative continuous random variable such that $Y \sim Juchez(y, \theta)$, then the cdf and pdf of the Power Juchez distribution are respectively obtained thus:

For the transformation $X = T^{\frac{1}{\varphi}} \rightarrow \mathrm{T} = X^{\varphi}$ of a random variable, whose cdf and pdf is given by:

UNDER PEER REVIEW

$$F(x) = G(x^{\varphi}) \tag{32}$$

$$f(x) = \frac{d}{dx}G(x^{\varphi}) = \varphi x^{\varphi-1}\, g(x^{\varphi})\,; \quad x > 0,\ \varphi > 0 \tag{33}$$

The pdf of Power Juchez distribution can be derived by direct substitution following equations (32) and (33);

$$p_{jd}(x,\theta,\varphi) = \varphi x^{\varphi-1}\frac{\theta^4}{\theta^3+\theta^2+6}\left(1 + x^{\varphi} + x^{3\varphi}\right)e^{-\theta x^{\varphi}} \tag{34}$$

$$P_{jd}(x,\theta,\varphi) = 1 - \left(1 + \frac{\theta^3 x^{\varphi} + \theta^3 x^{3\varphi} + 3\theta^2 x^{2\varphi} + 6\theta x^{\varphi}}{\theta^3 + \theta^2 + 6}\right)e^{-\theta x^{\varphi}} \tag{35}$$

Alternatively, it can be derived through mixture model as in equation (1):

$$f(x) = d_1 g_1(x,\theta,\varphi,1) + d_2\, g_2(x,\theta,\varphi,2) + d_3\, g_3(x,\theta,\varphi,4) \tag{36}$$

where $$d_1 = \frac{\theta^3}{\theta^3+\theta^2+6},\ d_2 = \frac{\theta^2}{\theta^3+\theta^2+6} + d_3 = \frac{6}{\theta^3+\theta^2+6} \tag{37}$$

$$g(x;\theta,\alpha) = \begin{cases} \frac{x^{\alpha-1}\theta^{\alpha}e^{-\theta x}}{\Gamma(\alpha)} & x > 0 \\ 0, & x < 0 \end{cases} \rightarrow g(x^{\varphi};\theta,\alpha) = \begin{cases} \frac{x^{\varphi(\alpha-1)}\theta^{\alpha}e^{-\theta x^{\varphi}}}{\Gamma(\alpha)} & x > 0 \\ 0, & x < 0 \end{cases} \tag{38}$$

But from equation (33)

$$\begin{aligned} \varphi x^{\varphi-1} \times g_1(x^{\varphi};\theta,1) &= \theta\varphi x^{\varphi-1}e^{-\theta x^{\varphi}} \\ \varphi x^{\varphi-1} \times g_2(x^{\varphi},\theta,2) &= \theta^2\varphi x^{2\varphi-1}\, e^{-\theta x^{\varphi}} \\ \varphi x^{\varphi-1} \times g_3(x^{\varphi},\theta,4) &= \frac{\theta^4\varphi x^{4\varphi-1}\, e^{-\theta x^{\varphi}}}{6} \end{aligned} \tag{39}$$

$$\rightarrow\ p_{jd}(x,\theta,\varphi) = \frac{\theta^3}{\theta^3+\theta^2+6}\left[\theta\varphi x^{\varphi-1}e^{-\theta x^{\varphi}}\right] + \frac{\theta^2}{\theta^3+\theta^2+6}\left[\theta^2\varphi x^{2\varphi-1}\, e^{-\theta x^{\varphi}}\right] + \frac{6}{\theta^3+\theta^2+6}\left[\frac{\theta^4\varphi x^{4\varphi-1}\, e^{-\theta x^{\varphi}}}{6}\right] \tag{40}$$

Simplifying the expression in equation (40) we have the pdf of Power Juchez distribution and the corresponding cdf.

$$p_{jd}(x,\theta,\varphi) = \frac{\varphi\theta^4}{\theta^3+\theta^2+6}\left(x^{\varphi-1} + x^{2\varphi-1} + x^{4\varphi-1}\right)e^{-\theta x^{\varphi}},\ x > 0,\ \theta > 0, \varphi > 0 \tag{41}$$

$$P_{jd}(x,\theta,\varphi) = 1 - \left(1 + \frac{\theta^3 x^{\varphi} + \theta^3 x^{3\varphi} + 3\theta^2 x^{2\varphi} + 6\theta x^{\varphi}}{\theta^3 + \theta^2 + 6}\right)e^{-\theta x^{\varphi}} \tag{42}$$

Using software, Mathematica, the validity of PJD was tested for $\int_0^{\infty} j(x,\theta)dx = 1$. It also suffices to state that: $\lim_{x\to\infty} F(x) = 1$

$$P_{jd}(x,\theta,\varphi) = 1 - \left(1 + \frac{\theta^3 x^{\varphi} + \theta^3 x^{3\varphi} + 3\theta^2 x^{2\varphi} + 6\theta x^{\varphi}}{\theta^3 + \theta^2 + 6}\right)e^{-\theta x^{\varphi}}$$

$$\lim_{x\to\infty} P_{jd}(x,\theta,\varphi) = 1 - (1 + \infty)0$$

$$\lim_{x\to\infty} P_{jd}(x,\theta,\varphi) = 1 \tag{43}$$

2.4 Properties of Power Juchez Distribution

UNDER PEER REVIEW

2.4.1 Shape of Power Juchez Distribution

The first derivative of PJD-Pdf at equation (41) gives the mode of the distribution:

$$\frac{d[p_{jd}(x,\theta,\varphi)]}{dx} = \frac{d\left[\frac{\varphi\theta^4}{\theta^3+\theta^2+6}\left(x^{\varphi-1}+x^{2\varphi-1}+x^{4\varphi-1}\right)e^{-\theta x^{\varphi}}\right]}{dx} \tag{44}$$

$$\{\theta^4\varphi[(\varphi-1)x^{\varphi-2} + (2\varphi-1)x^{2\varphi-2} + (4\varphi-1)x^{4\varphi-2}]\} - \{\theta^5\varphi^2 x^{\varphi-1}[x^{\varphi-1} + x^{2\varphi-1} + x^{4\varphi-1}]\} = 0 \tag{45}$$

$$\left.\frac{df(x)}{dx}\right|_{x\to Mode} = 0$$

2.4.2 Moments of Power Juchez Distribution

The r^{th} moment of the Power Juchez distribution is obtained

$$E(X^r) = \int_0^\infty x^r\, p_{jd}(x,\theta,\varphi)\ \ dx \tag{46}$$

$$E(X^r) = \frac{\theta^{-r/\varphi}\, r\left[(\theta^3+\theta^2+6)\varphi^3+(11+\theta^2)\varphi^2 r+6\varphi r^2+r^3\right]\Gamma\left({}^{r}/_{\varphi}\right)}{(\theta^3+\theta^2+6)\varphi^4} \tag{47}$$

$$\mu = \frac{\theta^{-1/\varphi}\left[(\theta^3+\theta^2+6)\varphi^3+(11+\theta^2)\varphi^2+6\varphi+1\right]\Gamma\left({}^{1}/_{\varphi}\right)}{(\theta^3+\theta^2+6)\varphi^4} \tag{48}$$

$$\mu_2' = \frac{\theta^{-2/\varphi}\, 2\left[(\theta^3+\theta^2+6)\varphi^3+(11+\theta^2)2\varphi^2+24\varphi+8\right]\Gamma\left({}^{2}/_{\varphi}\right)}{(\theta^3+\theta^2+6)\varphi^4} \tag{49}$$

$$\rightarrow \quad \mu_2 = \mu_2' - \mu^2 = \sigma^2 \tag{50}$$

2.4.3 Reliability Analysis of Power Juchez Distribution

The survival function $S_{pjd}(x,\theta,\varphi)$ and hazard rate function $H_{pjd}(x,\theta,\varphi)$ of PJD can be obtained thus:

$$S_{pjd}(x,\theta,\varphi) = 1 - P_{jd}(x,\theta,\varphi) = 1 - \left[1 - \left(1 + \frac{\theta^3 x^{\varphi}+\theta^3 x^{3\varphi}+3\theta^2 x^{2\varphi}+6\theta x^{\varphi}}{\theta^3+\theta^2+6}\right)e^{-\theta x^{\varphi}}\right] \tag{51}$$

$$= \left(1 + \frac{\theta^3 x^{\varphi}+\theta^3 x^{3\varphi}+3\theta^2 x^{2\varphi}+6\theta x^{\varphi}}{\theta^3+\theta^2+6}\right)e^{-\theta x^{\varphi}} \tag{52}$$

$$H_{pjd}(x,\theta,\varphi) = \frac{p_{jd}(x,\theta,\varphi)}{S_{pjd}(x,\theta,\varphi)} = \frac{\frac{\varphi\theta^4}{\theta^3+\theta^2+6}\left[x^{\varphi-1}+x^{2\varphi-1}+x^{4\varphi-1}\right]e^{-\theta x^{\varphi}}}{\left(1+\frac{\theta^3 x^{\varphi}+\theta^3 x^{3\varphi}+3\theta^2 x^{2\varphi}+6\theta x^{\varphi}}{\theta^3+\theta^2+6}\right)e^{-\theta x^{\varphi}}} \tag{53}$$

$$= \frac{\varphi\theta^4(x^{\varphi-1}+x^{2\varphi-1}+x^{4\varphi-1})}{(\theta^3+\theta^2+6)+[\theta^3 x^{\varphi}+\theta^3 x^{3\varphi}+3\theta^2 x^{2\varphi}+6\theta x^{\varphi}]} \tag{54}$$

2.4.4 Order Statistics (Power Juchez Distribution)

Let $X_1, X_2, \dots, X_n$ be a random sample of size n from Power Juchez Distribution. Let $X_1 < X_2 <, \dots, < X_n$ denote the corresponding order statistics. The pdf of the kth order statistics say $Y = X_k$ is given by:

$$f_{i:n} = \frac{n!}{(k-1)!(n-k)!}\sum_{l=0}^{n-k}(-1)^l\binom{n-k}{l}f(y)F^{k+l-1}(y) \tag{55}$$

Thus the pdf of kth order statistics can be expressed from equation (55) given by

UNDER PEER REVIEW

$$f_{i:n} = \left\{ \begin{array}{c} \frac{\mathrm{n}!\,\varphi\theta^4\left(x^{\varphi-1}+x^{2\varphi-1}+x^{4\varphi-1}\right)\,e^{-\theta x^{\varphi}}}{(\theta^3+\theta^2+6)(k-1)!(n-k)!} \sum_{l=0}^{n-k}\binom{n-k}{l}(-1)^l \\ \left[\left(1+\frac{\theta^3x^{\varphi}+\theta^3x^{3\varphi}+3\theta^2x^{2\varphi}+6\theta x^{\varphi}}{\theta^3+\theta^2+6}\right)e^{-\theta x^{\varphi}}\right] \end{array} \right\} \quad (56)$$

That implies that the pdf of minimum order statistics is obtained by substituting $j = k = 1$ in equation (56), to have:

$$f_{1:n} = \left\{ \begin{array}{c} \frac{\mathrm{n}\left[\theta^4\left(x^{\varphi-1}+x^{2\varphi-1}+x^{4\varphi-1}\right)\,e^{-\theta x^{\varphi}}\right]}{(\theta^3+\theta^2+6)} \sum_{l=0}^{n-1}\binom{n-1}{l}(-1)^l \\ \left[\left(1+\frac{\theta^3x^{\varphi}+\theta^3x^{3\varphi}+3\theta^2x^{2\varphi}+6\theta x^{\varphi}}{\theta^3+\theta^2+6}\right)e^{-\theta x^{\varphi}}\right] \end{array} \right\} \quad (57)$$

While the corresponding pdf of maximum order statistics is obtained by making the substitution of $j = k = n$ in equation (56)

$$f_{n:n} = \frac{\mathrm{n}\left[\theta^4\left(x^{\varphi-1}+x^{2\varphi-1}+x^{4\varphi-1}\right)\,e^{-\theta x^{\varphi}}\right]}{(\theta^3+\theta^2+6)}\left[\left(1+\frac{\theta^3x^{\varphi}+\theta^3x^{3\varphi}+3\theta^2x^{2\varphi}+6\theta x^{\varphi}}{\theta^3+\theta^2+6}\right)e^{-\theta x^{\varphi}}\right] \quad (58)$$

2.4.5 Quantile and Median of Power Juchez Distribution

The model in equation (23) also applies here; hence

$$u = 1-\left(1+\frac{\theta^3x^{\varphi}+\theta^3x^{3\varphi}+3\theta^2x^{2\varphi}+6\theta x^{\varphi}}{\theta^3+\theta^2+6}\right)e^{-\theta x^{\varphi}} \quad (59)$$

$$1-u = \left(1+\frac{\theta^3x^{\varphi}+\theta^3x^{3\varphi}+3\theta^2x^{2\varphi}+6\theta x^{\varphi}}{\theta^3+\theta^2+6}\right)e^{-\theta x^{\varphi}}$$

$$ln(1-u) = ln\left(1+\frac{\theta^3x^{\varphi}+\theta^3x^{3\varphi}+3\theta^2x^{2\varphi}+6\theta x^{\varphi}}{\theta^3+\theta^2+6}\right)-\theta x^{\varphi}$$

Thus the quantile for the simulation of samples from power juchez distribution is:

$$ln\left(1+\frac{\theta^3x^{\varphi}+\theta^3x^{3\varphi}+3\theta^2x^{2\varphi}+6\theta x^{\varphi}}{\theta^3+\theta^2+6}\right)-ln(1-u)-\theta x^{\varphi} = 0 \quad (60)$$

Where $u \sim uniform(0,1)$, and at $u = 0.5$, the quantile expression becomes the median of PJD.

2.4.6 Parameter Estimation for Power Juchez Distribution

Let $X_i, i = 1,2,3,\ldots,n$, be a random variable from Power Juchez Distribution, then the log-likelihood function $lnLf(x,\theta,\varphi)$, is obtained as:

$$Lf(x,\theta) = \prod_{i=1}^{n} f(x,\theta) \quad (61)$$

$$ln\mathrm{L}p_{jd}(x,\theta,\varphi) = 4\mathrm{n}ln\theta + \mathrm{n}ln\varphi - \mathrm{n}ln(\theta^3+\theta^2+6) + \sum_{i=1}^{n}\ln\left(x_i^{\varphi-1}+x_i^{2\varphi-1}+x_i^{4\varphi-1}\right)-\theta\sum_{i=1}^{n}x_i^{\varphi} \quad (62)$$

The MLE's are maximized at $\frac{\partial\, ln\mathrm{L}p_{jd}(x,\theta,\varphi)}{\partial\theta} = 0$ and $\frac{\partial\, ln\mathrm{L}p_{jd}(x,\theta,\varphi)}{\partial\varphi} = 0$

UNDER PEER REVIEW

$$\frac{\partial\, ln\mathrm{L}p_{jd}(x,\theta,\varphi)}{\partial\theta} = \frac{4n}{\theta} - \frac{n(3\theta^2+2\theta)}{\theta^3+\theta^2+6} - \sum_{i=1}^{n} x_i{}^{\varphi} \tag{63}$$

$$\frac{\partial\, ln\mathrm{L}f(x,\theta,\varphi)}{\partial\varphi} = \frac{n}{\varphi} + \sum_{i=1}^{n} \frac{x_i{}^{\varphi-1}\, lnx_i + x_i{}^{2\varphi-1} lnx_i + x_i{}^{4\varphi-1} lnx_i}{x_i{}^{\varphi-1} + x_i{}^{2\varphi-1} + x_i{}^{4\varphi-1}} - \theta\sum_{i=1}^{n} x_i{}^{\varphi}\ \ln x_i \tag{64}$$

Since $\frac{\partial(x^{\varphi})}{\partial\varphi} = x^{\varphi}\, lnx$

The likelihood equations in (63) and (64) can easily be solved iteratively using Fisher's scoring method; since at $\frac{\partial\, ln\mathrm{L}p_{jd}(x,\theta,\varphi)}{\partial\theta} = 0$ and $\frac{\partial\, ln\mathrm{L}p_{jd}(x,\theta,\varphi)}{\partial\varphi} = 0$ cannot be expressed as a closed form equation. We have thus:

$$\frac{\partial^2\, ln\mathrm{L}p_{jd}(x,\theta,\varphi)}{\partial\theta\,\partial\varphi} = \frac{\partial^2\, ln\mathrm{L}p_{jd}(x,\theta,\varphi)}{\partial\varphi\partial\theta} = \sum_{i=1}^{n} x_i{}^{\varphi}\ \ln x_i \tag{65}$$

$$\frac{\partial^2\, ln\mathrm{L}p_{jd}(x,\theta,\varphi)}{\partial\theta^2} = -\frac{4n}{\theta^2} + \left\{-\frac{n(3\theta^2+2\theta)^2}{[\theta^3+\theta^2+6]^2} + \frac{n(2+6\theta)^2}{\theta^3+\theta^2+6}\right\} \tag{66}$$

$$\frac{\partial^2\, ln\mathrm{L}p_{jd}(x,\theta,\varphi)}{\partial\varphi^2} = \sum_{i=1}^{n}\left\{\begin{matrix}\left[\frac{(x_i{}^{\varphi-1}\, lnx_i + x_i{}^{2\varphi-1} lnx_i + x_i{}^{4\varphi-1} lnx_i)(x_i{}^{\varphi-1}\, lnx_i + 2x_i{}^{2\varphi-1} lnx_i + 4x_i{}^{4\varphi-1} lnx_i)}{(x_i{}^{\varphi-1} + x_i{}^{2\varphi-1} + x_i{}^{4\varphi-1})^2}\right] + \\ \left[\frac{x_i{}^{\varphi-1}\, lnx_i{}^2 + x_i{}^{2\varphi-1} lnx_i{}^2 + x_i{}^{4\varphi-1} lnx_i{}^2}{x_i{}^{\varphi-1} + x_i{}^{2\varphi-1} + x_i{}^{4\varphi-1}}\right]\end{matrix}\right\} \tag{67}$$

Resolving the following matrix equations, the solutions of MLE $(\hat{\theta},\ \hat{\varphi})$ for $p_{jd}(x,\theta,\varphi)$ are obtained:

$$\begin{bmatrix}\frac{\partial^2\, ln\mathrm{L}p_{jd}(x,\theta,\varphi)}{\partial\theta^2} & \frac{\partial^2\, ln\mathrm{L}p_{jd}(x,\theta,\varphi)}{\partial\theta\,\partial\varphi} \\ \frac{\partial^2\, ln\mathrm{L}p_{jd}(x,\theta,\varphi)}{\partial\theta\,\partial\varphi} & \frac{\partial^2\, ln\mathrm{L}p_{jd}(x,\theta,\varphi)}{\partial\varphi^2}\end{bmatrix}\begin{bmatrix}\hat{\theta} = \theta_0 \\ \hat{\varphi} = \varphi_0\end{bmatrix} = \begin{bmatrix}\frac{\partial\, ln\mathrm{L}p_{jd}(x,\theta,\varphi)}{\partial\theta} \\ \frac{\partial\, ln\mathrm{L}p_{jd}(x,\theta,\varphi)}{\partial\varphi}\end{bmatrix} \tag{68}$$

where $\theta_0\ and\ \varphi_0$ are initial values of $\theta\ and\ \varphi$.

2.5 Juchez-G Family of Distribution

Proposition 3: Employing Gamma-G Type III or Weibull-G model Tahir (2016), with the link function $\frac{C(x)}{1-C(x)}$, for distributions with range $(0,\infty)$, Juchez-G family of distribution is derived thus:

$$F(x) = \int_0^{\frac{C(x)}{1-C(x)}} g(t)dt \tag{69}$$

$$f(x) = \frac{d}{dx}\{F(x)\} \tag{70}$$

Since $g(t)$ represents the baseline distribution in equation (3),

$$F(x) = \int_0^{\frac{C(x)}{1-C(x)}} \frac{\varphi^4}{\varphi^3+\varphi^2+6}\,(1+t+t^3)e^{-\varphi t}dt \tag{71}$$

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$$F(x) = 1 - \left(1 + \frac{\varphi^3\left[\frac{C(x)}{1-C(x)}\right] + \varphi^3\left[\frac{C(x)}{1-C(x)}\right]^3 + 3\varphi^2\left[\frac{C(x)}{1-C(x)}\right]^2 + 6\varphi\left[\frac{C(x)}{1-C(x)}\right]}{\varphi^3 + \varphi^2 + 6}\right) e^{-\varphi\left[\frac{C(x)}{1-C(x)}\right]} \tag{72}$$

Hence, the corresponding pdf derived from $\frac{d}{dx}\{F(x)\}$ is given thus:

$$f(x) = \frac{\varphi^4}{\varphi^3+\varphi^2+6}\left[\frac{c(x)}{[1-C(x)]^2}\right](1 + C(x) + [C(x)]^3)\, e^{-\varphi\left[\frac{C(x)}{1-C(x)}\right]}; \quad \varphi > 0 \tag{73}$$

where $c(x)$ and $C(x)$ can be pdf and pdf respectively, of any probability distribution of concern. More flexible distributions can further be developed from Juchez-G Distribution.

3. EMPIRICAL INVESTIGATIONS

Having proposed three different models, it is however necessary to note that the PJD has the baseline distribution as a sub-model at parameter value $\varphi = 1$.

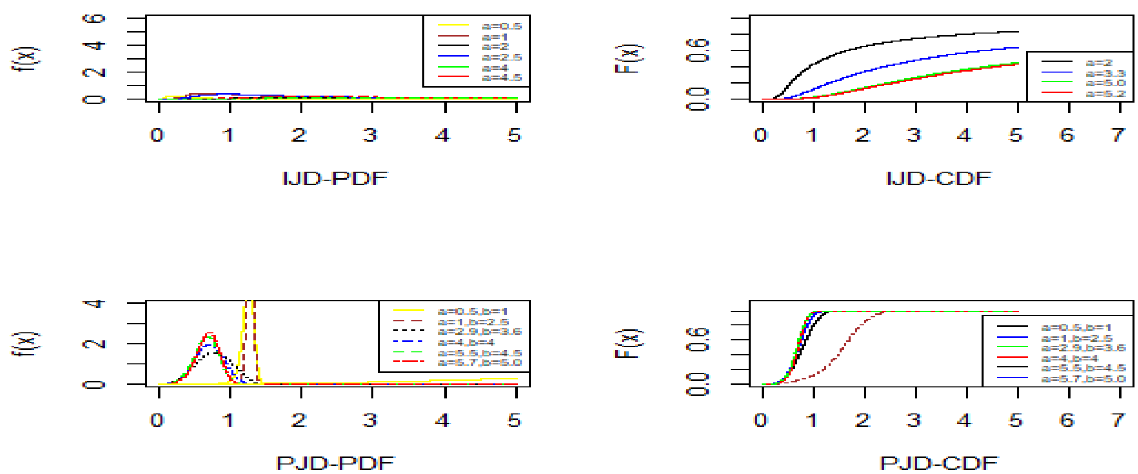

Figure 1. The PDF and CDF plot of the Juchez extended distributions

It could be deduced from figure 1, that the shape of the pdfs are unimodal and positively skewed, except for the inverse distribution that exhibits kurtosis at any parameter value; whereas the cdf is an increasing function, having its convergence at 1.

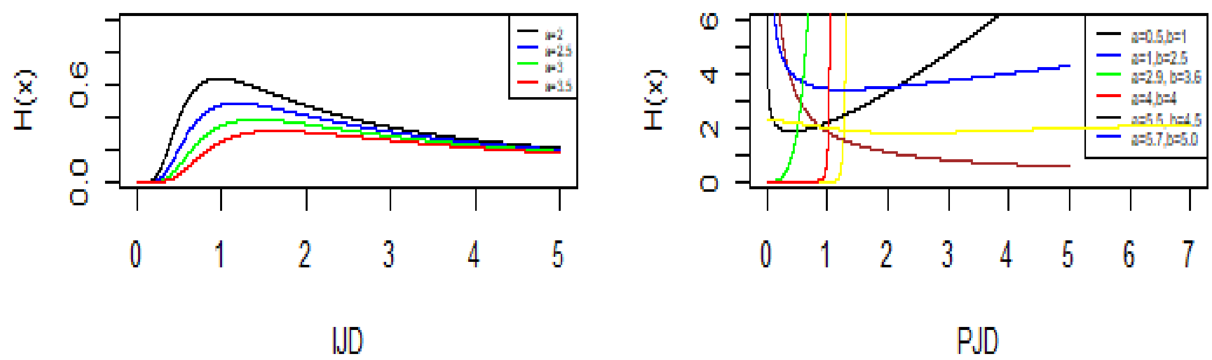

Figure 2. The Hazard Plot for the Juchez extended distributions

Figure 2, brings to light the hazard shapes of the IJD and PJD. The hazard function for IJD shows an inverted bathtub shape; whereas PJD reveals three different shapes namely: bathtub or unimodal shape, increasing and decreasing failure rate shape.

Empirically from equations (17), (18), (52) and (54), it can be deduced that

$$S_{ijd}(x,\theta) = \begin{cases} 0, & x = \infty \\ \blacksquare, & x = 0 \end{cases} \quad \text{and} \quad H_{ijd}(x,\theta) = \begin{cases} 0, & x = \infty \\ \blacksquare, & x = 0 \end{cases}. \tag{74}$$

More so,

$$S_{pjd}(x,\theta) = \begin{cases} 1, & x = 0 \\ \blacksquare, & x = \infty \end{cases} \quad \text{and} \quad H_{pjd}(x,\theta) = \begin{cases} 0, & x = 0 \\ \blacksquare, & x = 0 \end{cases}. \tag{75}$$

By careful observation, the statement: "if $S(\infty) = 0$, then $H(\infty) = \infty$" by Lawless (1982) applies for exponential distribution; but does not apply for these distributions likewise Lindley and Juchez distributions. However, equations (74) and (75) are also consistent with the relationship between hazard and survival function: $S(x) = e^{-[H(x)]}$, as given in equation (19).

Hence,

$$S_{ijd}(x) = \begin{cases} 0, & x = \infty \\ \blacksquare, & x = 0 \end{cases} \tag{76}$$

This refers to the time or settings where an individual or a system, fails. Following the same relationship,

$$S_{pjd}(x) = \begin{cases} 1, & x = 0 \\ \blacksquare, & x = \infty \end{cases} \tag{77}$$

where $\blacksquare$ implies "indeterminate".

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Table 1. Statistical Table for the PDF of IJD and PJD Distribution ($\theta = 0.1$ to $\theta = 0.5$)

	IJD				PJD $[\varphi = 0.5]$			
X	$\theta = 0.1$	$\theta = 0.25$	$\theta = 0.35$	$\theta = 0.5$	$\theta = 0.1$	$\theta = 0.25$	$\theta = 0.35$	$\theta = 0.5$
1	0.000045	0.001501	0.005146	0.01784	0.000022	0.000750	0.002573	0.008920
2	0.000008	0.000284	0.001021	0.00382	0.000027	0.000836	0.002750	0.008960
3	0.000003	0.000109	0.000401	0.00154	0.000032	0.000954	0.003038	0.009438
4	0.000002	0.000057	0.000209	0.00081	0.000037	0.001072	0.003324	0.009918
5	0.0000009	0.000034	0.000127	0.00050	0.000043	0.001184	0.003587	0.010332
6	0.0000006	0.000023	0.000085	0.00033	0.000048	0.001290	0.003825	0.010671
7	0.0000004	0.000016	0.000061	0.00024	0.000053	0.001389	0.004038	0.010939
8	0.0000003	0.000012	0.000046	0.00018	0.000059	0.001482	0.004230	0.011147
9	0.0000002	0.000009	0.000035	0.00014	0.000064	0.001568	0.004400	0.011302
10	0.0000002	0.000007	0.000028	0.00011	0.000069	0.001649	0.004553	0.011411
11	0.0000002	0.000006	0.000023	0.00009	0.000073	0.001725	0.004689	0.011485
12	0.0000001	0.000005	0.000019	0.00008	0.000078	0.001796	0.004811	0.011525
13	0.0000001	0.000004	0.000016	0.00006	0.000083	0.001862	0.004919	0.011537
14	0.00000009	0.000004	0.000014	0.00006	0.000087	0.001925	0.005015	0.011524
15	0.00000008	0.000003	0.000012	0.00005	0.000092	0.001984	0.005101	0.011491
16	0.00000007	0.000003	0.000010	0.00004	0.000096	0.002039	0.005177	0.011444
17	0.00000006	0.000002	0.0000092	0.00004	0.00010	0.002091	0.005244	0.011380
18	0.00000006	0.000002	0.0000082	0.00003	0.00010	0.002140	0.005303	0.011303
19	0.00000005	0.000002	0.0000073	0.00003	0.00011	0.002186	0.005354	0.011216
20	0.00000005	0.000002	0.0000066	0.00003	0.00011	0.002229	0.005399	0.011119
21	0.00000004	0.000002	0.0000059	0.00002	0.00012	0.002270	0.005437	0.011015
22	0.00000004	0.000001	0.0000054	0.00002	0.00012	0.002309	0.005471	0.010904
23	0.00000003	0.000001	0.0000049	0.00002	0.00012	0.002345	0.005499	0.010787
24	0.00000003	0.000001	0.0000045	0.00002	0.00013	0.002379	0.005522	0.010667
25	0.00000002	0.000001	0.0000041	0.00002	0.00013	0.002412	0.005541	0.010542

Carefully studying the patterns in Table 1, across the parameters, down the x-values, we deduce that the distributions are unimodal and positively skewed. These explicitly agree with the shape of their pdfs. More so, the probability outcomes at different values of x, is consistent with the probability axiom: $0 < P(x) < 1$.

The performance comparisons involve the baseline distribution and the extended distributions with other counterpart distributions. It is note-worthy to express that the distribution which corresponds to the lowest AIC, BIC and or highest log-likelihood is considered the best fit.

Data set I: The waiting time (in minutes) of one hundred (100) bank customers before service is being rendered, Ghitany (2008).

UNDER PEER REVIEW

0.8, 0.8, 1.3, 1.5, 1.8, 1.9, 1.9, 2.1, 2.6, 2.7, 2.9, 3.1, 3.2, 3.3, 3.5, 3.6, 4.0, 4.1, 4.2, 4.2, 4.3, 4.3, 4.4, 4.4, 4.6, 4.7, 4.7, 4.8, 4.9, 4.9, 5, 5.3, 5.5, 5.7, 5.7, 6.1, 6.2, 6.2, 6.2, 6.3, 6.7, 6.9, 7.1, 7.1, 7.1, 7.1, 7.4, 7.6, 7.7, 8, 8.2, 8.6, 8.6, 8.6, 8.8, 8.8, 8.9, 8.9, 9.5, 9.6, 9.7, 9.8, 10.7, 10.9, 11, 11, 11.1, 11.2, 11.2, 11.5, 11.9, 12.4, 12.5, 12.9, 13, 13.1, 13.3, 13.6, 13.7, 13.9, 14.1, 15.4, 15.4, 17.3, 17.3, 18.1, 18.2, 18.4, 18.9, 19, 19.9, 20.6, 21.3, 21.4, 21.9, 23.0, 27, 31.6, 33.1, 38.5

Table 2. Descriptive Statistics for Dataset 1

n	$\bar{x}$	σ^2	σ	Outlier Threshold $(3 \times \sigma) \pm \bar{x}$	Minimum	Maximum
100	9.877	52.37	7.24	(11.83, 31.59)	0.8	38.5

Data set II: The tensile strength, measured in GPa, of 69 carbon fibers tested under tension at gauge length of 20mm reported in Ghitany (2013).

1.312, 1.314, 1.479, 1.552, 1.700, 1.803, 1.861,1.865, 1.944,1.958,1.966,1.997,2.006,2.021,2.027, 2.055, 2.063, 2.098,2.14, 2.179, 2.224, 2.240, 2.253, 2.270, 2.272, 2.274, 2.301, 2.301, 2.359, 2.382, 2.382, 2.426,2.434, 2.435,2.478, 2.490, 2.511,2.514, 2.535, 2.554, 2.566, 2.57, 2.586, 2.629, 2.633, 2.642, 2.648, 2.684, 2.697, 2.726, 2.770,2.773,2.800, 2.809, 2.818, 2.821,2.848, 2.88,2.954, 3.012, 3.067, 3.084, 3.090, 3.096, 3.128, 3.233, 3.433,3.585, 4.594.

Table 3. Descriptive Statistics for Dataset 2

N	$\bar{x}$	σ^2	σ	Outlier Threshold $(3 \times \sigma) \pm \bar{x}$	Minimum	Maximum
69	2.4660	0.2936	0.5418	(+4.0941, - 0.8405)	1.312	4.594

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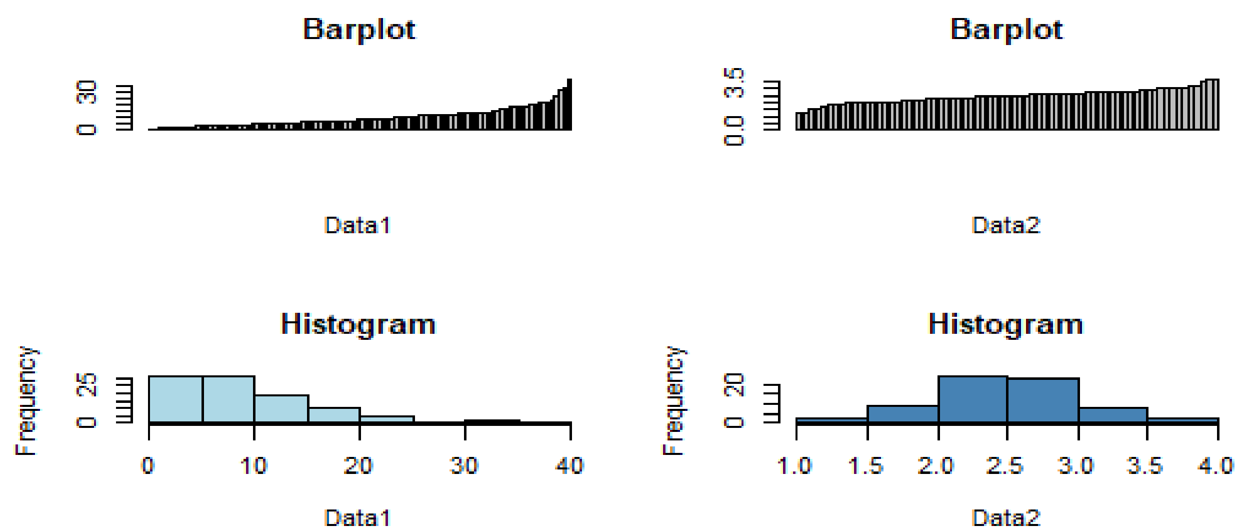

Figure 3. A graphical summary of the datasets 1 & 2

The bar-plot and the histogram used here as graphical summary, gives an insight that the datasets are heavily skewed to the right or positively skewed. This is suggestive that the two extension distribution could sustain the capacity to model them.

Table 4. Performance rating for the IJD and the baseline distribution

			Data Set 1		
Model	**Parameter Estimate**	**lnL**	**AIC**	**BIC**	**Rank**
Juchez	θ = 0.3968	-329.59	661.18	663.79	2
IJD	θ = 6.3736	-325.94	653.80	656.50	1

Table 5. Performance rating for the PJD and the baseline distribution

			Data Set 2		
Model	**Parameter Estimate**	**lnL**	**AIC**	**BIC**	**Rank**
Juchez	θ = 1.2429	-108.04	218.08	220.31	2
PJD	θ = 0.4016 φ = 2.4335	-55.037	114.07	118.54	1

For the datasets 1 & 2, as shown in Tables 4 and 5, the extended distributions showed to have better fit than the baseline distribution.

Table 6. Performance rating for the IJD and counterpart distributions

			Data Set 1		
Model	**Parameter Estimate**	**lnL**	**AIC**	**BIC**	**Rank**
IJD	θ = 6.3736	-325.94	653.80	656.50	1
ILD	θ = 6.1007	-336.62	675.24	677.85	3

UNDER PEER REVIEW

IED	θ = 5.3474	-336.56	675.12	677.72	2
IAD	θ = 2.3698	-399.45	800.91	803.51	4

The distributions compared alongside Inverse Juchez Distribution (IJD) are Inverse Lindley distribution (ILD), Inverse exponential distribution (IED) and Inverse Aishenawy distribution (IAD). Table 6 shows that the IJD exhibits better fit with respect to the applied data.

Table 7. The performance rating for the PJD and counterpart distributions

		Data Set 2			
Model	**Parameter Estimate**	**lnL**	**AIC**	**BIC**	**Rank**
PJD	θ = 0.4016 φ = 2.4335	-55.037	114.07	118.54	1
PLD	θ = 0.0739 φ = 3.4041	-56.557	117.11	121.58	3
PAD	θ = 0.2171 φ = 2.7661	-55.406	114.81	119.28	2
PED	θ = 0.0116 φ = 4.5229	-58.997	121.99	126.46	4

In this performance comparison, the baseline distribution and other counterpart power distributions were taken in consideration alongside PJD. These distributions are power Lindley distribution, power akash

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distribution and power exponential distribution (which is also known as Weibull distribution). In Table 7, it is clearly shown that PJD is a better fit compared to some power distributions and owing to the data used.

DISCUSSION AND CONCLUSION

The development of extensions of Juchez distribution was considered in this paper. The inverse and the power transformation approach were adopted to form new distributions termed "Inverse Juchez Distribution and Power Juchez Distribution". These distributions were proven both mathematically and empirically, to be valid distributions; where the baseline distribution is a special case of the power Juchez distribution at $\varphi = 1$. Properties like distribution mode, moment, mean and variance, survival and hazard functions, parameter estimation, quantile function, stress-strength reliability and order statistics were derived. The hazard shape for IJD is that of an inverted or upside-down bathtub shape; this implies that IJD can model outcomes that improve with time or use. PJD on the other hand exhibits three different shapes at various parameter values: bathtub shape, decreasing and increasing failure rate shape. This means that PJD is one such a flexible distribution that can model different sets of data. Inclusive, are such that wear out with time and vice versa, and finally, those that exhibit early failure. More so, the pdf of the PJD is highly skewed and this is suggestive that it can model effectively data from wild observations. Methodically, a generalized family generator called "Juchez-G" is proposed and could be used to generate other flexible distributions. Highlights on the performance comparison: the IJD and PJD show to be better fit than both the baseline distribution and or their counterpart distributions. PJD having extra parameter suffices for the limitation of the baseline in modeling data with outliers as observed in dataset 2 and Table 3 and 5.

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