Attractors and numerical simulations for a two-temperature phase transition system

Abstract

In this paper, we study a generalisation of the Caginalp phase field system obtained by considering the Fourier law as the heat conduction law and involving two temperatures. In the first section, we are interested in the existence and uniqueness of solutions. In section 2, we address the question of the asymptotic behaviour of the solutions, in particular with the demonstration of the existence of a global attractor. For this, we needed to show that the system is dissipative and to know the regularity of the solutions. Finally, in the last section, we turn to the numerical solution of our problem.

Keywords:

Caginalp phase-field system, two temperatures, global attractor, semigroup decomposition method.

MSC:35K55, 80A22, 35B40, 35B45, 35Q79(secondary)80A

Introduction

In this paper, we study a generalisation of the Caginalp phase field system based on a two-temperature theory and the usual Fourier law. It is presented as follows :

$$\frac{\partial u}{\partial t} - \Delta u + f(u) = \varphi - \Delta\varphi, \tag{0.1}$$

$$\frac{\partial \varphi}{\partial t} - \Delta\frac{\partial \varphi}{\partial t} - \Delta\varphi = -\frac{\partial u}{\partial t}, \tag{0.2}$$

$$u = \varphi = 0 \text{ on } \partial\Omega, \tag{0.3}$$

$$u|_{t=0} = u_0,\ \varphi|_{t=0} = \varphi_0. \tag{0.4}$$

This model is derived from the Caginalp phase-field system

$$\frac{\partial u}{\partial t} - \Delta u + f(u) = \theta, \tag{0.5}$$

$$\frac{\partial \theta}{\partial t} - \Delta \theta = -\frac{\partial u}{\partial t}, \tag{0.6}$$

where θ is the difference between the absolute temperature and the equilibrium melting temperature, u the phase-field or order parameter and f the derivative function of a double-well potential F. The system (0.5)-(0.6) was introduced in [8] to describe the phase transition phenomena in materials. Many works have been done on this model (see [1], [2], [6], [7], [12], [13], [14], [19], [20], [21], [22], [23],[28], [39] and [40],). The equations (0.5)-(0.6) are obtained by using the total Ginzburg-Landau free energy and the enthalpy of the system. Indeed, the total Ginzburg-Landau free energy is defined as follows:

$$\psi(u, \theta) = \int_\Omega \left(\frac{1}{2}|\nabla u|^2 + F(u) - u\theta - \frac{1}{2}\theta^2 \right) dx, \tag{0.7}$$

where Ω is the domain occupied by the system. It is assumed to be a bounded domain of $\mathbb{R}^n$, $n \leq 3$ with smooth boundary $\partial\Omega$. To obtain the evolution equation of the order parameter u, i.e. the equation (0.5), it is sufficient to calculate :

$$\frac{\partial u}{\partial t} = -\delta_u \psi, \tag{0.8}$$

where δ_u stands for the variational derivative with respect to u. To describe the state of such a system, we define the physical quantity called enthalpy. It is a thermodynamic potential that consists of the internal energy of the system (related to its temperature and momentum) and the work of the forces acting on its boundary. In our case, the enthalpy is given as follows:

$$H = u + \theta. \tag{0.9}$$

In addition, we have the energy equation

$$\frac{\partial H}{\partial t} = -\text{div}\, q, \tag{0.10}$$

where q is the heat flux. Assuming finally the classical Fourier law for heat conduction, which defines the heat flux as

$$q = -\nabla \theta, \tag{0.11}$$

we obtain (0.6).

It is important to point out that the use of Fourier's law in the model is a questionable choice. Indeed, Fourier's law leads to the propagation of heat at an infinite speed (see, for example, [15]). Several alternatives to Fourier's law have been explored. We can mention for example the Maxwell-Cattaneo law or laws from thermomechanics (see, for example, [4], [24], [25], [26], [27], [29], [30], [33], [34], [35], [36] and [37]).

Moreover, with the appearance of non-simple materials, a different approach to thermal conduction was proposed in the 1960s (see [9], [10] and [11]). Indeed, in this type of material, two temperatures intervene in the definition of the entropy: the conduction temperature θ, influencing the contribution of thermal conduction, and the thermodynamic temperature φ, appearing in the heat input part. In general, if in stationary models these two temperatures coincide in the absence of heat input, this is not the case in evolutionary models where they are generally different (see, for example, [16]). In this work, we make the hypothesis of being in a non-simple material. The two temperatures are then linked as follows (see [17], [18]):

$$\theta = \varphi - \Delta\varphi. \tag{0.12}$$

The paper is organized as follows in section 1, we give the existence and uniqueness of the solutions. In section 2, we address the question of the asysmptotic behaviour of the solutions, especially with the existence of attractors. Finally, in the last section, we present some numerical simulations.

In this article, the letters c, c', c'', and sometimes c''' denote constants which may change from one line to another. Similarly, $\|.\|_p$ will denote the norm L^p and $(.,.)$ the usual scalar product L^2. We will denote by $\|.\|_X$ the norm in the Banach space X. Finally, when there is no possible confusion, we will note $\|.\|$ instead of $\|.\|_2$.

1 Well-Posedness of the model

In this section we show the existence and uniqueness conditions of the solutions of our model. To do this, we will first make a number of assumptions.

Assumptions

$$-c_0 \leq F(s) \leq f(s)s + c_1, \quad c_0, c_1 \geq 0,\ s \in \mathbb{R},\ \text{and } F(s) = \int_0^s f(\tau)\,d\tau, \tag{1.1}$$

$$|f'(s)| \leq c_2(|s|^{2p} + 1), \quad c_2, p \geq 1, \quad s \in \mathbb{R}, \tag{1.2}$$

$$f' \geq -c_3, \quad c_3 \geq 0. \tag{1.3}$$

1.1 Existence

Theorem 1.1. *Let $T > 0$ be given. We assume that $u_0 \in H_0^1(\Omega)$, $\varphi_0 \in H^2(\Omega) \cap H_0^1(\Omega)$ and $F(u_0) < +\infty$. Then, under assumptions (1.1)-(1.2), the problem (0.1)-(0.4) admits at least one solution denoted (u, φ) such as $u \in L^\infty(\mathbb{R}_+; H_0^1(\Omega))$, $\varphi \in L^\infty(\mathbb{R}_+; H^2(\Omega) \cap H_0^1(\Omega))$, $\frac{\partial u}{\partial t} \in L^2((0,T) \times \Omega)$ and $\frac{\partial \varphi}{\partial t} \in L^2(0,T; H_0^1(\Omega))$.*

Proof. We simply perform formal calculations to obtain a priori estimates. The rigorous justification of these estimates can be obtained by the Galerkin scheme (see [19], [20] and [38] for more details).
Multiplying (0.1) by $\frac{\partial u}{\partial t}$ and integrating over Ω, we have

$$\frac{1}{2}\frac{d}{dt}\left(\|\nabla u\|^2 + 2\int_\Omega F(u)\,dx\right) + \left\|\frac{\partial u}{\partial t}\right\|^2 = \int_\Omega (\varphi - \Delta\varphi)\frac{\partial u}{\partial t}\,dx. \tag{1.4}$$

Multiplying (0.2) by $\varphi - \Delta\varphi$, we have, integrating through Ω

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\|\varphi\|^2 + 2\|\nabla\varphi\|^2 + \|\Delta\varphi\|^2\right) \\ &\qquad + \|\nabla\varphi\|^2 + \|\Delta\varphi\|^2 = -\int_\Omega (\varphi - \Delta\varphi)\frac{\partial u}{\partial t}\,dx. \end{aligned} \tag{1.5}$$

Summing up (1.4) and (1.5), we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\|\nabla u\|^2 + 2\int_\Omega F(u)\,dx + \|\varphi\|^2 + 2\|\nabla\varphi\|^2\right. \\ &\qquad \left. + \|\Delta\varphi\|^2\right) + \left\|\frac{\partial u}{\partial t}\right\|^2 + \|\nabla\varphi\|^2 + \|\Delta\varphi\|^2 = 0. \end{aligned} \tag{1.6}$$

Now, multplying (0.1) by u and integrating through Ω, one has

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 + \|\nabla u\|^2 + \int_\Omega f(u)u\,dx = \int_\Omega u\varphi\,dx + \int_\Omega \nabla u \nabla\varphi\,dx. \tag{1.7}$$

Applying Cauchy-Schwarz's, Poincare's and Young's inequalities, we obtain

$$\begin{aligned} \int_\Omega u\varphi\,dx &\leq \|u\|\|\varphi\| \\ &\leq c\|\nabla u\|\|\nabla\varphi\| \\ &\leq \frac{1}{4}\|\nabla u\|^2 + c\|\nabla\varphi\|^2 \end{aligned} \tag{1.8}$$

and

$$\begin{aligned}\int_\Omega \nabla u \nabla\varphi\,dx &\le \|\nabla u\|\|\nabla\varphi\| \\ &\le \frac{1}{4}\|\nabla u\|^2 + c\|\nabla\varphi\|^2.\end{aligned} \tag{1.9}$$

Collecting (1.7)-(1.9) and in view of (1.1), we arrive at

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 + \frac{1}{2}\|\nabla u\|^2 + \int_\Omega F(u)\,dx \le c\|\nabla\varphi\|^2 + c_1|\Omega|. \tag{1.10}$$

Multiplying (0.2) by $\frac{\partial\varphi}{\partial t}$ and integrating through Ω, we get

$$\frac{1}{2}\frac{d}{dt}\|\nabla\varphi\|^2 + \left\|\frac{\partial\varphi}{\partial t}\right\|^2 + \left\|\nabla\frac{\partial\varphi}{\partial t}\right\|^2 = -\int_\Omega \frac{\partial u}{\partial t}\frac{\partial\varphi}{\partial t}\,dx. \tag{1.11}$$

Cauchy-Schwarz's and Young's inequalities yield

$$\frac{1}{2}\frac{d}{dt}\|\nabla\varphi\|^2 + \frac{1}{2}\left\|\frac{\partial\varphi}{\partial t}\right\|^2 + \left\|\nabla\frac{\partial\varphi}{\partial t}\right\|^2 \le \frac{1}{2}\left\|\frac{\partial u}{\partial t}\right\|^2. \tag{1.12}$$

Summing (1.6), ε(1.10) and (1.12), where $\varepsilon > 0$ is as small as we need, we then get

$$\begin{aligned}&\frac{dE_1}{dt} + \varepsilon\|\nabla u\|^2 + 2\varepsilon\int_\Omega F(u)\,dx + 2\|\nabla\varphi\|^2 + 2\|\Delta\varphi\|^2 \\ &+\left\|\frac{\partial u}{\partial t}\right\|^2 + \left\|\frac{\partial\varphi}{\partial t}\right\|^2 + 2\left\|\nabla\frac{\partial\varphi}{\partial t}\right\|^2 \le 2\varepsilon c\|\nabla\varphi\|^2 + 2\varepsilon c_1|\Omega|,\end{aligned} \tag{1.13}$$

where

$$E_1 = \varepsilon\|u\|^2 + \|\nabla u\|^2 + 2\int_\Omega F(u)\,dx + \|\varphi\|^2 + 3\|\nabla\varphi\|^2 + \|\Delta\varphi\|^2 \tag{1.14}$$

satifies

$$E_1 \ge c(\|u\|^2_{H^1_0(\Omega)} + \|\varphi\|^2_{H^2(\Omega)}). \tag{1.15}$$

Now, choosing $\varepsilon > 0$ such that $1 - \varepsilon c > 0$, we are led to an inequality of the form

$$\frac{dE_1}{dt} + cE_1 + \left\|\frac{\partial u}{\partial t}\right\|^2 + c'\left\|\frac{\partial\varphi}{\partial t}\right\|^2_{H^1_0(\Omega)} \le c''. \tag{1.16}$$

An application of Gronwall's Lemma completes the proof. □

1.2 Uniqueness

Theorem 1.2. *We then have under conditions of Theorem 1.1 that the problem* (0.1)-(0.4) *possesses a unique solution* (u, φ) *with the above regularity.*

Remark 1.3. *In the case where the dimension* n *is equal to* 3*, we will assume that* $p = 1$ *in* (1.2)*.*

Proof. We suppose the existence of two solutions (u_1, φ_1) and (u_2, φ_2) to problem (0.1)-(0.3) associated to initial conditions (u_{01}, φ_{01}) and (u_{02}, φ_{02}), respectively. Setting $u = u_1 - u_2$, $\varphi = \varphi_1 - \varphi_2$, $u_0 = u_{01} - u_{02}$ and $\varphi_0 = \varphi_{01} - \varphi_{02}$. We then have that the couple (u, φ) solves the following problem

$$\frac{\partial u}{\partial t} - \Delta u + f(u_1) - f(u_2) = \varphi - \Delta\varphi, \tag{1.17}$$

$$\frac{\partial\varphi}{\partial t} - \Delta\frac{\partial\varphi}{\partial t} - \Delta\varphi = -\frac{\partial u}{\partial t}, \tag{1.18}$$

$$u|_{\partial\Omega} = \varphi|_{\partial\Omega} = 0, \tag{1.19}$$

$$u|_{t=0} = u_0, \quad \varphi|_{t=0} = \varphi_0 \tag{1.20}$$

Multiplying (1.17) by $\frac{\partial u}{\partial t}$ and integrating over Ω, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla u\|^2+\left\|\frac{\partial u}{\partial t}\right\|^2+\int_\Omega\Big(f(u_1)-f(u_2)\Big)\frac{\partial u}{\partial t}\,dx\\=\int_\Omega\Big(\varphi-\Delta\varphi\Big)\frac{\partial u}{\partial t}\,dx.\end{aligned}\tag{1.21}$$

Multiplying (1.18) by $\varphi-\Delta\varphi$, we have, integrating over Ω

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\Big(\|\varphi\|^2+2\|\nabla\varphi\|^2+\|\Delta\varphi\|^2\Big)+\|\nabla\varphi\|^2+\|\Delta\varphi\|^2\\=-\int_\Omega\Big(\varphi-\Delta\varphi\Big)\frac{\partial u}{\partial t}\,dx.\end{aligned}\tag{1.22}$$

Summing (1.21) and (1.22), one obtains

$$\frac{dE_2}{dt}+2\bigg(\left\|\frac{\partial u}{\partial t}\right\|^2+\|\nabla\varphi\|^2+\|\Delta\varphi\|^2\bigg)=-2\int_\Omega\Big(f(u_1)-f(u_2)\Big)\frac{\partial u}{\partial t}\,dx,\tag{1.23}$$

where

$$E_2=\|\nabla u\|^2+\|\varphi\|^2+2\|\nabla\varphi\|^2+\|\Delta\varphi\|^2.\tag{1.24}$$

Now, considering (1.2), and applying Hölder's inequality for $p=1$, when $n=3$, we can write

$$\begin{aligned}\int_\Omega\Big(f(u_1)-f(u_2)\Big)\frac{\partial u}{\partial t}\,dx&\le c\int_\Omega(|u_2|^{2p}+1)|u|\left|\frac{\partial u}{\partial t}\right|\,dx\\&\le c(\|\nabla u_2\|^{4p}+1)\|\nabla u\|^2+\left\|\frac{\partial u}{\partial t}\right\|^2.\end{aligned}\tag{1.25}$$

Finally, collecting (1.23)-(1.25), we are led to

$$\frac{dE_2}{dt}+\left\|\frac{\partial u}{\partial t}\right\|^2\le c(\|\nabla u_2\|^{4p}+1)E_2.\tag{1.26}$$

In particular,

$$\frac{dE_2}{dt}\le cE_2\tag{1.27}$$

and an application of the Gronwall's Lemma completes the proof. □

2 Asymptotic behaviour of solutions

In this section we particularly make the following assumption:

$$f(0)=0.\tag{2.1}$$

Dissipativity properties of the system

Remark 2.1. *It follows from theorems 1.1 and 1.2 that we can define the family solving operators*

$$\begin{aligned}S(t):&\Phi\longrightarrow\Phi\\&(u_0,\varphi_0)\longmapsto(u(t),\varphi(t)),\quad t\ge 0,\end{aligned}\tag{2.2}$$

where $\Phi=H_0^1(\Omega)\times H^2(\Omega)\cap H_0^1(\Omega)$ and (u,φ) is the unique solution to the problem (0.1)-(0.4). Besides, this family of solving operators forms a contnuous semigroup i.e. $S(0)=Id$ and $S(t+\tau)=S(t)\circ S(\tau)$, $\forall\, t,\tau\ge 0$.

Theorem 2.2. *Under the assumptions of Theorems 1.1 and 1.2, the semigroup $S(t)$ is dissipative on Φ. In other words, the semigroup $S(t)$ has a bounded absorbing set $\mathcal{B}$ in Φ.*

Proof. By Multiplying equation (0.1) by $\frac{\partial u}{\partial t}$ and integrating over Ω, we obtain

$$\int_\Omega |\frac{\partial u}{\partial t}|^2\,dx - \int_\Omega \Delta u\frac{\partial u}{\partial t}\,dx + \int_\Omega f(u)\frac{\partial u}{\partial t}\,dx = \int_\Omega (\varphi-\Delta\varphi)\frac{\partial u}{\partial t}\,dx. \tag{2.3}$$

So

$$-\int_\Omega \Delta u\frac{\partial u}{\partial t}\,dx = \int_\Omega \nabla u\nabla\frac{\partial u}{\partial t}\,dx. \tag{2.4}$$

Moreover

$$\int_\Omega \nabla u\nabla\frac{\partial u}{\partial t}\,dx = \frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla u|^2\,dx. \tag{2.5}$$

In the same way,

$$\frac{d}{dt}\int_\Omega F(u)\,dx = \int_\Omega \frac{\partial F(u)}{\partial u}\frac{\partial u}{\partial t}\,dx = \int_\Omega f(u)\frac{\partial u}{\partial t}\,dx. \tag{2.6}$$

The equalities (2.4), (2.5) and (2.6) transform equality (2.3) into

$$\int_\Omega |\frac{\partial u}{\partial t}|^2\,dx + \frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla u|^2\,dx + \frac{d}{dt}\int_\Omega F(u)\,dx = \int_\Omega (\varphi-\Delta\varphi)\frac{\partial u}{\partial t}\,dx. \tag{2.7}$$

By Multiplying equation (0.2) by $(\varphi-\Delta\varphi)$ and integrating over Ω, we get

$$\frac{1}{2}\frac{d}{dt}\left(\int_\Omega |\varphi|^2\,dx + 2\int_\Omega |\nabla\varphi|^2\,dx + \int_\Omega |\Delta\varphi|^2\,dx\right) + \int_\Omega |\nabla\varphi|^2\,dx + \int_\Omega |\Delta\varphi|^2\,dx = -\int_\Omega (\varphi-\Delta\varphi)\frac{\partial u}{\partial t}\,dx. \tag{2.8}$$

Summing up the equalities (2.7) and (2.8), we have

$$\frac{1}{2}\frac{d}{dt}\left(\int_\Omega |\varphi|^2\,dx + 2\int_\Omega |\nabla\varphi|^2\,dx + \int_\Omega |\Delta\varphi|^2\,dx + 2\int_\Omega F(u)\,dx\right) + \int_\Omega |\frac{\partial u}{\partial t}|^2\,dx + \int_\Omega |\nabla\varphi|^2\,dx + \int_\Omega |\Delta\varphi|^2\,dx = 0. \tag{2.9}$$

Also, multiplying (0.1) by u then integrating through Ω, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u|^2\,dx + \int_\Omega |\nabla u|^2\,dx + \int_\Omega f(u)u\,dx = \int_\Omega (\varphi-\Delta\varphi)u\,dx. \tag{2.10}$$

So

$$F(u) \le f(u)u + c_1. \tag{2.11}$$

Then (2.10) becomes

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u|^2\,dx + \int_\Omega |\nabla u|^2\,dx + \int_\Omega F(u)\,dx - c_1|\Omega| \le \int_\Omega (\varphi-\Delta\varphi)u\,dx. \tag{2.12}$$

In addition, we have

$$\int_\Omega (\varphi-\Delta\varphi)u\,dx = \int_\Omega \varphi u\,dx + \int_\Omega \nabla\varphi\nabla u\,dx. \tag{2.13}$$

In the following, by applying the Cauchy-Schwaz inequality, we obtain

$$\begin{aligned}\int_\Omega (\varphi-\Delta\varphi)u\,dx &\le \left(\int_\Omega |\varphi|^2\,dx\right)^{\frac{1}{2}}\left(\int_\Omega |u|^2\,dx\right)^{\frac{1}{2}} + \left(\int_\Omega |\nabla\varphi|^2\,dx\right)^{\frac{1}{2}}\left(\int_\Omega |\nabla u|^2\,dx\right)^{\frac{1}{2}} \\ . &\le \frac{1}{4}\|\nabla u\|^2 + c\|\nabla\varphi\|^2 + \frac{1}{4}\|\nabla u\|^2 + c'\|\nabla\varphi\|^2\end{aligned} \tag{2.14}$$

Hence (2.12) gives

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u|^2\,dx + \int_\Omega |\nabla u|^2\,dx + \int_\Omega F(u)\,dx - c_1|\Omega| \le \frac{1}{2}\|\nabla u\|^2 + c\|\nabla\varphi\|^2. \tag{2.15}$$

Or

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 + \frac{1}{2}\|\nabla u\|^2 + \int_\Omega F(u)\,dx \le c\|\nabla\varphi\|^2 + c_1|\Omega|. \tag{2.16}$$

Similarly, multiplying (0.2) by $\frac{\partial\varphi}{\partial t}$, gives

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla\varphi|^2\,dx + \int_\Omega |\frac{\partial\varphi}{\partial t}|^2\,dx + \int_\Omega |\nabla\frac{\partial\varphi}{\partial t}|^2\,dx = -\int_\Omega \frac{\partial u}{\partial t}\frac{\partial\varphi}{\partial t}\,dx. \tag{2.17}$$

Yet,

$$\int_\Omega \frac{\partial u}{\partial t}\frac{\partial\varphi}{\partial t}\,dx \le \|\frac{\partial u}{\partial t}\|\|\frac{\partial\varphi}{\partial t}\| \le \frac{1}{2}\|\frac{\partial u}{\partial t}\|^2 + \frac{1}{2}\|\frac{\partial\varphi}{\partial t}\|^2. \tag{2.18}$$

Thus

$$\frac{1}{2}\frac{d}{dt}\|\nabla\varphi\|^2 + \frac{1}{2}\|\frac{\partial\varphi}{\partial t}\|^2 + \|\nabla\frac{\partial\varphi}{\partial t}\|^2 \le \frac{1}{2}\|\frac{\partial u}{\partial t}\|^2 \tag{2.19}$$

For small enough $\varepsilon > 0$, we sum (2.9), ε(2.16) and (2.19), we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left(\varepsilon\|u\|^2 + \|\nabla u\|^2 + \|\varphi\|^2 + 3\|\nabla\varphi\|^2 + \|\Delta\varphi\|^2 + 2\int_\Omega F(u)\,dx\right) + \|\frac{\partial u}{\partial t}\|^2 + \|\nabla\varphi\|^2 + \|\Delta\varphi\|^2 +\\ \frac{\varepsilon}{2}\|\nabla u\|^2 + \varepsilon\int_\Omega F(u)\,dx + \frac{1}{2}\|\frac{\partial\varphi}{\partial t}\|^2 + \|\nabla\frac{\partial\varphi}{\partial t}\|^2 \le C\varepsilon\|\nabla\varphi\|^2 + c_1\varepsilon|\Omega| + \frac{1}{2}\|\frac{\partial u}{\partial t}\|^2\end{aligned} \tag{2.20}$$

Let's put

$$E_1 = \varepsilon\left(\|u\|^2 + \|\nabla u\|^2 + \|\varphi\|^2 + 3\|\nabla\varphi\|^2 + \|\Delta\varphi\|^2 + 2\int_\Omega F(u)\,dx\right) \tag{2.21}$$

Then (2.20) becomes

$$\frac{1}{2}\frac{dE_1}{dt} + cE_1 + c'\left(\|\frac{\partial u}{\partial t}\|^2 + \|\nabla\frac{\partial\varphi}{\partial t}\|^2_{H^1(\Omega)}\right) \le c_1\varepsilon|\Omega| \tag{2.22}$$

From (2.22), we derive that

$$\frac{1}{2}\frac{dE_1}{dt} + cE_1 \le c_1\varepsilon|\Omega| \tag{2.23}$$

By multiplying (2.23) by e^{Kt}, we obtain From (2.22), we derive that

$$\frac{d}{dt}(E_1e^{Kt}) \le 2c_1\varepsilon|\Omega|e^{Kt} \tag{2.24}$$

Then, we integrate (2.24) on $[0,t]$, we have the following result

$$E_1(t) \le E_1(0)e^{-Kt} + C \tag{2.25}$$

Let u_0 and $\varphi_0 \in B(0,R)$, then

$$E_1(t) \le \mu(R)e^{-Kt} + C \tag{2.26}$$

Let ρ such that

$$\mu(R)e^{-Kt} + C \le \rho \tag{2.27}$$

From (2.27), we derive that

$$t_0 = max(0, \frac{1}{K}\ln(\frac{\mu(R)}{\rho - C})) \tag{2.28}$$

□

2.1 Regularity

Theorem 2.3. *Under conditions of Theorem1.1 and assuming that* (1.3) *and* (2.1) *hold true. Then* $u \in L^2(0,T;H^2(\Omega))$, $T>0$.

Proof. Multiplying this time (0.1) by $-\Delta u$ and integrating over Ω, we get

$$\frac{1}{2}\frac{d}{dt}\|\nabla u\|^2 + \|\Delta u\|^2 + \int_\Omega f'(u)|\nabla u|^2\,dx = \int_\Omega \nabla u \nabla\varphi\,dx + \int_\Omega \Delta u\Delta\varphi\,dx \tag{2.29}$$

and Cauchy-Schwarz's inequality yields taking into account (1.3)

$$\frac{1}{2}\frac{d}{dt}\|\nabla u\|^2 + \|\Delta u\|^2 \le c'\|\nabla u\|^2 + c''(\|\nabla\varphi\|^2 + \|\Delta\varphi\|^2). \tag{2.30}$$

Summing (1.16) and δ(2.30), where $\delta>0$ is small enough and then choosing δ such that $c-\delta c', c-\delta c''$, we arrive at an inequality of the type

$$\frac{dE_3}{dt} + cE_3 + c'\|\Delta u\|^2 + \left\|\frac{\partial u}{\partial t}\right\|^2 + \left\|\frac{\partial\varphi}{\partial t}\right\|^2 + \left\|\nabla\frac{\partial\varphi}{\partial t}\right\|^2 \le c'', \tag{2.31}$$

where

$$E_3 = E_1 + \delta\|\nabla u\|^2 \tag{2.32}$$

enjoys similar estimate as E_1. It clearly appears from (2.31) that $u\in L^2(0,T;H^2(\Omega))$. □

Theorem 2.4. *We suppose that* (1.1) *and* (2.1) *hold true and that* $p=1$ *in* (1.2) *when* $n=3$*. Then, the solution* (u,φ) *to problem* (0.1)-(0.4) *has the following regularity* $u\in L^\infty(\mathbb{R}_+;H^2(\Omega)\cap H_0^1(\Omega))$, $\varphi\in L^\infty(\mathbb{R}_+;H^3(\Omega)\cap H_0^1(\Omega))$, $\frac{\partial u}{\partial t}\in L^2(0,T;H_0^1(\Omega))$ *and* $\frac{\partial\varphi}{\partial t}\in L^2(0,T;H^2(\Omega)\cap H_0^1(\Omega))$*, provived that* $(u_0,\varphi_0)\in H^2(\Omega)\cap H_0^1(\Omega)\times H^3(\Omega)\cap H_0^1(\Omega)$ *and* $F(u_0)<+\infty$.

Proof. The proof leans on Gronwall's lemma. Indeed, thanks to (1.16), we observe that

$$\int_t^{t+r}\|\Delta u(s)\|^2\,ds \le a(r),\ \forall\, t\ge t_0,\ r>0, \tag{2.33}$$

for $(u_0,\varphi_0)\in B$, where $B\subset H^2(\Omega)\cap H_0^1(\Omega)$ is a bounded set and t_0 is such that $t\ge t_0$ implies $S(t)B\subset\mathbb{B}_0$, $\mathbb{B}_0$ being the bounded and absorbing set obtained in the theorem 1.1. It appears always from theorem 1.1 that

$$u\in L^\infty(\mathbb{R}_+;H_0^1(\Omega)). \tag{2.34}$$

We also have from theorem 2.4 that

$$u\in L^2(0,T;H^2(\Omega)),\ T>0. \tag{2.35}$$

In the other hand, multiplying (0.1) by $-\Delta\frac{\partial u}{\partial t}$ and (0.2) by $\Delta^2\varphi$, we have, integrating over Ω

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\Delta u\|^2 + \left\|\nabla\frac{\partial u}{\partial t}\right\|^2 &+ \int_\Omega f'(u)\nabla u\nabla\frac{\partial u}{\partial t}\,dx\\ &= \int_\Omega \nabla\varphi\nabla\frac{\partial u}{\partial t}\,dx + \int_\Omega \Delta\varphi\Delta\frac{\partial u}{\partial t}\,dx\end{aligned} \tag{2.36}$$

and

$$\frac{1}{2}\frac{d}{dt}(\|\Delta\varphi\|^2 + \|\nabla\Delta\varphi\|^2) + \|\nabla\Delta\varphi\|^2 = -\int_\Omega \Delta\varphi\Delta\frac{\partial u}{\partial t}, \tag{2.37}$$

respectively.

Now, the sum of (2.36) and (2.37) yields

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\|\Delta u\|^2+\|\Delta\varphi\|^2+\|\nabla\Delta\varphi\|^2)+\|\nabla\Delta\varphi\|^2+\left\|\nabla\frac{\partial u}{\partial t}\right\|^2\\ =\int_\Omega \nabla\varphi\nabla\frac{\partial u}{\partial t}\,dx-\int_\Omega f'(u)\nabla u\nabla\frac{\partial u}{\partial t}\,dx.\end{aligned} \tag{2.38}$$

Thanks to (1.2) and Hölder's inequality

$$\begin{aligned}\int_\Omega f'(u)\nabla u\nabla\frac{\partial u}{\partial t}\,dx &\le c\int_\Omega(|u|^{2p}+1)|\nabla u|\left|\nabla\frac{\partial u}{\partial t}\right|\,dx\\ &\le c\|u\|^{4p}_{H^1(\Omega)}(\|u\|^2_{H^2(\Omega)}+1)+\frac{1}{3}\left\|\nabla\frac{\partial u}{\partial t}\right\|^2,\end{aligned} \tag{2.39}$$

as a result

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\|\Delta u\|^2+\|\Delta\varphi\|^2+\|\nabla\Delta\varphi\|^2)+\|\nabla\Delta\varphi\|^2+\frac{1}{2}\left\|\nabla\frac{\partial u}{\partial t}\right\|^2\\ \le c\|u\|^{4p}_{H^1(\Omega)}(\|u\|^2_{H^2(\Omega)}+1)+c'\|\nabla\varphi\|^2.\end{aligned} \tag{2.40}$$

Adding (2.31) and (2.40), one gets

$$\frac{dE_4}{dt}+\left\|\frac{\partial u}{\partial t}\right\|^2+\left\|\nabla\frac{\partial u}{\partial t}\right\|^2+\left\|\frac{\partial\varphi}{\partial t}\right\|^2+\left\|\nabla\frac{\partial\varphi}{\partial t}\right\|^2\le c\|u\|^{4p}_{H^1(\Omega)}(E_4+1), \tag{2.41}$$

where

$$E_4=E_3+\|\Delta u\|^2+\|\Delta\varphi\|^2+\|\nabla\Delta\varphi\|^2 \tag{2.42}$$

enjoys

$$E_4\ge c(\|u\|^2_{H^2(\Omega)}+\|\varphi\|^2_{H^3(\Omega)}+\int_\Omega F(u)\,dx). \tag{2.43}$$

Multiplying, finally, (0.2) by $-\Delta\frac{\partial\varphi}{\partial t}$, we have, integrating over Ω

$$\frac{1}{2}\frac{d}{dt}\|\Delta\varphi\|^2+\left\|\nabla\frac{\partial\varphi}{\partial t}\right\|^2+\left\|\Delta\frac{\partial\varphi}{\partial t}\right\|^2=\int_\Omega\frac{\partial u}{\partial t}\Delta\frac{\partial\varphi}{\partial t}\,dx \tag{2.44}$$

and Cauchy-Schwarz's inequality implies

$$\frac{d}{dt}\|\Delta\varphi\|^2+\left\|\nabla\frac{\partial\varphi}{\partial t}\right\|^2+\left\|\Delta\frac{\partial\varphi}{\partial t}\right\|^2\le\left\|\nabla\frac{\partial u}{\partial t}\right\|^2. \tag{2.45}$$

Summing up (2.41) and η(2.45), with $\eta>0$ small enough, we then get a similar inequality as (2.41) (choosing $\eta<1$)

$$\begin{aligned}\frac{dE_5}{dt}+c\left\|\frac{\partial u}{\partial t}\right\|^2+\left\|\nabla\frac{\partial u}{\partial t}\right\|^2+\left\|\frac{\partial\varphi}{\partial t}\right\|^2\\ +\left\|\Delta\frac{\partial\varphi}{\partial t}\right\|^2+\left\|\frac{\partial\varphi}{\partial t}\right\|^2\le c'\|u\|^{4p}_{H^1(\Omega)}(E_5+1),\end{aligned} \tag{2.46}$$

where

$$E_5=E_4+\eta\|\Delta\varphi\|^2 \tag{2.47}$$

satisfies similar estimates as E_4. An application of Gronwall's lemma to (2.46) and in view of (2.34) and (2.35), we then have

$$u\in L^\infty(0,T;H^2(\Omega)\cap H^1_0(\Omega)),\ \forall T>0. \tag{2.48}$$

It follows from (2.40) that

$$\int_t^{t+r} \|\nabla\Delta\varphi(s)\|^2\, ds \le c(r),\ \forall\, t \ge t_0 + r. \tag{2.49}$$

Setting $y = E_5$ and $g = c'\|u\|^{4p}_{H^1(\Omega)}$, we deduce from (2.46) that $y' \le gy + g$, $t \ge t_0 + r$ and it is clear in view of previous estimates that y and g satisfy assumptions of the uniform Gronwall lemma (see [18]), for $t \ge t_0 + r$, which yields

$$\int_t^{t+r} E_5(s)\, ds \le c,\ t \ge t_0 + r, \tag{2.50}$$

that is to say

$$u \in L^\infty((t_0 + r, +\infty); H^2(\Omega) \cap H_0^1(\Omega)). \tag{2.51}$$

Analogously, we prove that $\varphi \in L^\infty(\mathbb{R}_+; H^2(\Omega) \cap H_0^1(\Omega))$. Finally, from (2.46), we draw that

$$\frac{\partial u}{\partial t} \in L^2(0, T; H_0^1(\Omega)) \text{ and } \frac{\partial \varphi}{\partial t} \in L^2(0, T; H^2(\Omega) \cap H_0^1(\Omega)), \tag{2.52}$$

which completes the proof. □

Remark 2.5. *The theorem 2.4 states that we can thus define the semigroup $S(t)$ on the more regular space $\Phi_1 := H^2(\Omega) \cap H_0^1(\Omega) \times H^3(\Omega) \cap H_0^1(\Omega)$. Furthermore, the latter possesses a bounded absorbing set $\mathbb{B}_1$ in Φ_1 i.e. $\forall B \subset \Phi_1$ (bounded), $\exists\, t_1 = t_1(B)$ such that $t \ge t_1$ implies $S(t)B \subset \mathbb{B}_1$.*

2.2 Global attractor

The global attractor is, by definition, the smallest compact (for inclusion) set of the phase space which is invariant by the semigroup $S(t)$, $t \ge 0$, and attracting all bounded sets of the initial data when t tends to infinity. It thus appears to be a good object for describing the asymptotic behaviour of our dynamical system.

Theorem 2.6. *Under conditions of theorem 1.2 and taking into account (1.2)-(2.1). Then the semigroup $S(t)$ defined onto $H_0^1(\Omega) \times H^2(\Omega)$ possesses the global attractor denoted $\mathcal{A}$ which is bounded in $H^2(\Omega) \times H^3(\Omega)$.*

Remark 2.7. *The absence of a regularising effect of the initial data related to the term $\Delta\frac{\partial\varphi}{\partial t}$ will lead us to decompose the semigroup into a sum of two families of operators, the first family tending towards 0 when t tends to infinity and the second family being asymptotically compact in the sense of Kuratowski's noncompactness measure.*

Proof. The proof is based on a decomposition argument (see for example [18] and [29]). Indeed, the key idea is to split the semigroup $S(t)$, $t \ge 0$, into the sum of two operators families: $S(t) = S_1(t) + S_2(t)$, where operators $S_1(t)$ go to zero as t tends to infinity while operators $S_2(t)$ are compact, in a sense that we will clarify later. This corresponds to the following solution decomposition: $(u, \varphi) = (u^d, \varphi^d) + (u^c, \varphi^c)$, where (u^d, φ^d) and (u^c, φ^c) solve, respectively

$$\frac{\partial u^d}{\partial t} - \Delta u^d = \varphi^d - \Delta\varphi^d, \tag{2.53}$$

$$\frac{\partial \varphi^d}{\partial t} - \Delta\frac{\partial \varphi^d}{\partial t} - \Delta\varphi^d = -\frac{\partial u^d}{\partial t}, \tag{2.54}$$

$$u^d = \varphi^d = 0 \text{ on } \partial\Omega, \tag{2.55}$$

$$u^d|_{t=0} = u_0,\ \varphi^d|_{t=0} = \varphi_0 \tag{2.56}$$

and

$$\frac{\partial u^c}{\partial t} - \Delta u^c + f(u) = \varphi^c - \Delta\varphi^c, \tag{2.57}$$

$$\frac{\partial \varphi^c}{\partial t} - \Delta\frac{\partial \varphi^c}{\partial t} - \Delta\varphi^c = -\frac{\partial u^c}{\partial t}, \tag{2.58}$$

$$u^c = \varphi^c = 0 \text{ on } \partial\Omega, \tag{2.59}$$

$$u^c|_{t=0} = 0,\ \varphi^c|_{t=0} = 0. \tag{2.60}$$

We consider the problem (2.53)-(2.56). Repeating estimates that led us to (1.4)-(1.16) and, noting that $f \equiv 0$, in that case, we have an inequality of the form

$$\frac{dE_6}{dt} + cE_6 + c'\left(\left\| \frac{\partial u^d}{\partial t} \right\|^2 + \left\| \frac{\partial \varphi^d}{\partial t} \right\|^2 + \left\| \nabla \frac{\partial \varphi^d}{\partial t} \right\|^2 \right) \leq 0, \tag{2.61}$$

where $E_6 = \varepsilon \|u^d\|^2 + \|\nabla u^d\|^2 + \|\varphi^d\|^2 + 3\|\nabla \varphi^d\|^2 + \|\Delta \varphi^d\|^2$ satisfies $E_6 \geq c(\|u^d\|^2_{H^1_0(\Omega)} + \|\varphi^d\|^2_{H^2(\Omega)})$. As a result, the Gronwall's lemma yields

$$\|u^d(t)\|^2_{H^1_0(\Omega)} + \|\varphi^d(t)\|^2_{H^2(\Omega)} \leq e^{-ct}(\|u_0\|^2_{H^1_0(\Omega)} + \|\varphi_0\|^2_{H^2(\Omega)}). \tag{2.62}$$

We can see that $S_1(u_0, \varphi_0) = (u^d(t), \varphi^d(t))$ tends to zero when t goes to infinity.

Now, we consider the problem (2.57)-(2.60), and repeating estimates that led in turn to (1.16), then to (2.38), we have, in particular

$$\frac{dE_7}{dt} \leq c\|f'(u)\nabla u\|^2, \tag{2.63}$$

where $E_7 = \varepsilon\|u^c\|^2 + \|\nabla u^c\|^2 + \|\Delta u^c\|^2 + \|\varphi^c\|^2 + 3\|\nabla\varphi^c\|^2 + \|\Delta\varphi^c\|^2 + \|\nabla\Delta\varphi^c\|^2$ enjoys $E_7 \geq c(\|u^c\|^2_{H^2(\Omega)} + \|\varphi^c\|^2_{H^3(\Omega)})$. Thanks to Cauchy-Schwarz's inequality and assumption (1.2), with $p = 1$, when the space dimension is $n = 3$, we write

$$\begin{aligned} \left| \int_\Omega f'(u)\nabla u \, dx \right| &\leq c \int_\Omega (|u|^{2p} + 1)|\nabla u| \, dx \\ &\leq c(\|u\|^{2p}_{H^1_0(\Omega)} + 1)\|\nabla u\|. \end{aligned} \tag{2.64}$$

It clearly appears from (2.31) that

$$\|f'(u)\nabla u\|^2_{L^2(0,T;L^2(\Omega)^n)} \leq Q(T, \|u_0\|_{H^1_0(\Omega)} + \|\varphi_0\|_{H^2(\Omega)}), \tag{2.65}$$

where Q is a monotone function independent of initial data. Collecting (2.63)-(2.65), we deduce that

$$\|u^c(t)\|^2_{H^2(\Omega)} + \|\varphi^c(t)\|^2_{H^3(\Omega)} \leq Q(T, \|u_0\|_{H^1_0(\Omega)} + \|\varphi_0\|_{H^2(\Omega)}), \, t \in [0, T]. \tag{2.66}$$

Consequently, the operator $S_2(t)(u_0, \varphi_0) = (u^c(t), \varphi^c(t))$ is asymptotically compact in the sense of the Kuratowski measure of noncompactness (see [18] and [32]). We then conclude from (2.62) and (2.66) the existence of compact attracting set, which achieves the proof. □

Remark 2.8. *The global attractor is, by definition, the smallest set (for the inclusion) compact of the phase space which is invariant by the semigroup $S(t)$, $t \geq 0$, and attracts all bounded sets of initial data when the time t goes to infinity. It thus appears as a good object for the description of the asymptotic behavior of our dynamical system. For more details on the subject, we can see for example [18].*

3 Numerical simulations

In this section, we will present the discretization of the problem which will lead us to obtain a semi-implicit first order scheme. The discretization in space is based on finite elements $\mathbb{P}_1$, i.e. piecewise continuous linear functions.

3.1 Resolution scheme

Let Ω be a convex domain of $\mathbb{R} = 1, 2$ and τ_h the uniform triangulation of Ω. We define by $V_h = \left\{ v_h \in C^0(\overline{\Omega}); v_{h|_{\tau_h}} \in \mathbb{P}_1(\tau_h), \forall \tau \in \tau_h \right\}$ the conformal subspace of finite elements of $H^1(\Omega)$. The discretized problem is then : for $(u_{0,h}, \phi_{0,h}) \in V_h \times V_h$, find $(u_h, \phi_h) \in V_h \times V_h$ such that

$$\left(\frac{\partial u_h}{\partial t}, \psi \right) - (\Delta u_h, \psi) + (f(u_h), \psi) = (\phi_h, \psi) - (\Delta \phi_h, \psi), \, \forall \psi \in V_h \tag{3.1}$$

$$\left(\frac{\partial \phi}{\partial t}, \psi \right) - \left(\Delta \frac{\partial \phi}{\partial t}, \psi \right) - (\Delta \phi, \psi) = -\left(\frac{\partial u}{\partial t}, \psi \right), \, \forall \psi \in V_h \tag{3.2}$$

$$u_h(0) = u_{0,h}, \; \phi_h(0) = \phi_{0,h}. \tag{3.3}$$

Let $(\varphi_k)_{1\leq i\leq N}$ be a basis of the space V_h. We write the unknowns in this basis and obtain $u_h(t) = \sum_{i=1}^{N} u_i(t)\varphi_i$, $\phi_h(t) = \sum_{i=1}^{N} \phi_i(t)\varphi_i$. Let's put

$$(M)_{ij} = (\varphi_i, \varphi_j)\,,\ (R)_{ij} = (\nabla\varphi_i, \nabla\varphi_j)\,, \forall 1 \leq i,\, j \leq N \tag{3.4}$$

and

$$U(t) = \begin{pmatrix} u_1(t) \\ \vdots \\ u_N(t) \end{pmatrix},\ \Phi(t) = \begin{pmatrix} \phi_1(t) \\ \vdots \\ \phi_N(t) \end{pmatrix},\ F_h(U) = \begin{pmatrix} (f(u_h), \varphi_1) \\ \vdots \\ (f(u_h), \varphi_N) \end{pmatrix}. \tag{3.5}$$

Setting $\psi = \varphi_j,\ j = 1, \ldots, N$ in (3.1)-(3.2) and using (3.4), (3.5), we obtain the following system with two equations and two unknowns

$$M\frac{\partial U(t)}{\partial t} + RU(t) + F_h(U) = M\Phi(t) + R\Phi(t) \tag{3.6}$$

$$M\frac{\partial \Phi(t)}{\partial t} + R\frac{\partial \Phi(t)}{\partial t} + R\Phi(t) = -M\frac{\partial U(t)}{\partial t}. \tag{3.7}$$

We note by (U^{n+1}, Φ^{n+1}) and by (U^n, Φ^n) the approximate values at time $t = t^{n+1}$ and $t = t^n$ respectively. Taking into account its notations, the solution scheme of the problem is obtained by applying an Euler scheme to (3.6)-(3.7)

$$M\frac{U^{n+1} - U^n}{\delta t} + RU^{n+1} + F_h(U^{n+1}) = M\Phi^{n+1} + R\Phi^{n+1}, \tag{3.8}$$

$$M\frac{\Phi^{n+1} - \Phi^n}{\delta t} + R\frac{\Phi^{n+1} - \Phi^n}{\delta t} + R\Phi^{n+1} = -M\frac{U^{n+1} - U^n}{\delta t}, \tag{3.9}$$

where δt is time step. The solution of (3.8)-(3.9) can be written in the following matrix form

$$(\frac{1}{\delta t}M + R)U^{n+1} - F_h(U^{n+1}) - (M + R)\Phi^{n+1} = \frac{1}{\delta t}MU^n \tag{3.10}$$

$$\frac{1}{\delta t}MU^{n+1} + (\frac{1}{\delta t}(M + R) + R)\Phi^{n+1} = \frac{1}{\delta t}MU^n + \frac{1}{\delta t}(M + R)\Phi^n. \tag{3.11}$$

By setting

$$\mathcal{W} = \begin{pmatrix} U^{n+1} \\ \Phi^{n+1} \end{pmatrix}, \mathcal{A} = \begin{bmatrix} (\frac{1}{\delta t}M + R) & -(M + R) \\ \frac{1}{\delta t}M & (\frac{1}{\delta t}M + R) + R \end{bmatrix}$$
$$\text{and } \mathcal{V} = \begin{pmatrix} \frac{1}{\delta t}MU^n - F_h(U^{n+1}) \\ \frac{1}{\delta t}(M + R)\Phi^n + MU^n \end{pmatrix}$$

the system (3.10)-(3.11) can be written in the following form $\mathcal{A}\mathcal{W} = \mathcal{V}$.

3.2 Numerical Results

In this section, we take $\Omega = (0, 1)$, $\delta t = 1/N,\ N \in \mathbb{N}^\star$, $x_k = k\delta t,\ k = 0, \ldots,\ N$ and

$$V_h = \left\{ v_h \in C^0([0, 1]); v_h(0) = v_h(1),\ v_{h|_{[x_k, x_{k+1}]}} \text{ is affine } , \forall k = 0, \ldots, N-1 \right\}$$

the conforming P^1 finite element space. The presence of the term $F_h(U^{n+1})$ in the system (3.10)-(3.11) is the raison of its nonlinearity. For its resolution, we used a Newton algorithm. For numerical tests, we used the cubic nonlinear term $f(s) = 0.83[s^3 - (0.5)^2 s]$ and the logarithmic one $f(s) = -2k_0 s + k_1 \ln(\dfrac{1+s}{1-s})$ with $(k_0, k_1) = (\ln(6), 1)$. The Caginalp phase-field model that we use for numericals tests is

$$\frac{\partial u}{\partial t} - \alpha\Delta u + f(u) = \phi - \Delta\phi \tag{3.12}$$

$$\frac{\partial \phi}{\partial t} - a\Delta\frac{\partial \phi}{\partial t} - b\Delta\phi = -\frac{\partial u}{\partial t} \tag{3.13}$$

$$u(0) = u_0,\ \phi(0) = \phi_0. \tag{3.14}$$

In the following tests, we will vary the coefficients, α, a and b. The initial values are $u(x,0) = 0.1\sin(2\pi x)\cos(\pi x)$ and $\phi(x,0) = 0.9\cos(2\pi x)\sin(\pi x)$.

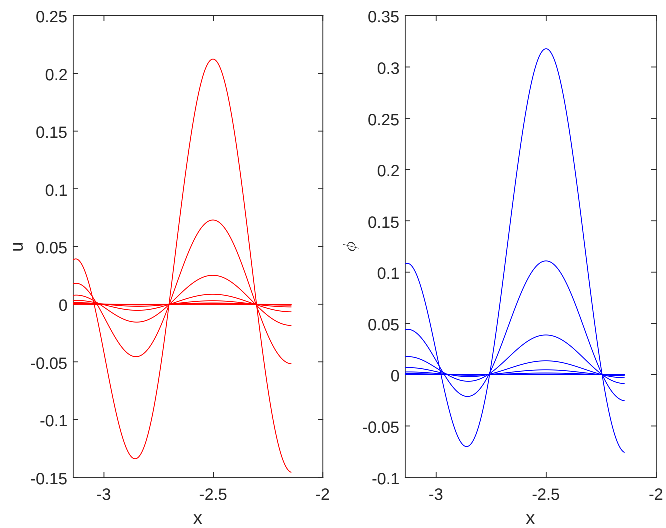

Figure 1: Cubic case with $\alpha = a = b = 1$

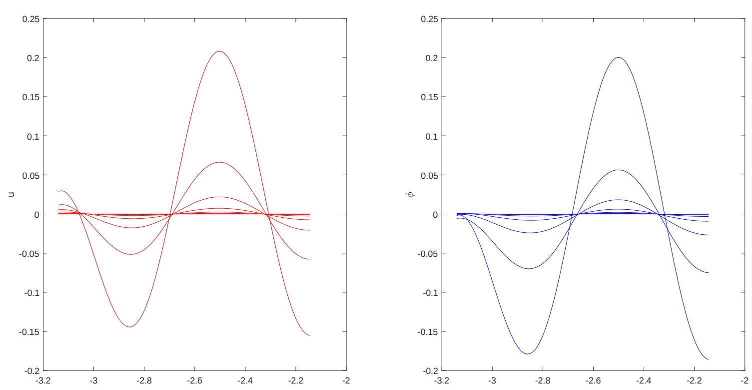

Figure 2: Logarithmic case with $\alpha = a = b = 1$

In figures 1-4, we show the influence of the parameters α, a and b.

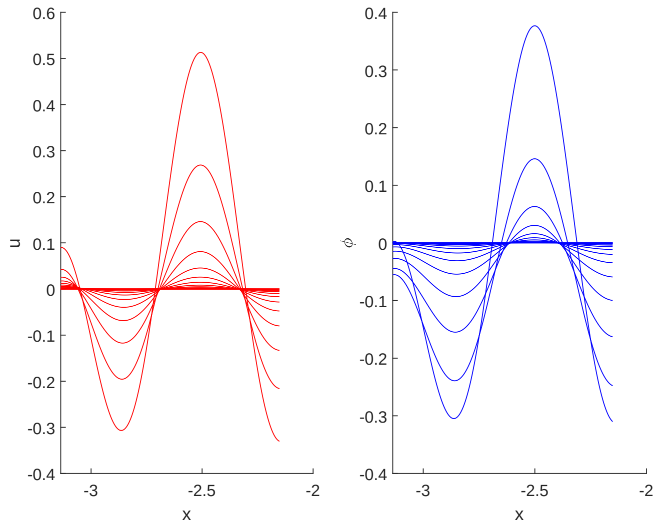

Figure 3: Logarithmic case with $\alpha = 0.75,\ a = b = 1$

The results of tests 1-4, show that all solutions tend towards the null solution both in the polynomial and logarithmic case when the time becomes large.

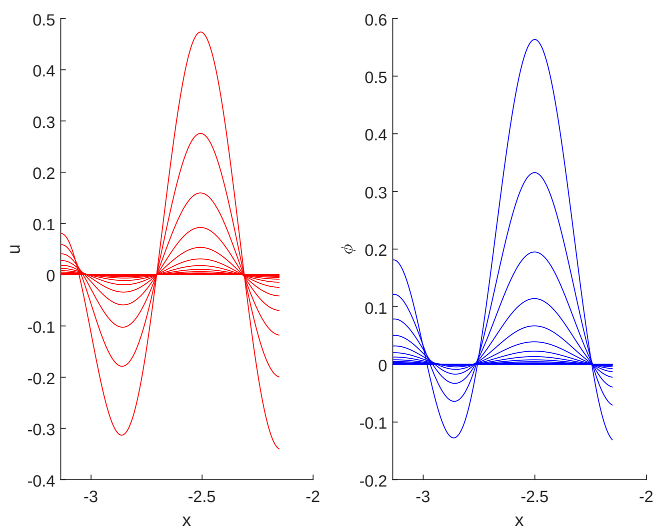

Figure 4: Cubic case with $\alpha = 0.75,\ a = b = 1$

For the rest of the tests, we change the two initial conditions and opt for a hyperbolic tangent profile. Thus we define $u_{0,\beta}(x) = 0.1\tanh(\dfrac{x-0.5}{\sqrt{\beta}})$ and $\phi_{0,\beta}(x) = 0.9\tanh(\dfrac{x-0.25}{\sqrt{\beta}})$. We are inspired here by the tests made by Bail et al in [5]. Indeed, $u_0(x) = \tanh(\frac{x}{\sqrt{2}})$ is a good approximation of the solution of the Allen-Cahn equation in the stationary case given by $-u''(x) + u^3(x) - u(x) = 0$ associated with neumann boundary conditions in the interval $[-L, L]$ with L large. It is thus interesting for us despite the fact that our boundary conditions are of the Dirichlet type to do this test.

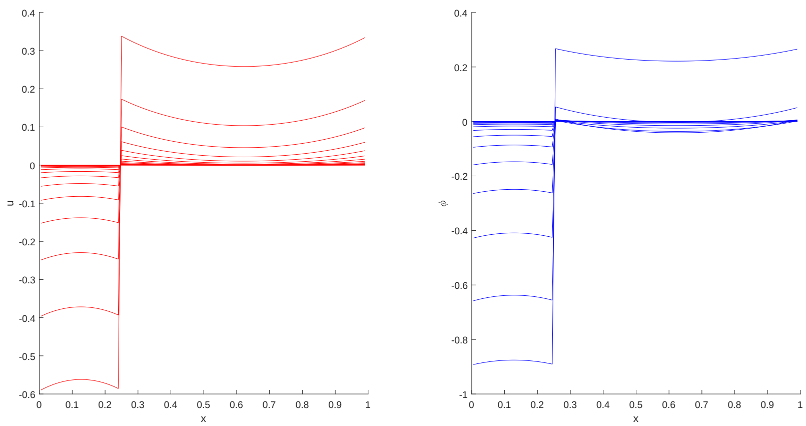

Figure 5: Logarithmic case with $\beta = 0.000001,\ a = b = 1$

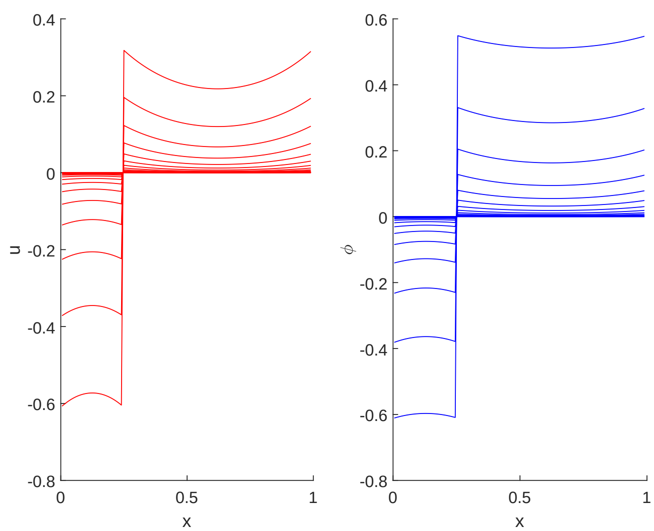

Figure 6: Cubic case with $\beta = 0.000001,\ a = b = 1$

The figures 5-6 show quasi-stationary solutions. These solutions as in the previous tests tend towards the null solution when the time becomes large.

Remark 3.1. *Numerical results have been obtained both for a potential with polynomial growth and within the framework of a logarithmic potential. For the mathematical analysis of this initial and boundary value problem in the case of a potential of logarithmic type see [3].*

Conclusion

In this paper, we have shown the existence and uniqueness of solutions of a Calginalp-type phase transition system using the Fourier law as the thermal conduction law in a material with two temperatures. The demonstration of the existence of a global attractor allows us to characterise the asymptotic behaviour of the solutions in a bounded domain. The various numerical tests carried out have confirmed this fact. As a perspective to this work, it would be interesting to see the influence of the thermal conduction law in such a system by performing the same work but with a different thermal conduction law.

conflict of interest.

On behalf of all authors, the corresponding author states that there is no conflict of interest.

References

[1] S. Aizicovici, E. Feireisl, *Long-time stabilization of solutions to a phase-field model with memory*, J. Evol. Equ. 1 (2001) 69-84.

[2] S. Aizicovici, E. Feireisl, *Long-time convergence of solutions to a phase-field system*, Math. Methods Appl. Sci. 24 (2001) 277-287.

[3] A. Andami Ovono, B. L. Doumbé Bangola and M. A. Ipopa, *Global attractor in the framework of two temperatures theory with a logarithmic potential.*, to appear.

[4] A. Andami Ovono, B. L. Doumbé Bangola and M. A. Ipopa, *On the Caginalp phase-field system based on type III with two temperatures and nonlinear coupling*, Journal of Applied Analysis and Computation. Volume 9, Number 1, February 2019. 345–362.

[5] F. Bai, C. M. Elliott, A. Gardiner, A. Spence, M. Stuart, *The Viscous Cahn-Hilliard Equation. Part I: Computations*, Nonlinearity 8 (1995) 131-160.

[6] D. Brochet, X. Chen, D. Hilhorst, *Finite-dimensional exponential attractors for the phase-field model*, Appl. Anal. 49 (1993) 197-212.

[7] M. Brokate, J. Sprekels, *Hysteresis and Phase Transitions*, Springer, New York, 1996.

[8] G. Caginalp, *An analysis of a phase field model of a free boundary*, Arch. Ration. Mech. Anal. 92 (1986) 205-245.

[9] P. J. Chen and M. E. Gurtin, On a theory of heat conduction involving two temperatures, J. Appl. Math. Phys. (ZAMP) 19 (1968) 614-627.

[10] P. J. Chen, M. E. Gurtin, and W. O. Williams, *A note on non-simple heat conduction*, J. Appl. Math. Phys. (ZAMP) 19 (1968) 969-970.

[11] P. J. Chen, M. E. Gurtin, and W. O. Williams, *On the thermodynamics of non-simple materials with two temperatures*, J. Appl. Math. Phys. (ZAMP) 20 (1969) 107-112.

[12] L. Cherfils, A. Miranville, *Some results on the asymptotic behavior of the Caginalp system with singular potentials*, Adv. Math. Sci. Appl. 17 (2007) 107-129.

[13] L. Cherfils, A. Miranville, *On the Caginalp system with dynamic boundary conditions and singular potentials*, Appl. Math. 54 (2009) 89-115.

[14] R. Chill, E. Fasangová, J. Prüss, *Convergence to steady states of solutions of the Cahn-Hilliard equation with dynamic boundary conditions*, Math. Nachr. 279 (2006) 1448-1462.

[15] C.I. Christov, P.M. Jordan, *Heat conduction paradox involving second-sound propagation in moving media*, Phys. Rev. Lett. 94 (2005) 154-301.

[16] B. L. DOUMBÉ BANGOLA, *Phase-field system with two temperatures and a nonlinear coupling term*, AIMS MATHEMATICS 3(2) (2018) 298-315.

[17] B. L. DOUMBÉ BANGOLA, *Global and exponential attractors for a Caginalp type phase-field problem*, CENT. EUR. J. MATH., 11(9) (2013) 1651-1676.

[18] B. L. DOUMBÉ BANGOLA, *Étude de modèles de champ de phase de type Caginalp*, PHD THESIS, UNIVERSITÉ DE POITIERS, 2013

[19] S. GATTI, A. MIRANVILLE, *Asymptotic behavior of a phase-field system with dynamic boundary conditions*, IN: DIFFERENTIAL EQUATIONS: INVERSE AND DIRECT PROBLEMS (PROCEEDINGS OF THE WORKSHOP "EVOLUTION EQUATIONS: INVERSE AND DIRECT PROBLEMS ", CORTONA, JUNE 21-25, 2004), IN A. FAVINI, A. LORENZI (EDS), A SERIES OF LECTURE NOTES IN PURE AND APPLIED MATHEMATICS, VOL. 251, CHAPMAN HALL, 2006, PP. 149-170.

[20] C. GIORGI, M. GRASSELLI, V. PATA, *Uniform attractors for a phase-field model with memory and quadratic nonlinearity*, INDIANA UNIV. MATH. J. 48 (1999) 1395-1446.

[21] M. GRASSELI, A. MIRANVILLE, V. PATA, S. ZELIK, *Well-posedness and long time behavior of a parabolic-hyperbolic phase-field system with singular potentials*, MATH. NACHR. 280 (2007) 1475-1509.

[22] M. GRASSELLI, *On the large time behavior of a phase-field system with memory*, ASYMPTOT. ANAL. 56 (2008) 229-249.

[23] M. GRASSELLI, A. MIRANVILLE, AND G. SHIMPERNA, *The Caginalp phase-field system with coupled dynamic boundary conditions and singular potentials*, DISCRETE CONTIN. DYN. SYST. 28 (2010), NO. 1 67-98.

[24] A.E. GREEN, P.M. NAGHDI, *A re-examination of the basic postulates of thermomechanics*, PROC. ROY. SOC. LOND. A 432 (1991) 171-194.

[25] A.E. GREEN, P.M. NAGHDI, *On undamped heat waves in an elastic solid*, J. THERMAL STRESSES 15 (1992) 253-264.

[26] J. JIANG, *Convergence to equilibrium for a parabolic-hyperbolic phase-field model with Cattaneo heat flux law*, J. MATH. ANAL. APPL. 341 (2008) 149-169.

[27] J. JIANG, *Convergence to equilibrium for a fully hyperbolic phase field model with Cattaneo heat flux law*, MATH. METHODS APPL. SCI. 32 (2009)1156-1182.

[28] PH. LAURENÇOT, *Long-time behaviour for a model of phase-field type*, PROC. ROY. SOC. EDINBURGH SECT. A 126 (1996) 167-185.

[29] A. MIRANVILLE, R. QUINTANILLA, *Some generalizations of the Caginalp phase-field system*, APPL. ANAL. 88 (2009) 877-894.

[30] A. MIRANVILLE, R. QUINTANILLA, *A generalization of the Caginalp phase-field system based on the Cattaneo law*, NONLINEAR ANAL. TMA 71 (2009) 2278-2290.

[31] A. MIRANVILLE, R. QUINTANILLA, *A Caginalp phase-field system with a nonlinear coupling*, NONLINEAR ANAL: REAL WORLD APPLICATIONS 11 (2010) 2849-2861.

[32] A. MIRANVILLE, S. ZELIK, *Attractors for dissipative partial differential equations in bounded and unbounded domains*, IN: C.M. DAFERMOS, M. POKORNY (EDS.), IN: HANDBOOK OF DIFFERENTIAL EQUATIONS, EVOLUTIONARY PARTIAL DIFFERENTIAL EQUATIONS, ELSEVIER, AMSTERDAM, 2008.

[33] A. MIRANVILLE AND R. QUINTANILLA, *A phase-field model based on a three-phase-lag heat conduction*, APPL. MATH. OPTIM. 63 (2011), NO. 1, 133-150.

[34] A. Novick-Cohen, *A phase field system with memory: Global existence*, J. Int. Equ. Appl. 14 (2002) 73-107.

[35] R. Quintanilla, *On existence in thermoelasticity without energy dissipation*, J. Thermal Stresses 25 (2002) 195-202.

[36] R. Quintanilla, *End effects in thermoelasticity*, Math. Methods Appl. Sci. 24 (2001) 93-102.

[37] R. Quintanilla, R. Racke, *Stability in thermoelasticity of type III*, Discrete Contin. Dyn. Syst. B 3 (2003) 383-400.

[38] R. Temam, *Infinite-dimensional Dynamical Systems in Mechanics and Physics, second edition*, in: Applied Mathematical Sciences, vol. 68, Springer-Verlag, New York, 1997.

[39] Z. Zhang, *Asymptotic behavior of solutions to the phase-field equations with Neumann boundary conditions*, Comm. Pure Appl. Anal. 4 (2005) 683-693.

[40] Z. Zhang, *Asymptotic behavior of solutions to the phase-field equations with Neumann boundary conditions*, Commun. Pure Appl. Anal. 4 (2005), no. 3, 683-693.