

A novel ansatz method for solving the neutron diffusion system in Cartesian geometry

Abstract

This paper analyzes the system of partial differential equations (PDEs) describing the diffusion kinetic problem with one delayed neutron precursor concentration in Cartesian geometry. The neutron diffusion kinetic equation is a popular problem in the fundamental Physics which is of practical applications in both nuclear physics and reactor design. For safety considerations, accurate solution of the this fundamental problem is required and mandatory. However, many difficulties arise when dealing with the current model using various numerical/analytical approaches as can be noticed in the literature. So, it is the objective of this paper to develop a new ansatz method to directly solve such fundamental model. It is shown in this work that our approach is straightforward and simpler when compared with other approaches in the relevant literature.

Keywords: Neutron diffusion; partial differential equations; analytic solution; ansatz method.

1 Introduction

In this paper, we consider the following system of coupled PDEs ([1]-[2]):

$$\frac{1}{V} \frac{\partial \phi}{\partial t} = D \frac{\partial^2 \phi}{\partial x^2} + \left(-\sum_a + (1 - \beta)\nu \sum_f \right) \phi(x, t) + \lambda C(x, t), \quad (1)$$

$$\frac{\partial C}{\partial t} = \beta\nu \sum_f \phi(x, t) - \lambda C(x, t), \quad (2)$$

subject to the boundary conditions (BCs):

$$\phi(0, t) = 0, \quad \phi(L, t) = 0, \quad t > 0, \quad (3)$$

$$\phi(x, 0) = \phi_0, \quad C(x, 0) = \frac{\beta\nu \sum_f}{\lambda} \phi_0, \quad 0 < x < L, \quad (4)$$

where $\phi(x, t)$ and $C(x, t)$ are respectively, the neutron flux and the delayed neutron concentration. The involved parameters in the system (1)-(4) are well-known standard for the neutron diffusion problem and well described in Refs. ([1]-[2]). The present system is of practical interest in both nuclear physics and reactor design. For safety considerations, accurate solution of the above system is required and mandatory. It can be seen that various analytical and numerical techniques have been applied in the literature [1-8]. Ceolin et al. [1] implemented the General Integral Transform Technique (GITTT) to deal with the system (1)-(4). Their approach was based on imposing an artificial auxiliary parameter (small +ve parameter) into Eq. (2) and then utilizing series expansions for $\phi(x, t)$ and $C(x, t)$ in terms of eigenfunctions. Very recently, Khaled [2] solved the current model through applying the Laplace transform method (LT) combined with the method of residues. He obtained explicit forms for the neutron flux $\phi(x, t)$ and the delayed neutron concentration $C(x, t)$. Other analytical and numerical approaches have been also investigated by several authors for similar models such as Nahla and Al-Ghamdi [3], Tumelero et al. [4], Nahla et al. [5], Khaled [6], Khaled and Mutairi [7], and Dulla et al. [8].

Although the LT method was shown as an effective and efficient approach to solve several scientific models [9-23], it needs massive computational calculations when applied on the present system, especially when using the residues method as can be observed in Khaled [2], Bakodah and Ebaid [16], and Ebaid et al. [20]. However, researchers in the field of Mathematics and Physics would prefer simple but effective methods to solve the governing system of the studied phenomena.

Although the above approaches may be viewed as effective tools to treat the neutron diffusion kinetic equation, they missed the direct sense to reaching the desired simplest method of solution. The preceding discussion formed the main purpose of the present work. So, it is the objective of this paper is to introduce a direct approach to obtain a closed-form solution for Eqs. (1)-(4). Our task will be achieved via developing anastz method to quickly and simply solve the current system.

The paper is structured as follows. In section 2, the ansatz method is stated/introduced. Section 3 is devoted to the application of the developed ansatz method to deriving the required coefficients involved in the ansatz expression. Section 4 introduces the solution in explicit forms. Section 5 focuses on the verification of the present solution. Conclusion is outlined the section 6. The paper is ended by Appendix.

2 Ansatz method

Before launching to the main purpose of this section, we rewrite the system (1)-(2) as

$$\frac{\partial \phi}{\partial t} = VD \frac{\partial^2 \phi}{\partial x^2} + \omega \phi(x, t) + \lambda VC(x, t), \quad (5)$$

$$\frac{\partial C}{\partial t} = \alpha \phi(x, t) - \lambda C(x, t), \quad (6)$$

where

$$\omega = V \left(- \sum_a + (1 - \beta)\nu \sum_f \right), \quad \alpha = \beta\nu \sum_f. \quad (7)$$

The initial condition (IC) $C(x, 0)$ becomes

$$C(x, 0) = h\phi_0, \quad \text{where} \quad h = \frac{\beta\nu \sum_f}{\lambda} = \frac{\alpha}{\lambda}. \quad (8)$$

Assume $\phi(x, t)$ and $C(x, t)$ in the forms:

$$\phi(x, t) = \sum_{n=0}^{\infty} \sin(\gamma_n x) (A_n e^{\epsilon_n t} + B_n e^{\delta_n t}), \quad (9)$$

$$C(x, t) = \sum_{n=0}^{\infty} \sin(\gamma_n x) (E_n e^{\epsilon_n t} + F_n e^{\delta_n t}), \quad (10)$$

where $\gamma_n = (2n + 1)\frac{\pi}{L}$ and $A_n, B_n, E_n, F_n, \epsilon_n, \delta_n$ are unknown constants and to be determined later. At $x = 0$, it should be noted that the BC $\phi(0, t) = 0$ is automatically satisfied. Also, at $x = L$, the ansatz (9) satisfies the BC $\phi(0, L) = 0$, where $\sin(\gamma_n L) = \sin((2n + 1)\pi) = 0 \forall n \in \mathbb{N}$. The satisfaction of the ICs (4) will also be verified in a subsequent section.

3 Coefficients determination

From (9) and (10), we have

$$\frac{\partial \phi(x, t)}{\partial t} = \sum_{n=0}^{\infty} \sin(\gamma_n x) (\epsilon_n A_n e^{\epsilon_n t} + \delta_n B_n e^{\delta_n t}), \quad (11)$$

$$\frac{\partial^2 \phi(x, t)}{\partial x^2} = \sum_{n=0}^{\infty} -\gamma_n^2 \sin(\gamma_n x) (\epsilon_n A_n e^{\epsilon_n t} + \delta_n B_n e^{\delta_n t}), \quad (12)$$

$$\frac{\partial C(x, t)}{\partial t} = \sum_{n=0}^{\infty} \sin(\gamma_n x) (\epsilon_n E_n e^{\epsilon_n t} + \delta_n F_n e^{\delta_n t}). \quad (13)$$

Substituting Eqs. (9-13) into the system (1)-(2), gives

$$\begin{aligned} & \sum_{n=0}^{\infty} \sin(\gamma_n x) [(\epsilon_n - \omega + VD\gamma_n^2) A_n - \lambda V E_n] e^{\epsilon_n t} + \\ & \sum_{n=0}^{\infty} \sin(\gamma_n x) [(\delta_n - \omega + VD\gamma_n^2) B_n - \lambda V E_n] e^{\delta_n t} = 0, \end{aligned} \quad (14)$$

and

$$\sum_{n=0}^{\infty} \sin(\gamma_n x) [(\epsilon_n + \lambda) E_n - \alpha A_n] e^{\epsilon_n t} + \sum_{n=0}^{\infty} \sin(\gamma_n x) [(\delta_n + \lambda) F_n - \alpha B_n] e^{\delta_n t} = 0. \quad (15)$$

According to (14) and (15), we have the following system

$$(\epsilon_n - \omega + VD\gamma_n^2) A_n - \lambda V E_n = 0, \quad (16)$$

$$(\delta_n - \omega + VD\gamma_n^2) B_n - \lambda V F_n = 0, \quad (17)$$

$$(\epsilon_n + \lambda) E_n - \alpha A_n = 0, \quad (18)$$

$$(\delta_n + \lambda) F_n - \alpha B_n = 0. \quad (19)$$

Applying the IC $\phi(x, 0) = \phi_0$ on Eq. (9), yields

$$\sum_{n=0}^{\infty} \sin(\gamma_n x) (A_n + B_n) = \phi_0, \quad (20)$$

and hence [24],

$$A_n + B_n = \frac{4\phi_0}{\gamma_n L}. \quad (21)$$

Similarly, applying the IC $C(x, 0) = h\phi_0$ on Eq. (10) leads to

$$\sum_{n=0}^{\infty} \sin(\gamma_n x) (E_n + F_n) = h\phi_0, \quad (22)$$

and therefore [24],

$$E_n + F_n = \frac{4h\phi_0}{\gamma_n L}. \quad (23)$$

Based on above, the constants A , B , E , F , ϵ_n , and δ_n can be evaluated by solving the following algebraic system:

$$(\epsilon_n - \omega + VD\gamma_n^2) A_n - \lambda V E_n = 0, \quad (24)$$

$$(\delta_n - \omega + VD\gamma_n^2) B_n - \lambda V F_n = 0, \quad (25)$$

$$(\epsilon_n + \lambda) E_n - \alpha A_n = 0, \quad (26)$$

$$(\delta_n + \lambda) F_n - \alpha B_n = 0, \quad (27)$$

$$A_n + B_n = \frac{4\phi_0}{\gamma_n L}, \quad (28)$$

$$E_n + F_n = \frac{4h\phi_0}{\gamma_n L}. \quad (29)$$

From (24) and (26), one can get

$$\epsilon_n^2 - (\omega - \lambda - VD\gamma_n^2) \epsilon_n - \lambda (\omega + \alpha V - VD\gamma_n^2) = 0, \quad (30)$$

which can be written as

$$\epsilon_n^2 - a_n \epsilon_n - b_n = 0, \quad (31)$$

where

$$a_n = \omega - \lambda - VD\gamma_n^2, \quad b_n = \lambda (\omega + \alpha V - VD\gamma_n^2). \quad (32)$$

Solving Eq. (31) for ϵ_n , then

$$\epsilon_n = \frac{a_n \pm \sqrt{a_n^2 + 4b_n}}{2}. \quad (33)$$

Combining Eq. (25) and Eq. (27) gives an equivalent equation, similar to Eq. (31), for δ_n

$$\delta_n^2 - a_n \delta_n - b_n = 0, \quad (34)$$

which has the same solution formula in (33). However, the consistence of the system (24-29) requires that $\epsilon_n \neq \delta_n$, hence, ϵ_n and δ_n may be chosen as

$$\epsilon_n = \frac{a_n + \sqrt{a_n^2 + 4b_n}}{2}, \quad \delta_n = \frac{a_n - \sqrt{a_n^2 + 4b_n}}{2}. \quad (35)$$

In order to simplifying the calculations of the constants A_n , B_n , E_n and F_n , we may put

$$\Omega_n = \omega - VD\gamma_n^2, \quad (36)$$

and thus we have from (24) and (25) that

$$A_n = \frac{\lambda V E_n}{\epsilon_n - \Omega_n}, \quad (37)$$

and

$$B_n = \frac{\lambda V F_n}{\delta_n - \Omega_n}. \quad (38)$$

Inserting A_n and B_n into (28) implies

$$\frac{\lambda V E_n}{\epsilon_n - \Omega_n} + \frac{\lambda V F_n}{\delta_n - \Omega_n} = \frac{4\phi_0}{\gamma_n L}. \quad (39)$$

By solving (29) and (39) for E_n and F_n with simplifying, we obtain

$$E_n = \frac{4\phi_0}{\gamma_n L \lambda V} \left[\frac{(\epsilon_n - \Omega_n)(\delta_n - \Omega_n - \lambda V h)}{\delta_n - \epsilon_n} \right], \quad \delta_n \neq \epsilon_n. \quad (40)$$

However, the expression $(\epsilon_n - \Omega_n)(\delta_n - \Omega_n - \lambda V h)$ can be further simplified, see Appendix, to give

$$(\epsilon_n - \Omega_n)(\delta_n - \Omega_n - \lambda V h) = -\lambda V (\alpha + h(\epsilon_n - \Omega_n)). \quad (41)$$

Therefore, Eq. (40) becomes

$$E_n = \frac{4\phi_0}{\gamma_n L} \left[\frac{\alpha + h(\epsilon_n - \Omega_n)}{\epsilon_n - \delta_n} \right], \quad \delta_n \neq \epsilon_n. \quad (42)$$

Substituting (42) into (29) and simplifying we obtain

$$F_n = \frac{4\phi_0}{\gamma_n L} \left[\frac{-\alpha + h(\Omega_n - \delta_n)}{\epsilon_n - \delta_n} \right], \quad \delta_n \neq \epsilon_n. \quad (43)$$

It is clear from (42) and (43) that the sum of the two coefficients E_n and F_n equals $\frac{4h\phi_0}{\gamma_n L}$ which is equivalent to Eq. (29). From (42) and (37), we get

$$A_n = \frac{4\lambda V \phi_0}{\gamma_n L} \left[\frac{\alpha + h(\epsilon_n - \Omega_n)}{(\epsilon_n - \Omega_n)(\epsilon_n - \delta_n)} \right]. \quad (44)$$

The coefficient B_n is obtained from (43) and (38) as

$$B_n = \frac{4\lambda V \phi_0}{\gamma_n L} \left[\frac{-\alpha + h(\Omega_n - \delta_n)}{(\delta_n - \Omega_n)(\epsilon_n - \delta_n)} \right], \quad (45)$$

In the Appendix, it is shown the product $(\epsilon_n - \Omega_n)(\delta_n - \Omega_n)$ is simply given by

$$(\epsilon_n - \Omega_n)(\delta_n - \Omega_n) = -\alpha\lambda V. \quad (46)$$

Using this relation one can obtain A_n and B_n in the form:

$$A_n = \frac{4\phi_0}{\gamma_n L} \left[\frac{\alpha V - \delta_n + \Omega_n}{\epsilon_n - \delta_n} \right], \quad (47)$$

and

$$B_n = \frac{4\phi_0}{\gamma_n L} \left[\frac{\epsilon_n - \Omega_n - \alpha V}{\epsilon_n - \delta_n} \right], \quad (48)$$

respectively. It is clear from (47) and (48) that the sum of the two coefficients A_n and B_n equals $\frac{4\phi_0}{\gamma_n L}$ which is equivalent to Eq. (28).

4 Explicit solution

Based on above, we get $\phi(x, t)$ as

$$\phi(x, t) = 4\phi_0 \sum_{n=0}^{\infty} \frac{\sin(\gamma_n x)}{\gamma_n L} u_1(t), \quad (49)$$

where

$$u_1(t) = \frac{1}{(\epsilon_n - \delta_n)} [(\alpha V - \delta_n + \Omega_n)e^{\epsilon_n t} + (\epsilon_n - \Omega_n - \alpha V)e^{\delta_n t}]. \quad (50)$$

In view of ϵ_n and δ_n in (35), we can rewrite (50) as

$$u_1(t) = \frac{e^{\frac{1}{2}a_n t}}{(\epsilon_n - \delta_n)} [(\alpha V - \delta_n + \Omega_n)e^{\frac{1}{2}\sqrt{a_n^2+4b_n t}} + (\epsilon_n - \Omega_n - \alpha V)e^{-\frac{1}{2}\sqrt{a_n^2+4b_n t}}], \quad (51)$$

or

$$\begin{aligned} u_1(t) &= \frac{e^{\frac{1}{2}a_n t}}{(\epsilon_n - \delta_n)} \left[(\alpha V + \Omega_n) \left(e^{\frac{1}{2}\sqrt{a_n^2+4b_n t}} - e^{-\frac{1}{2}\sqrt{a_n^2+4b_n t}} \right) + \epsilon_n e^{-\frac{1}{2}\sqrt{a_n^2+4b_n t}} - \delta_n e^{\frac{1}{2}\sqrt{a_n^2+4b_n t}} \right] \\ &= \frac{e^{\frac{1}{2}a_n t}}{(\epsilon_n - \delta_n)} \left[2(\alpha V + \Omega_n) \sinh \left(\frac{1}{2}\sqrt{a_n^2+4b_n t} \right) + \epsilon_n e^{-\frac{1}{2}\sqrt{a_n^2+4b_n t}} - \delta_n e^{\frac{1}{2}\sqrt{a_n^2+4b_n t}} \right]. \end{aligned} \quad (52)$$

The expression combining the last two terms in (52) can be also written as

$$\begin{aligned} \epsilon_n e^{-\frac{1}{2}\sqrt{a_n^2+4b_n t}} - \delta_n e^{\frac{1}{2}\sqrt{a_n^2+4b_n t}} &= -\frac{a_n}{2} \left(e^{\frac{1}{2}\sqrt{a_n^2+4b_n t}} - e^{-\frac{1}{2}\sqrt{a_n^2+4b_n t}} \right) + \\ &\quad \frac{1}{2}\sqrt{a_n^2+4b_n} \left(e^{\frac{1}{2}\sqrt{a_n^2+4b_n t}} + e^{-\frac{1}{2}\sqrt{a_n^2+4b_n t}} \right) \\ &= -a_n \sinh \left(\frac{1}{2}\sqrt{a_n^2+4b_n t} \right) + \sqrt{a_n^2+4b_n} \cosh \left(\frac{1}{2}\sqrt{a_n^2+4b_n t} \right). \end{aligned} \quad (53)$$

Substituting (53) into (52) and simplifying, we obtain

$$u_1(t) = \left[\frac{2\alpha V + \lambda + \Omega_n}{\sqrt{(\lambda + \Omega_n)^2 + 4\lambda\alpha V}} \sinh \left(\frac{1}{2}\sqrt{(\lambda + \Omega_n)^2 + 4\lambda\alpha V t} \right) + \cosh \left(\frac{1}{2}\sqrt{(\lambda + \Omega_n)^2 + 4\lambda\alpha V t} \right) \right] \times e^{\frac{1}{2}(\Omega_n - \lambda)t}, \quad (54)$$

where

$$a_n^2 + 4b_n = (\lambda + \Omega_n)^2 + 4\lambda\alpha V, \quad (55)$$

is used. One can use a similar analysis to obtain the concentration $C(x, t)$ as

$$C(x, t) = 4h\phi_0 \sum_{n=0}^{\infty} \frac{\sin(\gamma_n x)}{\gamma_n L} u_2(t), \quad (56)$$

where

$$u_2(t) = \left[\frac{\lambda - \Omega_n}{\sqrt{(\lambda + \Omega_n)^2 + 4\lambda\alpha V}} \sinh\left(\frac{1}{2}\sqrt{(\lambda + \Omega_n)^2 + 4\lambda\alpha V}t\right) + \cosh\left(\frac{1}{2}\sqrt{(\lambda + \Omega_n)^2 + 4\lambda\alpha V}t\right) \right] \times e^{\frac{1}{2}(\Omega_n - \lambda)t}. \quad (57)$$

5 Verification

The above obtained explicit solutions for the neutron flux $\phi(x, t)$ and the delayed neutron concentration $C(x, t)$ can be easily verified by direct substitution into the governing equations (1) and (2). To justify such verification, the infinity in Eq. (49) and Eq. (56) may be replaced with a certain finite number of terms. This can be easily done through any software such as MATHEMATICA. Regarding the BCs (3), they are automatically satisfied as discussed in section 2. In addition, our solution satisfies the ICs (4) which can be declared as follows. From Eq. (49), we have

$$\phi(x, 0) = 4\phi_0 \sum_{n=0}^{\infty} \frac{\sin(\gamma_n x)}{\gamma_n L} u_1(0). \quad (58)$$

From Eq. (54), we note that the initial value $u_1(0) = 1$. Substituting $u_1(0) = 1$ and $\gamma_n = (2n+1)\frac{\pi}{L}$ into Eq. (58), yields

$$\phi(x, 0) = \frac{4}{\pi}\phi_0 \sum_{n=0}^{\infty} \frac{\sin\left((2n+1)\pi\frac{x}{L}\right)}{2n+1}. \quad (59)$$

However, the sum of the infinite series in the last equation for $0 < x < L$ is given by

$$\sum_{n=0}^{\infty} \frac{\sin\left((2n+1)\pi\frac{x}{L}\right)}{2n+1} = \frac{\pi}{4}. \quad (60)$$

Thus, Eq. (59) becomes

$$\phi(x, 0) = \phi_0, \quad (61)$$

and hence the first IC in (4) at the initial time $t = 0$ is satisfied. Similarly, the concentration $C(x, t)$ at $t = 0$ is given from Eq. (56) by

$$C(x, 0) = 4h\phi_0 \sum_{n=0}^{\infty} \frac{\sin(\gamma_n x)}{\gamma_n L} u_2(0). \quad (62)$$

From Eq. (57), we also get $u_2(0) = 1$. Therefore

$$C(x, 0) = \frac{4}{\pi} h\phi_0 \sum_{n=0}^{\infty} \frac{\sin\left((2n+1)\pi\frac{x}{L}\right)}{2n+1}. \quad (63)$$

Employing the result (60) leads to

$$C(x, 0) = h\phi_0, \quad (64)$$

which is the second IC in Eqs. (4) with the help of Eq. (8). Finally, the simplicity of the present method over other methods in the relevant literature is obvious and clear.

6 Conclusion

The neutron diffusion kinetic equation is a well known problem in Mathematical Physics. In this paper, a direct anastz method was introduced to analytically solve the system describing the neutron flux in Cartesian geometry. Explicit forms were obtained in closed forms for the neutron flux and the delayed neutron concentration. In comparison to the other approaches in the relevant literature, our method not only efficient but also direct and simpler. The simplicity achieved in this paper deserves further extension to other diffusion kinetic equations with another set of appropriate physical boundary conditions.

Appendices

A Simplified expressions

The result introduced in Eq. (41) is deduced as follows. In view of (32) and (36) we may write a_n and b_n as

$$a_n = \Omega_n - \lambda, \quad b_n = \lambda(\alpha V + \Omega_n) = \lambda(\lambda + \alpha V) + \lambda a_n. \quad (\text{A.1})$$

From Eqs. (35), we have

$$\epsilon_n \delta_n = -b_n, \quad \epsilon_n + \delta_n = a_n, \quad (\text{A.2})$$

i.e.,

$$\epsilon_n \delta_n = -\lambda(\alpha V + \Omega_n), \quad \epsilon_n + \delta_n = \Omega_n - \lambda. \quad (\text{A.3})$$

Therefore, the left hand side in Eq. (41) is

$$\begin{aligned} (\epsilon_n - \Omega_n)(\delta_n - \Omega_n - \lambda V h) &= \epsilon_n \delta_n - \epsilon_n \Omega_n - \delta_n \Omega_n + \Omega_n^2 - \epsilon_n h \lambda V + \Omega_n h \lambda V \\ &= \epsilon_n \delta_n - \Omega_n(\epsilon_n + \delta_n) + \Omega_n^2 - h \lambda V(\epsilon_n - \Omega_n) \\ &= -\lambda(\alpha V + \Omega_n) - \Omega_n(\Omega_n - \lambda) + \Omega_n^2 - h \lambda V(\epsilon_n - \Omega_n) \\ &= -\lambda \alpha V - h \lambda V(\epsilon_n - \Omega_n) \\ &= -\lambda V(\alpha + h(\epsilon_n - \Omega_n)). \end{aligned} \quad (\text{A.4})$$

Also, for the product $(\epsilon_n - \Omega_n)(\delta_n - \Omega_n)$ we have

$$\begin{aligned} (\epsilon_n - \Omega_n)(\delta_n - \Omega_n) &= \epsilon_n \delta_n - \Omega_n(\epsilon_n + \delta_n) + \Omega_n^2 \\ &= -\lambda(\alpha V + \Omega_n) - \Omega_n(\Omega_n - \lambda) + \Omega_n^2 \\ &= -\lambda \alpha V - \lambda \Omega_n - \Omega_n^2 + \lambda \Omega_n + \Omega_n^2 \\ &= -\alpha \lambda V. \end{aligned} \quad (\text{A.5})$$

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