Hyers-Ulam-Rassias stability of a general septic functional equation

Abstract. In this paper, we investigate the stability of the following general septic functional equation

$$\sum_{i=0}^{8} {}_8C_i(-1)^{8-i}f(x+(i-4)y)=0$$

which is a generalization of many functional equations such as the additive functional equation, the quadratic functional equation, the cubic functional equation, the quartic functional equation, the quintic functional equation, and the sextic functional equation. The equation is analysed from the perspective of Hyers-Ulam-Rassias stability.

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Key Words: stability of a functional equation; general septic functional equation; general septic mapping.

1 Introduction

It is well known that the study on the stability of functional equations began as an attempt to solve Ulam's question in [28] about the stability of the group homomorphisms. As a partial answer to this question, Hyers [7] solved the stability of the Cauchy functional equation in the following year. Since then, many mathematicians have generalized Hyers' results by showing the stability of various kind of functional equations, see [2, 5, 6, 8, 14, 26]. Today the term 'Hyers-Ulam-Rassias stability' refers to the generalization introduced by Rassias [26].

Throughout this paper, V, X, and Y are a real vector space, a real normed space, and a real Banach space, respectively. For a mapping f from V to Y, we consider the functional equation

$$\sum_{i=0}^{k} {}_kC_i(-1)^{k-i}f(x+iy)=0, \tag{1.1}$$

where $k=2,3,4,5,6,7,8$. Observe that a solution mapping $f:V\to Y$ of (1.1) is a "generalized polynomial mapping of degree at most 7" in the sense of J. Baker in

[1]. So the functional equation (1.1) is called a Jensen, a general quadratic, a general cubic, a general quartic, a general quintic, a general sextic, and a general septic functional equation, for $k = 2, 3, 4, 5, 6, 7, 8$, respectively. Also each solution mapping of (1.1), for $k = 2, 3, 4, 5, 6, 7, 8$, is called as a Jensen, a general quadratic, a general cubic, a general quartic, a general quintic, a general sextic, and a general septic mapping, respectively.

Recall, the stability problems for the functional equation (1.1) were studied in many ways. In the case of a Jensen functional equation, K.-W. Jun et al. [18] showed the stability result. The stability of a general quadratic function equation was obtained by Y. H. Lee [13], Y. H. Lee et al. [20], and S. S. Jin et al. [9]. On the other hand, the stability of a general cubic function equation was studied by Y. H. Lee [12, 17], S. M. Jun et al. [11], and Y. H. Lee et al. [25, 21], and the stability of the general quartic function equation are discussed in Y. H. Lee [15] and Y. H. Lee et al. [19, 21, 22, 23, 24]. Moreover, the stability of a general quintic functional equation has been studied by S. S. Jin et al. [10], and the stability of the general sextic function equation has been obtained by Y. H. Lee [16], I. S, Chang et al. [3], and J. Roh et al. [27].

In this article, we investigate the stability of the following general septic functional equation

$$Df(x,y) := \sum_{i=0}^{8} {}_8C_i(-1)^{8-i} f(x + (i-4)y) = 0 \tag{1.2}$$

in the sense of Hyers-Ulam-Rassias. Prior to this paper, in [4], I. S, Chang et al. used the method of Găvruta to prove the stability of a general septic functional equation, i.e., if the function $f : V \to Y$ satisfies the inequality

$$\|Df(x,y)\| \le \phi(x,y)$$

where a function $\phi : V^2 \to [0,\infty)$ satisfies the condition

$$\sum_{i=0}^{\infty} 128^n \phi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) < \infty,$$

for all $x, y \in V$, then there exists a unique general septic mapping F near the function f. On the other hand, in this paper, we use Theorem 2.2 to improve the stability result of the general septic functional equation. Precisely, for a real number $\theta > 0$ and a non-negative real number $p \ne 1, 2, 3, 4, 5, 6, 7$, let $f : X \to Y$ satisfy

$$\|Df(x,y)\| \le \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$, then it is proved that there exists a unique septic mapping F, i.e., $DF(x,y) = 0$ for all $x, y \in X$, such that $F(0) = 0$ and

$$\|f(x) - f(0) - F(x)\| \le \epsilon_p \theta \|x\|^p$$

for all $x \in X$, where the constant ϵ_p depends only on p, see (2.14).

2 Stability of a general septic functional equation

Definition 2.1 *For a given mapping $f : V \to Y$, we define the mappings $Df : V^2 \to Y$, $\tilde{f}$, f_o, f_e, Γf, $\Delta f : V \to Y$ as*

$$\begin{aligned}
Df(x,y) :=& \sum_{i=0}^{8} {}_8C_i(-1)^{8-i} f(x+(i-4)y),\\
\tilde{f}(x) :=& f(x) - f(0), \quad f_o(x) := \frac{f(x)-f(-x)}{2}, \quad f_e(x) := \frac{f(x)+f(-x)}{2},\\
\Gamma f(x) :=& Df_o(8x,2x) + 8Df_o(6x,2x) + 36Df_o(4x,2x) + 120Df_o(2x,2x)\\
&+ 160Df_o(4x,x) + 1280Df_o(3x,x) + 4032Df_o(2x,x) + 5376Df_o(x,x),\\
\Delta f(x) :=& Df_e(4x,x) + 8Df_e(3x,x) + 36Df_e(2x,x) + 120Df_e(x,x) + 123Df_e(0,x)
\end{aligned}$$

for all $x, y \in V$.

Theorem 2.2 *For $f : V \to Y$, let us define f_1, f_2, f_3, f_4, f_5, f_6, $f_7 : V \to Y$ as follows;*

$$f_1(x) := \frac{1}{M}\begin{vmatrix} f_o(x) & 1 & 1 & 1\\ f_o(2x) & 8 & 32 & 128\\ f_o(4x) & 8^2 & 32^2 & 128^2\\ f_o(8x) & 8^3 & 32^3 & 128^3 \end{vmatrix}, \quad f_2(x) := \frac{1}{M'}\begin{vmatrix} f_e(x) & 1 & 1\\ f_e(2x) & 16 & 64\\ f_e(4x) & 16^2 & 64^2 \end{vmatrix},$$

$$f_3(x) := \frac{1}{M}\begin{vmatrix} 1 & f_o(x) & 1 & 1\\ 2 & f_o(2x) & 32 & 128\\ 2^2 & f_o(4x) & 32^2 & 128^2\\ 2^3 & f_o(8x) & 32^3 & 128^3 \end{vmatrix}, \quad f_4(x) := \frac{1}{M'}\begin{vmatrix} 1 & f_e(x) & 1\\ 4 & f_e(2x) & 64\\ 4^2 & f_e(4x) & 64^2 \end{vmatrix},$$

$$f_5(x) := \frac{1}{M}\begin{vmatrix} 1 & 1 & f_o(x) & 1\\ 2 & 8 & f_o(2x) & 128\\ 2^2 & 8^2 & f_o(4x) & 128^2\\ 2^3 & 8^3 & f_o(8x) & 128^3 \end{vmatrix}, \quad f_6(x) := \frac{1}{M'}\begin{vmatrix} 1 & 1 & f_e(x)\\ 4 & 16 & f_e(2x)\\ 4^2 & 16^2 & f_e(4x) \end{vmatrix},$$

$$f_7(x) := \frac{1}{M}\begin{vmatrix} 1 & 1 & 1 & f_o(x)\\ 2 & 8 & 32 & f_o(2x)\\ 2^2 & 8^2 & 32^2 & f_o(4x)\\ 2^3 & 8^3 & 32^3 & f_o(8x) \end{vmatrix}$$

for all $x \in V$, where

$$M := \begin{vmatrix} 1 & 1 & 1 & 1\\ 2 & 8 & 32 & 128\\ 2^2 & 8^2 & 32^2 & 128^2\\ 2^3 & 8^3 & 32^3 & 128^3 \end{vmatrix} \quad \textit{and} \quad M' := \begin{vmatrix} 1 & 1 & 1\\ 4 & 16 & 64\\ 4^2 & 16^2 & 64^2 \end{vmatrix}.$$

Then

$$f(x) = f_o(x) + f_e(x) = \sum_{i=1}^{7} f_i(x) \tag{2.1}$$

for all $x \in V$.

Proof. It can be noted that $M \neq 0$ and $M' \neq 0$. The uniqueness of solution (stated in Cramer's rule) implies that the family $\{f_1(x), f_3(x), f_5(x), f_7(x)\}$ is the only solution to the system of non-homogeneous linear equations

$$\begin{cases} f_1(x) + f_3(x) + f_5(x) + f_7(x) = f_o(x) \\ 2f_1(x) + 8f_3(x) + 32f_5(x) + 128f_7(x) = f_o(2x) \\ 2^2 f_1(x) + 8^2 f_3(x) + 32^2 f_5(x) + 128^2 f_7(x) = f_o(4x) \\ 2^3 f_1(x) + 8^3 f_3(x) + 32^3 f_5(x) + 128^3 f_7(x) = f_o(8x) \end{cases}$$

for all $x \in V$. Similarly, we have $f_e(x) = f_2(x) + f_4(x) + f_6(x)$ for all $x \in V$. □

By laborious computation we can get the following equalities;

$$\begin{aligned} \Gamma \tilde{f}(x) &= f_o(16x) - 170 f_o(8x) + 5712 f_o(4x) - 43520 f_o(2x) + 65536 f_o(x), \\ \Delta \tilde{f}(x) &= \tilde{f}_e(8x) - 84 \tilde{f}_e(4x) + 1344 \tilde{f}_e(2x) - 4096 \tilde{f}_e(x), \end{aligned}$$

and

$$f_1(x) = \frac{32768 f_o(x) - 5376 f_o(2x) + 168 f_o(4x) - f_o(8x)}{22680}, \tag{2.2}$$

$$f_2(x) = \frac{1024 f_e(x) - 80 f_e(2x) + f_e(4x)}{720}, \tag{2.3}$$

$$f_3(x) = -\frac{8192 f_o(x) - 4416 f_o(2x) + 162 f_o(4x) - f_o(8x)}{17280}, \tag{2.4}$$

$$f_4(x) = -\frac{256 f_e(x) - 68 f_e(2x) + f_e(4x)}{576}, \tag{2.5}$$

$$f_5(x) = \frac{2048 f_o(x) - 1296 f_o(2x) + 138 f_o(4x) - f_o(8x)}{69120}, \tag{2.6}$$

$$f_6(x) = \frac{64 f_e(x) - 20 f_e(2x) + f_e(4x)}{2880}, \tag{2.7}$$

$$f_7(x) = -\frac{512 f_o(x) - 336 f_o(2x) + 42 f_o(4x) - f_o(8x)}{1451520}, \tag{2.8}$$

as well as

$$\tilde{f}_1(x) - \frac{\tilde{f}_1(2x)}{2} = \frac{\Gamma \tilde{f}_o(x)}{45360}, \qquad \tilde{f}_2(x) - \frac{\tilde{f}_2(2x)}{4} = -\frac{\Delta \tilde{f}_e(x)}{2880}, \tag{2.9}$$

$$\tilde{f}_3(x) - \frac{\tilde{f}_3(2x)}{8} = -\frac{\Gamma \tilde{f}_o(x)}{138240}, \qquad \tilde{f}_4(x) - \frac{\tilde{f}_4(2x)}{16} = \frac{\Delta \tilde{f}_e(x)}{9216}, \tag{2.10}$$

$$\tilde{f}_5(x) - \frac{\tilde{f}_5(2x)}{32} = \frac{\Gamma \tilde{f}_o(x)}{2211840}, \qquad \tilde{f}_6(x) - \frac{\tilde{f}_6(2x)}{64} = -\frac{\Delta \tilde{f}_e(x)}{184320}, \tag{2.11}$$

$$\tilde{f}_7(x) - \frac{\tilde{f}_7(2x)}{128} = -\frac{\Gamma \tilde{f}_o(x)}{185794560} \tag{2.12}$$

for all $x \in V$.

Now, the stability of the general septic functional equation (1.2) is computed.

Theorem 2.3 *Let $p \neq 1,2,3,4,5,6,7$ be a non-negative real number, and let $\theta > 0$. Suppose that $f : X \to Y$ satisfies*

$$\|Df(x,y)\| \leq \theta(\|x\|^p + \|y\|^p) \tag{2.13}$$

for all $x, y \in X$, then there exists a unique mapping F such that $F(0) = 0$, $DF(x,y) = 0$ for all $x, y \in X$, and

$$\begin{aligned}\|\tilde{f}(x) - F(x)\| \leq & \frac{K'\theta\|x\|^p}{22680 \cdot |2-2^p|} + \frac{K\theta\|x\|^p}{720 \cdot |4-2^p|} + \frac{K'\theta\|x\|^p}{17280 \cdot |8-2^p|} + \frac{K\theta\|x\|^p}{576 \cdot |16-2^p|} \\ & + \frac{K'\theta\|x\|^p}{69120 \cdot |32-2^p|} + \frac{K\theta\|x\|^p}{2880 \cdot |64-2^p|} + \frac{K'\theta\|x\|^p}{1451520 \cdot |2^p-128|}\end{aligned} \tag{2.14}$$

for all $x \in X$, where K and K' are constants given by $K := \big(4^p + 8 \cdot 3^p + 36 \cdot 2^p + 408\big)$ and $K' := \big(8^p + 8 \cdot 6^p + 196 \cdot 4^p + 1280 \cdot 3^p + 4317 \cdot 2^p + 16224\big)$.

Proof. Notice that $\tilde{f}(0) = 0$, $D\tilde{f}(x,y) = Df(x,y)$, and

$$\|Df_o(x,y)\|, \ \|D\tilde{f}_e(x,y)\| \leq \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$, by (2.13). Then, together with the definitions of $\Gamma\tilde{f}$ and $\Delta\tilde{f}$, we get

$$\begin{aligned}\|\Gamma\tilde{f}_o(x)\| = & \big\|Df_o(8x,2x) + 8Df_o(6x,2x) + 36Df_o(4x,2x) + 120Df_o(2x,2x) \\ & + 160Df_o(4x,x) + 1280Df_o(3x,x) + 4032Df_o(2x,x) + 5376Df_o(x,x)\| \\ \leq & \big(8^p + 2^p + 8 \cdot 6^p + 8 \cdot 2^p + 36 \cdot 4^p + 36 \cdot 2^p + 240 \cdot 2^p + 160 \cdot 4^p \\ & + 160 + 1280 \cdot 3^p + 1280 + 4032 \cdot 2^p + 4032 + 10752\big)\theta\|x\|^p \\ \leq & \ K'\theta\|x\|^p, \end{aligned} \tag{2.15}$$

$$\begin{aligned}\|\Delta\tilde{f}_e(x)\| = & \big\|D\tilde{f}_e(4x,x) + 8D\tilde{f}_e(3x,x) + 36D\tilde{f}_e(2x,x) + 120D\tilde{f}_e(x,x) + 123D\tilde{f}_e(0,x)\big\| \\ \leq & \ K\theta\|x\|^p \end{aligned} \tag{2.16}$$

for all $x \in X$. The theorem can be proved in seven steps in the following manner:

Step 1. For $p \neq 1$, there exists a mapping $F^{(1)} : X \to Y$ satisfying $F^{(1)}(0) = 0$, $DF^{(1)}(x,y) = 0$ for all $x, y \in X$, and

$$\|\tilde{f}_1(x) - F^{(1)}(x)\| \leq \frac{K'\theta\|x\|^p}{22680 \cdot |2-2^p|} \tag{2.17}$$

for all $x \in X$.

(1) If $0 \leq p < 1$, then it follows from (2.9) we obtain that

$$\begin{aligned}\left\|\frac{\tilde{f}_1(2^n x)}{2^n} - \frac{\tilde{f}_1(2^{n+m}x)}{2^{n+m}}\right\| &= \left\|\sum_{i=n}^{n+m-1}\left(\frac{\tilde{f}_1(2^i x)}{2^i} - \frac{\tilde{f}_1(2^{i+1}x)}{2^{i+1}}\right)\right\| \\ &\leq \sum_{i=n}^{n+m-1}\left\|\frac{\Gamma\tilde{f}_o(2^i x)}{45360 \cdot 2^i}\right\| \leq \sum_{i=n}^{n+m-1}\frac{K'\theta 2^{ip}\|x\|^p}{45360 \cdot 2^i}\end{aligned} \tag{2.18}$$

for all $x \in X$ and $n, m \in \mathbb{N} \cup \{0\}$. So $\left\{\frac{\tilde{f}_1(2^n x)}{2^n}\right\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, it converges and we can define a mapping $F^{(1)} : X \to Y$ by

$$F^{(1)}(x) := \lim_{n\to\infty} \frac{\tilde{f}_1(2^n x)}{2^n}$$

for all $x \in X$. Moreover, letting $n = 0$ and passing the limit $m \to \infty$ in (2.18), the following inequality is obtained

$$\|\tilde{f}_1(x) - F^{(1)}(x)\| \le \frac{K'\theta\|x\|^p}{22680(2-2^p)}$$

for all $x \in X$, and, together with (2.2), it holds that

$$\begin{aligned}
\|DF^{(1)}(x,y)\| &= \lim_{n\to\infty} \left\|\frac{D\tilde{f}_1(2^n x, 2^n y)}{2^n}\right\| \le \lim_{n\to\infty} \left(\left\|\frac{32768 D\tilde{f}_o(2^n x, 2^n y)}{22680 \cdot 2^n}\right\| \right. \\
&\quad \left. + \left\|\frac{5376 D\tilde{f}_o(2^{n+1} x, 2^{n+1} y)}{22680 \cdot 2^n}\right\| + \left\|\frac{168 D\tilde{f}_o(2^{n+2} x, 2^{n+2} y)}{22680 \cdot 2^n}\right\| + \left\|\frac{D\tilde{f}_o(2^{n+3} x, 2^{n+3} y)}{22680 \cdot 2^n}\right\| \right) \\
&\le \lim_{n\to\infty} \left(32768 \cdot 2^{np} + 5376 \cdot 2^{(n+1)p} + 168 \cdot 2^{(n+2)p} + 2^{(n+3)p}\right) \frac{\theta(\|x\|^p + \|y\|^p)}{22680 \cdot 2^n} \\
&= 0
\end{aligned}$$

for all $x, y \in X$.

(2) If $p > 1$, then it follows from (2.9) and (2.15) that

$$\begin{aligned}
\left\|2^n \tilde{f}_1(2^{-n}x) - 2^{n+m}\tilde{f}_1(2^{-n-m}x)\right\| &= \left\|\sum_{i=n}^{n+m-1} \left(2^i \tilde{f}_1(2^{-i}x) - 2^{i+1}\tilde{f}_1(2^{-i-1}x)\right)\right\| \\
&\le \sum_{i=n}^{n+m-1} \left\|\frac{2^{i+1}\Gamma\tilde{f}_o(2^{-i-1}x)}{45360}\right\| \le \sum_{i=n}^{n+m-1} \frac{2^i K'\theta\|x\|^p}{22680 \cdot 2^{(i+1)p}} \qquad (2.19)
\end{aligned}$$

for all $x \in X$ and $n, m \in \mathbb{N} \cup \{0\}$. So $\{2^n \tilde{f}_1(2^{-n}x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence converges and we can define a mapping $F^{(1)} : X \to Y$ by

$$F^{(1)}(x) := \lim_{n\to\infty} 2^n \tilde{f}_1(2^{-n}x)$$

for all $x \in X$. Moreover, letting $n = 0$ and passing the limit $m \to \infty$ in (2.19), the following inequality is obtained

$$\|\tilde{f}_1(x) - F^{(1)}(x)\| \le \frac{K'\theta\|x\|^p}{22680(2^p - 2)}$$

for all $x \in X$, and, using (2.2), we have

$$\begin{aligned}
\|DF^{(1)}(x,y)\| &= \lim_{n\to\infty} 2^n \left\|D\tilde{f}_1(2^{-n}x, 2^{-n}y)\right\| \le \lim_{n\to\infty} 2^n \left(\left\|\frac{32768 D\tilde{f}_o(2^{-n}x, 2^{-n}y)}{22680}\right\| \right. \\
&\quad \left. + \left\|\frac{5376 D\tilde{f}_o(2^{-n+1}x, 2^{-n+1}y)}{22680}\right\| + \left\|\frac{168 D\tilde{f}_o(2^{-n+2}x, 2^{-n+2}y)}{22680}\right\| + \left\|\frac{D\tilde{f}_o(2^{-n+3}x, 2^{-n+3}y)}{22680}\right\| \right) \\
&\le \lim_{n\to\infty} \left(32768 \cdot 2^{-np} + 5376 \cdot 2^{-np+p} + 168 \cdot 2^{-np+2p} + 2^{-np+3p}\right) \frac{2^n\theta(\|x\|^p + \|y\|^p)}{22680} \\
&= 0
\end{aligned}$$

for all $x, y \in X$.

Step 2. For $p \neq 2$, there exists a mapping $F^{(2)} : X \to Y$ satisfying $F^{(2)}(0) = 0$, $DF^{(2)}(x,y) = 0$ for all $x, y \in X$, and

$$\|\tilde{f}_2(x) - F^{(2)}(x)\| \leq \frac{K\theta\|x\|^p}{720 \cdot |4 - 2^p|} \tag{2.20}$$

for all $x \in X$.

(1) If $p < 2$, then it follows from (2.9) and (2.16) that

$$\begin{aligned}\left\|\frac{\tilde{f}_2(2^n x)}{4^n} - \frac{\tilde{f}_2(2^{n+m} x)}{4^{n+m}}\right\| &= \left\|\sum_{i=n}^{n+m-1}\left(\frac{\tilde{f}_2(2^i x)}{4^i} - \frac{\tilde{f}_2(2^{i+1} x)}{4^{i+1}}\right)\right\| \\ &\leq \sum_{i=n}^{n+m-1}\left\|\frac{\Delta\tilde{f}_e(2^i x)}{2880 \cdot 4^i}\right\| \leq \sum_{i=n}^{n+m-1}\frac{K\theta 2^{ip}\|x\|^p}{2880 \cdot 4^i}\end{aligned} \tag{2.21}$$

for all $x \in X$ and $n, m \in \mathbb{N} \cup \{0\}$. So $\left\{\frac{\tilde{f}_2(2^n x)}{4^n}\right\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, it converges and we can define a mapping $F^{(2)} : X \to Y$ by

$$F^{(2)}(x) := \lim_{n\to\infty}\frac{\tilde{f}_2(2^n x)}{4^n}$$

for all $x \in X$. Moreover, letting $n = 0$ and passing the limit $m \to \infty$ in (2.21), the following inequality is obtained

$$\|\tilde{f}_2(x) - F^{(2)}(x)\| \leq \frac{K\theta\|x\|^p}{720(4 - 2^p)}$$

for all $x \in X$, and by (2.3) it holds that

$$\begin{aligned}\|DF^{(2)}(x,y)\| &= \lim_{n\to\infty}\left\|\frac{D\tilde{f}_2(2^n x, 2^n y)}{4^n}\right\| \\ &= \lim_{n\to\infty}\left\|\frac{1024 D\tilde{f}_e(2^n x, 2^n y)}{720 \cdot 4^n} - \frac{80 D\tilde{f}_e(2^{n+1} x, 2^{n+1} y)}{720 \cdot 4^n} + \frac{D\tilde{f}_e(2^{n+2} x, 2^{n+2} y)}{720 \cdot 4^n}\right\| \\ &\leq \lim_{n\to\infty}\left(1024 \cdot 2^{np} + 80 \cdot 2^{(n+1)p} + 2^{(n+2)p}\right)\frac{\theta(\|x\|^p + \|y\|^p)}{720 \cdot 4^n} \\ &= 0\end{aligned}$$

for all $x, y \in X$.

(2) If $p > 2$, then it follows from (2.9) and (2.16) that

$$\begin{aligned}\left\|4^n \tilde{f}_2(2^{-n} x) - 4^{n+m}\tilde{f}_2(2^{-n-m} x)\right\| &= \left\|\sum_{i=n}^{n+m-1}\left(4^i \tilde{f}_2(2^{-i} x) - 4^{i+1}\tilde{f}_2(2^{-i-1} x)\right)\right\| \\ &\leq \sum_{i=n}^{n+m-1}\left\|\frac{4^{i+1}\Delta\tilde{f}_e(2^{-i-1} x)}{2880}\right\| \leq \sum_{i=n}^{n+m-1}\frac{4^i K\theta\|x\|^p}{720 \cdot 2^{(i+1)p}}\end{aligned} \tag{2.22}$$

for all $x \in X$ and $n, m \in \mathbb{N} \cup \{0\}$. So $\{4^n \tilde{f}_2(2^{-n}x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, it converges and we can define a mapping $F^{(2)} : X \to Y$ by

$$F^{(2)}(x) := \lim_{n\to\infty} 4^n \tilde{f}_2(2^{-n}x)$$

for all $x \in X$. Moreover, letting $n = 0$ and passing the limit $m \to \infty$ in (2.22), the following inequality is obtained

$$\|\tilde{f}_2(x) - F^{(2)}(x)\| \le \frac{K\theta\|x\|^p}{720(2^p - 4)}$$

and

$$\begin{aligned}
\|DF^{(2)}(x,y)\| &= \lim_{n\to\infty} 4^n \left\|D\tilde{f}_2(2^{-n}x, 2^{-n}y)\right\| \\
&= \lim_{n\to\infty} 4^n \left\|\frac{1024 D\tilde{f}_e(2^{-n}x, 2^{-n}y)}{720} - \frac{80 D\tilde{f}_e(2^{-n+1}x, 2^{-n+1}y)}{720} + \frac{D\tilde{f}_e(2^{-n+2}x, 2^{-n+2}y)}{720}\right\| \\
&\le \lim_{n\to\infty} \left(1024 \cdot 2^{-np} + 80 \cdot 2^{-np+p} + 2^{-np+2p}\right) \frac{4^n\theta(\|x\|^p + \|y\|^p)}{720} \\
&= 0
\end{aligned}$$

for all $x, y \in X$.

Step 3. For $p \ne 3$, there exists a mapping $F^{(3)} : X \to Y$ satisfying $F^{(3)}(0) = 0$, $DF^{(3)}(x,y) = 0$ for all $x, y \in X$, and

$$\|\tilde{f}_3(x) - F^{(3)}(x)\| \le \frac{K'\theta\|x\|^p}{17280 \cdot |8 - 2^p|} \tag{2.23}$$

for all $x \in X$.

(1) If $p < 3$, then it follows from (2.10) and (2.15) that

$$\begin{aligned}
\left\|\frac{\tilde{f}_3(2^n x)}{8^n} - \frac{\tilde{f}_3(2^{n+m}x)}{8^{n+m}}\right\| &= \left\|\sum_{i=n}^{n+m-1}\left(\frac{\tilde{f}_3(2^i x)}{8^i} - \frac{\tilde{f}_3(2^{i+1}x)}{8^{i+1}}\right)\right\| \\
&\le \sum_{i=n}^{n+m-1}\left\|\frac{\Gamma\tilde{f}_o(2^i x)}{138240 \cdot 8^i}\right\| \le \sum_{i=n}^{n+m-1}\frac{K'\theta 2^{ip}\|x\|^p}{138240 \cdot 8^i}
\end{aligned} \tag{2.24}$$

for all $x \in X$ and $n, m \in \mathbb{N} \cup \{0\}$. Then $\left\{\frac{\tilde{f}_3(2^n x)}{8^n}\right\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, it converges and we can define a mapping $F^{(3)} : X \to Y$ by

$$F^{(3)}(x) := \lim_{n\to\infty} \frac{\tilde{f}_3(2^n x)}{8^n}$$

for all $x \in X$. Moreover, letting $n = 0$ and passing the limit $m \to \infty$ in (2.24), the following inequality is obtained

$$\|\tilde{f}_3(x) - F^{(3)}(x)\| \le \frac{K'\theta\|x\|^p}{17280(8 - 2^p)}$$

for all $x \in X$, and by (2.4) it holds that

$$\begin{aligned}
\|DF^{(3)}(x,y)\| &= \lim_{n\to\infty} \left\| \frac{D\tilde{f}_3(2^n x, 2^n y)}{8^n} \right\| = \lim_{n\to\infty} \left(\left\| \frac{8192 D\tilde{f}_o(2^n x, 2^n y)}{17280 \cdot 8^n} \right\| \right. \\
&\quad + \left\| \frac{4416 D\tilde{f}_o(2^{n+1}x, 2^{n+1}y)}{17280 \cdot 8^n} \right\| + \left\| \frac{162 D\tilde{f}_o(2^{n+2}x, 2^{n+2}y)}{17280 \cdot 8^n} \right\| + \left. \left\| \frac{D\tilde{f}_o(2^{n+3}x, 2^{n+3}y)}{17280 \cdot 8^n} \right\| \right) \\
&\le \lim_{n\to\infty} \left(8192 \cdot 2^{np} + 4416 \cdot 2^{(n+1)p} + 162 \cdot 2^{(n+2)p} + 2^{(n+3)p} \right) \frac{\theta(\|x\|^p + \|y\|^p)}{17280 \cdot 8^n} \\
&= 0
\end{aligned}$$

for all $x, y \in X$.

(2) If $p > 3$, then it follows from (2.10) and (2.15) that

$$\begin{aligned}
\left\| 8^n \tilde{f}_3(2^{-n}x) - 8^{n+m} \tilde{f}_3(2^{-n-m}x) \right\| &= \left\| \sum_{i=n}^{n+m-1} \left(8^i \tilde{f}_3(2^{-i}x) - 8^{i+1} \tilde{f}_3(2^{-i-1}x) \right) \right\| \\
&\le \sum_{i=n}^{n+m-1} \left\| \frac{8^{i+1} \Gamma \tilde{f}_o(2^{-i-1}x)}{138240} \right\| \le \sum_{i=n}^{n+m-1} \frac{8^i K' \theta \|x\|^p}{17280 \cdot 2^{(i+1)p}} \qquad (2.25)
\end{aligned}$$

for all $x \in X$ and $n, m \in \mathbb{N} \cup \{0\}$. So $\{8^n \tilde{f}_3(2^{-n}x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence converges and we can define a mapping $F^{(3)} : X \to Y$ by

$$F^{(3)}(x) := \lim_{n\to\infty} 8^n \tilde{f}_3(2^{-n}x)$$

for all $x \in X$. Moreover, letting $n = 0$ and passing the limit $m \to \infty$ in (2.25), the following inequality is obtained

$$\|\tilde{f}_3(x) - F^{(3)}(x)\| \le \frac{K' \theta \|x\|^p}{17280(2^p - 8)}$$

for all $x \in X$, and by (2.4) it holds that

$$\begin{aligned}
\|DF^{(3)}(x,y)\| &= \lim_{n\to\infty} 8^n \left\| D\tilde{f}_3(2^{-n}x, 2^{-n}y) \right\| = \lim_{n\to\infty} 8^n \left(\left\| \frac{8192 D\tilde{f}_o(2^{-n}x, 2^{-n}y)}{17280} \right\| \right. \\
&\quad + \left\| \frac{4416 D\tilde{f}_o(2^{-n+1}x, 2^{-n+1}y)}{17280} \right\| + \left\| \frac{162 D\tilde{f}_o(2^{-n+2}x, 2^{-n+2}y)}{17280} \right\| + \left. \left\| \frac{D\tilde{f}_o(2^{-n+3}x, 2^{-n+3}y)}{17280} \right\| \right) \\
&\le \lim_{n\to\infty} \left(8192 \cdot 2^{-np} + 4416 \cdot 2^{-np+p} + 162 \cdot 2^{-np+2p} + 2^{-np+3p} \right) \frac{8^n \theta(\|x\|^p + \|y\|^p)}{17280} \\
&= 0
\end{aligned}$$

for all $x, y \in X$.

Step 4. For $p \neq 4$, there exists a mapping $F^{(4)} : X \to Y$ satisfying $F^{(4)}(0) = 0$, $DF^{(4)}(x,y) = 0$ for all $x, y \in X$, and

$$\|\tilde{f}_4(x) - F^{(4)}(x)\| \le \frac{K \theta \|x\|^p}{576 \cdot |16 - 2^p|} \qquad (2.26)$$

for all $x \in X$.

(1) If $p < 4$, then it follows from (2.10) and (2.16) that

$$\begin{aligned}\left\|\frac{\tilde{f}_4(2^n x)}{16^n} - \frac{\tilde{f}_4(2^{n+m} x)}{16^{n+m}}\right\| &= \left\|\sum_{i=n}^{n+m-1}\left(\frac{\tilde{f}_4(2^i x)}{16^i} - \frac{\tilde{f}_4(2^{i+1} x)}{16^{i+1}}\right)\right\| \\ &\leq \sum_{i=n}^{n+m-1}\left\|\frac{\Delta\tilde{f}_e(2^i x)}{9216\cdot 16^i}\right\| \leq \sum_{i=n}^{n+m-1}\frac{K\theta 2^{ip}\|x\|^p}{9216\cdot 16^i} \end{aligned} \tag{2.27}$$

for all $x \in X$ and $n, m \in \mathbb{N}\cup\{0\}$. So $\left\{\frac{\tilde{f}_4(2^n x)}{16^n}\right\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, it converges and we can define a mapping $F^{(4)} : X \to Y$ by

$$F^{(4)}(x) := \lim_{n\to\infty}\frac{\tilde{f}_4(2^n x)}{16^n}$$

for all $x \in X$. Moreover, letting $n = 0$ and passing the limit $m \to \infty$ in (2.27), the following inequality is obtained

$$\|\tilde{f}_4(x) - F^{(4)}(x)\| \leq \frac{K\theta\|x\|^p}{576(16-2^p)}$$

for all $x \in X$, and by (2.5) it holds that

$$\begin{aligned}\|DF^{(4)}(x,y)\| &= \lim_{n\to\infty}\left\|\frac{D\tilde{f}_4(2^n x, 2^n y)}{16^n}\right\| \\ &= \lim_{n\to\infty}\left\| -\frac{256D\tilde{f}_e(2^n x, 2^n y)}{576\cdot 16^n} + \frac{68D\tilde{f}_e(2^{n+1}x, 2^{n+1}y)}{576\cdot 16^n} - \frac{D\tilde{f}_e(2^{n+2}x, 2^{n+2}y)}{576\cdot 16^n}\right\| \\ &\leq \lim_{n\to\infty}\left(256\cdot 2^{np} + 68\cdot 2^{(n+1)p} + 2^{(n+2)p}\right)\frac{\theta(\|x\|^p + \|y\|^p)}{576\cdot 16^n} \\ &= 0\end{aligned}$$

for all $x, y \in X$.

(2) If $p > 4$, then it follows from (2.10) and (2.16) that

$$\begin{aligned}\left\|16^n\tilde{f}_4(2^{-n}x) - 16^{n+m}\tilde{f}_4(2^{-n-m}x)\right\| &= \left\|\sum_{i=n}^{n+m-1}\left(16^i\tilde{f}_4(2^{-i}x) - 16^{i+1}\tilde{f}_4(2^{-i-1}x)\right)\right\| \\ &\leq \sum_{i=n}^{n+m-1}\left\|\frac{16^{i+1}\Delta\tilde{f}_e(2^{-i-1}x)}{9216}\right\| \leq \sum_{i=n}^{n+m-1}\frac{K\theta 16^i\|x\|^p}{576\cdot 2^{(i+1)p}}\end{aligned} \tag{2.28}$$

for all $x \in X$ and $n, m \in \mathbb{N}\cup\{0\}$. So $\{16^n\tilde{f}_4(2^{-n}x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, it converges and we can define a mapping $F^{(4)} : X \to Y$ by

$$F^{(4)}(x) := \lim_{n\to\infty}16^n\tilde{f}_4(2^{-n}x)$$

for all $x \in X$. Moreover, letting $n = 0$ and passing the limit $m \to \infty$ in (2.28), the following inequality is obtained

$$\|\tilde{f}_4(x) - F^{(4)}(x)\| \le \frac{K\theta\|x\|^p}{576(2^p - 16)}$$

for all $x \in X$, and by (2.5) it holds that

$$\begin{aligned}
\|DF^{(4)}(x,y)\| &= \lim_{n\to\infty} 16^n \left\|D\tilde{f}_4(2^{-n}x, 2^{-n}y)\right\| \\
&= \lim_{n\to\infty} 16^n \left\| -\frac{256D\tilde{f}_e(2^{-n}x, 2^{-n}y)}{576} + \frac{68D\tilde{f}_e(2^{-n+1}x, 2^{-n+1}y)}{576} - \frac{D\tilde{f}_e(2^{-n+2}x, 2^{-n+2}y)}{576} \right\| \\
&\le \lim_{n\to\infty} \left(256 \cdot 2^{-np} + 68 \cdot 2^{-np+p} + 2^{-np+2p}\right) \frac{16^n\theta(\|x\|^p + \|y\|^p)}{576} \\
&= 0
\end{aligned}$$

for all $x, y \in X$.

Step 5. For $p \neq 5$, there exists a mapping $F^{(5)} : X \to Y$ satisfying $F^{(5)}(0) = 0$, $DF^{(5)}(x,y) = 0$ for all $x, y \in X$, and

$$\|\tilde{f}_5(x) - F^{(5)}(x)\| \le \frac{K'\theta\|x\|^p}{69120 \cdot |32 - 2^p|} \tag{2.29}$$

for all $x, y \in X$.

(1) If $p < 5$, then it follows from (2.11) and (2.15) that

$$\begin{aligned}
\left\|\frac{\tilde{f}_5(2^n x)}{32^n} - \frac{\tilde{f}_5(2^{n+m}x)}{32^{n+m}}\right\| &= \left\|\sum_{i=n}^{n+m-1}\left(\frac{\tilde{f}_5(2^i x)}{32^i} - \frac{\tilde{f}_5(2^{i+1}x)}{32^{i+1}}\right)\right\| \\
&\le \sum_{i=n}^{n+m-1}\left\|\frac{\Gamma\tilde{f}_o(2^i x)}{2211840 \cdot 32^i}\right\| \le \sum_{i=n}^{n+m-1}\frac{K'\theta 2^{ip}\|x\|^p}{2211840 \cdot 32^i}
\end{aligned} \tag{2.30}$$

for all $x \in X$ and $n, m \in \mathbb{N} \cup \{0\}$. Then $\left\{\frac{\tilde{f}_5(2^n x)}{32^n}\right\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, it converges and we can define a mapping $F^{(5)} : X \to Y$ by

$$F^{(5)}(x) := \lim_{n\to\infty}\frac{\tilde{f}_5(2^n x)}{32^n}$$

for all $x \in X$. Moreover, letting $n = 0$ and passing the limit $m \to \infty$ in (2.30), the following inequality is obtained

$$\|\tilde{f}_5(x) - F^{(5)}(x)\| \le \frac{K'\theta\|x\|^p}{69120(32 - 2^p)}$$

for all $x \in X$, and by (2.6) it holds that

$$\|DF^{(5)}(x,y)\| = \lim_{n\to\infty}\left\|\frac{D\tilde{f}_5(2^n x, 2^n y)}{32^n}\right\| \le \lim_{n\to\infty}\left(\left\|\frac{2048 D\tilde{f}_o(2^n x, 2^n y)}{69120\cdot 32^n}\right\|\right.$$
$$\left. + \left\|\frac{1296 D\tilde{f}_o(2^{n+1} x, 2^{n+1} y)}{69120\cdot 32^n}\right\| + \left\|\frac{138 D\tilde{f}_o(2^{n+2} x, 2^{n+2} y)}{69120\cdot 32^n}\right\| + \left\|\frac{D\tilde{f}_o(2^{n+3} x, 2^{n+3} y)}{69120\cdot 32^n}\right\|\right)$$
$$\le \lim_{n\to\infty}\left(2048\cdot 2^{np} + 1296\cdot 2^{(n+1)p} + 138\cdot 2^{(n+2)p} + 2^{(n+3)p}\right)\frac{\theta(\|x\|^p + \|y\|^p)}{69120\cdot 32^n}$$
$$= 0$$

for all $x, y \in X$.

(2) If $p > 5$, then it follows from (2.11) and (2.15) that

$$\left\|32^n \tilde{f}_5(2^{-n}x) - 32^{n+m}\tilde{f}_5(2^{-n-m}x)\right\| = \left\|\sum_{i=n}^{n+m-1}\left(32^i \tilde{f}_5(2^{-i}x) - 32^{i+1}\tilde{f}_5(2^{-i-1}x)\right)\right\|$$
$$\le \sum_{i=n}^{n+m-1}\left\|\frac{32^{i+1}\Gamma\tilde{f}_o(2^{-i-1}x)}{2211840}\right\| \le \sum_{i=n}^{n+m-1}\frac{32^i K'\theta\|x\|^p}{69120\cdot 2^{(i+1)p}} \tag{2.31}$$

for all $x \in X$ and $n, m \in \mathbb{N}\cup\{0\}$. So $\{32^n \tilde{f}_5(2^{-n}x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, it converges and we can define a mapping $F^{(5)} : X \to Y$ by

$$F^{(5)}(x) := \lim_{n\to\infty} 32^n \tilde{f}_5(2^{-n}x)$$

for all $x \in X$. Moreover, letting $n = 0$ and passing the limit $m \to \infty$ in (2.31), the following inequality is obtained

$$\|\tilde{f}_5(x) - F^{(5)}(x)\| \le \frac{K'\theta\|x\|^p}{69120(2^p - 32)}$$

for all $x \in X$, and, using (2.6), we have

$$\|DF^{(5)}(x,y)\| = \lim_{n\to\infty} 32^n\left\|D\tilde{f}_5(2^{-n}x, 2^{-n}y)\right\| \le \lim_{n\to\infty} 32^n\left(\left\|\frac{2048 D\tilde{f}_o(2^{-n}x, 2^{-n}y)}{69120}\right\|\right.$$
$$\left. + \left\|\frac{1296 D\tilde{f}_o(2^{-n+1}x, 2^{-n+1}y)}{69120}\right\| + \left\|\frac{138 D\tilde{f}_o(2^{-n+2}x, 2^{-n+2}y)}{69120}\right\| + \left\|\frac{D\tilde{f}_o(2^{-n+3}x, 2^{-n+3}y)}{69120}\right\|\right)$$
$$\le \lim_{n\to\infty}\left(2048\cdot 2^{-np} + 1296\cdot 2^{-np+p} + 138\cdot 2^{-np+2p} + 2^{-np+3p}\right)\frac{32^n\theta(\|x\|^p + \|y\|^p))}{69120}$$
$$= 0$$

for all $x, y \in X$.

Step 6. For $p \neq 6$, there exists a mapping $F^{(6)} : X \to Y$ satisfying $F^{(6)}(0) = 0$, $DF^{(6)}(x,y) = 0$ for all $x, y \in X$, and

$$\|\tilde{f}_6(x) - F^{(6)}(x)\| \le \frac{K\theta\|x\|^p}{2880\cdot|64 - 2^p|} \tag{2.32}$$

for all $x, y \in X$.

(1) If $p < 6$, then it follows from (2.11) and (2.16) that

$$\begin{aligned}\left\|\frac{\tilde{f}_6(2^n x)}{64^n} - \frac{\tilde{f}_6(2^{n+m} x)}{64^{n+m}}\right\| &= \left\|\sum_{i=n}^{n+m-1}\left(\frac{\tilde{f}_6(2^i x)}{64^i} - \frac{\tilde{f}_6(2^{i+1} x)}{64^{i+1}}\right)\right\| \\ &\le \sum_{i=n}^{n+m-1}\left\|\frac{\Delta \tilde{f}_e(2^i x)}{184320 \cdot 64^i}\right\| \le \sum_{i=n}^{n+m-1}\frac{K 2^{ip}\theta\|x\|^p}{184320 \cdot 64^i} \qquad (2.33)\end{aligned}$$

for all $x \in X$ and $n, m \in \mathbb{N} \cup \{0\}$. So $\left\{\frac{\tilde{f}_6(2^n x)}{64^n}\right\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, it converges and we can define a mapping $F^{(6)} : X \to Y$ by

$$F^{(6)}(x) := \lim_{n\to\infty}\frac{\tilde{f}_6(2^n x)}{64^n}$$

for all $x \in X$. Moreover, letting $n = 0$ and passing the limit $m \to \infty$ in (2.33), the following inequality is obtained

$$\|\tilde{f}_6(x) - F^{(6)}(x)\| \le \frac{K\theta\|x\|^p}{2880(64 - 2^p)}$$

for all $x \in X$, and by (2.7) it holds that

$$\begin{aligned}\|DF^{(6)}(x,y)\| &= \lim_{n\to\infty}\left\|\frac{D\tilde{f}_6(2^n x, 2^n y)}{64^n}\right\| \\ &= \lim_{n\to\infty}\left\|\frac{64 D\tilde{f}_e(2^n x, 2^n y)}{2880 \cdot 64^n} - \frac{5376 D\tilde{f}_o(2^{n+1} x, 2^{n+1} y)}{2880 \cdot 64^n} + \frac{168 D\tilde{f}_o(2^{n+2} x, 2^{n+2} y)}{2880 \cdot 64^n}\right\| \\ &\le \lim_{n\to\infty}\left(64 \cdot 2^{np} + 20 \cdot 2^{(n+1)p} + 2^{(n+2)p}\right)\frac{\theta(\|x\|^p + \|y\|^p)}{2880 \cdot 64^n} \\ &= 0\end{aligned}$$

for all $x, y \in X$.

(2) If $p > 6$, then it follows from (2.11) and (2.16) that

$$\begin{aligned}\left\|64^n \tilde{f}_6(2^{-n} x) - 64^{n+m}\tilde{f}_6(2^{-n-m} x)\right\| &= \left\|\sum_{i=n}^{n+m-1}\left(64^i \tilde{f}_6(2^{-i} x) - 64^{i+1}\tilde{f}_6(2^{-i-1} x)\right)\right\| \\ &\le \sum_{i=n}^{n+m-1}\left\|\frac{64^{i+1}\Delta \tilde{f}_e(2^{-i-1} x)}{184320}\right\| \le \sum_{i=n}^{n+m-1}\frac{64^i K\theta\|x\|^p}{2880 \cdot 2^{(i+1)p}} \qquad (2.34)\end{aligned}$$

for all $x \in X$ and $n, m \in \mathbb{N} \cup \{0\}$. So $\{64^n \tilde{f}_6(2^{-n} x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, it converges and we can define a mapping $F^{(6)} : X \to Y$ by

$$F^{(6)}(x) := \lim_{n\to\infty} 64^n \tilde{f}_6(2^{-n} x)$$

for all $x \in X$. Moreover, letting $n = 0$ and passing the limit $m \to \infty$ in (2.34), the following inequality is obtained

$$\|\tilde{f}_6(x) - F^{(6)}(x)\| \leq \frac{K\theta\|x\|^p}{2880(2^p - 64)}$$

for all $x \in X$, and by (2.7) it holds that

$$\begin{aligned}
\|DF^{(6)}(x,y)\| &= \lim_{n\to\infty} 64^n \left\|D\tilde{f}_6(2^{-n}x, 2^{-n}y)\right\| \\
&= \lim_{n\to\infty} \frac{64^n}{2880} \left\|64D\tilde{f}_e(2^{-n}x, 2^{-n}y) - 20D\tilde{f}_e(2^{-n+1}x, 2^{-n+1}y) + D\tilde{f}_e(2^{-n+2}x, 2^{-n+2}y)\right\| \\
&\leq \lim_{n\to\infty} \left(64 \cdot 2^{-np} + 20 \cdot 2^{-np+p} + 2^{-np+2p}\right) \frac{64^n\theta(\|x\|^p + \|y\|^p)}{2880} \\
&= 0
\end{aligned}$$

for all $x, y \in X$.

Step 7. For $p \neq 7$, there exists a mapping $F^{(7)} : X \to Y$ satisfying $F^{(7)}(0) = 0$, $DF^{(7)}(x,y) = 0$ for all $x, y \in X$, and

$$\|\tilde{f}_7(x) - F^{(7)}(x)\| \leq \frac{K'\theta\|x\|^p}{1451520 \cdot |128 - 2^p|} \tag{2.35}$$

for all $x \in X$.

(1) If $p < 7$, then it follows from (2.12) and (2.15) that

$$\begin{aligned}
\left\|\frac{\tilde{f}_7(2^n x)}{128^n} - \frac{\tilde{f}_7(2^{n+m}x)}{128^{n+m}}\right\| &= \left\|\sum_{i=n}^{n+m-1}\left(\frac{\tilde{f}_7(2^i x)}{128^i} - \frac{\tilde{f}_7(2^{i+1}x)}{128^{i+1}}\right)\right\| \\
&\leq \sum_{i=n}^{n+m-1}\left\|\frac{\Gamma\tilde{f}_o(2^i x)}{185794560 \cdot 128^i}\right\| \leq \sum_{i=n}^{n+m-1}\frac{K'\theta 2^{ip}\|x\|^p}{185794560 \cdot 128^i}
\end{aligned} \tag{2.36}$$

for all $x \in X$ and $n, m \in \mathbb{N} \cup \{0\}$. So $\left\{\frac{\tilde{f}_7(2^n x)}{128^n}\right\}$ becomes a Cauchy sequence for all $x \in X$. Since Y is complete, it converges, and hence we can define a mapping $F^{(7)} : X \to Y$ by

$$F^{(7)}(x) := \lim_{n\to\infty} \frac{\tilde{f}_7(2^n x)}{128^n}$$

for all $x \in X$. Moreover, letting $n = 0$ and passing the limit $m \to \infty$ in (2.36), the following inequality is obtained

$$\|\tilde{f}_7(x) - F^{(7)}(x)\| \leq \frac{K'\theta\|x\|^p}{1451520(128 - 2^p)}$$

for all $x \in X$, and, using (2.8), we have

$$\|DF^{(7)}(x,y)\| = \lim_{n\to\infty} \|\frac{D\tilde{f}_7(2^n x, 2^n y)}{128^n}\| = \lim_{n\to\infty} \Big\| -\frac{512 D\tilde{f}_o(2^n x, 2^n y)}{65536\cdot 128^n}$$
$$+ \frac{336 D\tilde{f}_o(2^{n+1}x, 2^{n+1}y)}{65536\cdot 128^n} - \frac{42 D\tilde{f}_o(2^{n+2}x, 2^{n+2}y)}{65536\cdot 128^n} + \frac{D\tilde{f}_o(2^{n+3}x, 2^{n+3}y)}{65536\cdot 128^n}\Big\|$$
$$\le \lim_{n\to\infty} \Big(512\cdot 2^{np} + 336\cdot 2^{(n+1)p} + 42\cdot 2^{(n+2)p} + 2^{(n+3)p}\Big) \frac{\theta(\|x\|^p + \|y\|^p)}{65536\cdot 128^n}$$
$$= 0$$

for all $x, y \in X$.

(2) If $p > 7$, then it follows from (2.12) and (2.15) that

$$\Big\|128^n \tilde{f}_7(2^{-n}x) - 128^{n+m}\tilde{f}_7(2^{-n-m}x)\Big\| = \Big\|\sum_{i=n}^{n+m-1}\Big(128^i \tilde{f}_7(2^{-i}x) - 128^{i+1}\tilde{f}_7(2^{-i-1}x)\Big)\Big\|$$
$$\le \sum_{i=n}^{n+m-1}\Big\| -\frac{128^{i+1}\Gamma \tilde{f}_o(2^{-i-1}x)}{185794560}\Big\| \le \sum_{i=n}^{n+m-1} \frac{128^i K'\theta\|x\|^p}{1451520\cdot 2^{(i+1)p}} \tag{2.37}$$

for all $x \in X$ and $n, m \in \mathbb{N}\cup\{0\}$. So $\{128^n \tilde{f}_7(2^{-n}x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, it converges and we can define a mapping $F^{(7)} : X \to Y$ by

$$F^{(7)}(x) := \lim_{n\to\infty} 128^n \tilde{f}_7(2^{-n}x)$$

for all $x \in X$. Moreover, letting $n = 0$ and passing the limit $m \to \infty$ in (2.37), the following inequality is obtained

$$\|\tilde{f}_7(x) - F^{(7)}(x)\| \le \frac{K'\theta\|x\|^p}{1451520(2^p - 128)}$$

for all $x \in X$, and, using (2.8), we have

$$\|DF^{(7)}(x,y)\| = \lim_{n\to\infty} 128^n \Big\|D\tilde{f}_7(2^{-n}x, 2^{-n}y)\Big\| = \lim_{n\to\infty} 128^n \Big\|\frac{-512 D\tilde{f}_o(2^{-n}x, 2^{-n}y)}{2835\cdot 512}$$
$$+ \frac{336 D\tilde{f}_o(2^{-n+1}x, 2^{-n+1}y)}{2835\cdot 512} - \frac{42 D\tilde{f}_o(2^{-n+2}x, 2^{-n+2}y)}{2835\cdot 8} + \frac{D\tilde{f}_o(2^{-n+3}x, 2^{-n+3}y)}{2835\cdot 512}\Big\|$$
$$\le \lim_{n\to\infty} \big(512\cdot 2^{-np} + 336\cdot 2^{-np+p} + 42\cdot 2^{-np+2p} + 2^{-np+3p}\big) \frac{128^n\theta(\|x\|^p + \|y\|^p))}{1451520}$$
$$= 0$$

for all $x, y \in X$.

Now, using the functions $F^{(1)}, F^{(2)}, \cdots, F^{(7)} : X \to Y$ of *Step1, Step 2*, ..., and *Step 7*, respectively, we put

$$F(x) := \sum_{i=1}^{7} F^{(i)}(x)$$

for all $x \in X$. Since $\|\tilde{f}(x) - F(x)\| \le \sum_{i=1}^{7} \|\tilde{f}_i(x) - F^{(i)}(x)\|$ for all $x \in X$, together with (2.17), (2.20), (2.23), (2.26), (2.29), (2.32), (2.35), the property (2.14) is obtained. It is obvious that

$$DF(x,y) = \sum_{i=1}^{7} DF^{(i)}(x,y) = 0$$

for all $x, y \in X$. Finally, to prove the uniqueness of F, let $G : X \to Y$ be another mapping, which satisfies the property (2.14), $G(0) = 0$, and $DG(x,y) = 0$ for all $x, y \in X$. And let $G_1, G_2, \cdots, G_7 : X \to Y$ be defined as in Definition 2.1. Since $DG_o(x,y) = DG_e(x,y) = 0$ for all $x, y \in X$,

$$DG_i(x,y) = 0, \quad i = 1, 2, \cdots, 7$$

for all $x, y \in X$, Additionally, by (2.14) with the definitions of the odd function and the even function of Definition 2.1, we get

$$\|\tilde{f}_e(x) - G_e(x)\|, \quad \|\tilde{f}_e(x) - G_e(x)\| \le N\theta\|x\|^p \tag{2.38}$$

for all $x \in X$, where

$$\begin{aligned} N := &\frac{K'}{22680 \cdot |2 - 2^p|} + \frac{K}{720 \cdot |4 - 2^p|} + \frac{K'}{17280 \cdot |8 - 2^p|} + \frac{K}{576 \cdot |16 - 2^p|} \\ &+ \frac{K'}{69120 \cdot |32 - 2^p|} + \frac{K}{2880 \cdot |64 - 2^p|} + \frac{K'}{1451520 \cdot |2^p - 128|}. \end{aligned}$$

It is notable that $\Gamma G(x) = \Delta G(x) = 0$ for all $x \in X$ by (2.9)-(2.12), it can be proved that $G_i(2x) = 2^i G_i(x)$, $i = 1, 2, \cdots, 7$, for all $x \in X$. Particularly, they hold that

$$G_2(x) = 2^2 G_2\left(\frac{x}{2}\right) = \cdots\cdots = 4^n G_2\left(\frac{x}{2^n}\right) \tag{2.39}$$

$$G_3(2^n x) = 2^3 G_3(2^{n-1}x) = \cdots\cdots = 8^n G_3(x) \tag{2.40}$$

for all $x \in X$ and $n \in \mathbb{N}$. Now, it can be noted that, in the case of $2 < p < 3$, we have

$$\begin{aligned} \left\|4^n \tilde{f}_2\left(\frac{x}{2^n}\right) - G_2(x)\right\| &= \left\|4^n \tilde{f}_2\left(\frac{x}{2^n}\right) - 4^n G_2\left(\frac{x}{2^n}\right)\right\| \le \frac{1024 \cdot 4^n}{720}\left\|\tilde{f}_e\left(\frac{x}{2^n}\right) - G_e\left(\frac{x}{2^n}\right)\right\| \\ &\quad + \frac{80 \cdot 4^n}{720}\left\|\tilde{f}_e\left(\frac{2x}{2^n}\right) - G_e\left(\frac{2x}{2^n}\right)\right\| + \frac{4^n}{720}\left\|\tilde{f}_e\left(\frac{4x}{2^n}\right) - G_e\left(\frac{4x}{2^n}\right)\right\| \\ &\le (1024 + 80 \cdot 2^p + 4^p)\frac{4^n}{720 \cdot 2^{np}} N\theta\|x\|^p, \end{aligned}$$

and

$$\begin{aligned} \left\|\frac{\tilde{f}_3(2^n x)}{8^n} - G_3(x)\right\| &= \left\|\frac{\tilde{f}_3(2^n x)}{8^n} - \frac{G_3(2^n x)}{8^n}\right\| \\ &\le \left\|\frac{8192(\tilde{f}_o(2^n x) - G_o(2^n x))}{17280 \cdot 8^n}\right\| + \left\|\frac{4416(\tilde{f}_o(2^{n+1}x) - G_o(2^{n+1}x))}{17280 \cdot 8^n}\right\| \\ &\quad + \left\|\frac{162(\tilde{f}_o(2^{n+2}x) - G_o(2^{n+2}x))}{17280 \cdot 8^n}\right\| + \left\|\frac{(\tilde{f}_o(2^{n+3}x) - G_o(2^{n+3}x))}{17280 \cdot 8^n}\right\| \\ &\le \left(8192 + 4416 \cdot 2^p + 162 \cdot 2^{2p} + 2^{3p}\right)\frac{2^{np} N\theta\|x\|^p}{172808^n} \end{aligned}$$

for all $x \in X$ and all positive integers n. Taking the limit in the above inequalities as $n \to \infty$, we obtain

$$G_2(x) = \lim_{n\to\infty} \tilde{4}^n \tilde{f}_2\left(\frac{x}{2^n}\right) = F^{(2)}(x), \qquad G_3(x) = \lim_{n\to\infty} \frac{\tilde{f}_3(2^n x)}{8^n} = F^{(3)}(x)$$

for all $x \in X$. Also, in the same way, it can be shown that $G_i(x) = F^{(i)}(x)$, $i = 1, 4, 5, 6, 7$, for all $x \in X$. Therefore, it can be shown that $G(x) = F(x)$ for all $x \in X$ in the case of $2 < p < 3$. Similarly, the uniqueness of F can be proved in other cases of p. □

3 Conclusion

In this work, we have investigated the stability of the following general septic functional equation

$$D(x,y) := \sum_{i=0}^{8} {}_8C_i(-1)^{8-i} f(x + (i-4)y) = 0,$$

from the perspective of Hyers-Ulam-Rassias stability. Precisely, for a non-negative real number $p \neq 1, 2, 3, 4, 5, 6, 7$ and a real number $\theta > 0$, if the mapping $f : X \to Y$ satisfies

$$\|Df(x,y)\| \leq \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$, then there exists a unique septic mapping F, i.e., $DF(x,y) = 0$ for all $x, y \in X$, such that $F(0) = 0$ and

$$\|f(x) - f(0) - F(x)\| \leq \epsilon_p \theta \|x\|^p$$

for all $x \in X$, where the constant ϵ_p depends only on p. To prove it, we use the mappings f_1, f_2, f_3, f_4, f_5, f_6, $f_7 : X \to Y$ which are defined in Theorem 2.2, such that

$$f(x) = \sum_{i=1}^{7} f_i(x)$$

for all $x \in X$. And then, it is possible to construct the septic mappings $F^{(1)}, F^{(2)}, \cdots, F^{(7)} : X \to Y$ satisfying $F^{(i)}(0) = 0$ and

$$\|f_i(x) - f_i(0) - F^{(i)}(x)\| \leq \epsilon_{i,p} \theta \|x\|^p,$$

for all $x \in X$, where $i = 1, 2, \cdots, 7$ and $\epsilon_{i,p}$ depends only on i and p. Then, putting

$$F(x) := \sum_{i=1}^{7} F^{(i)}(x)$$

for all $x \in X$, we have shown that F is the unique solution of the general septic functional equation $D(x,y) = 0$ such that $\|f(x) - f(0) - F(x)\| \leq \epsilon_p \theta \|x\|^p$ for all $x \in X$, where $\epsilon_p := \sum_{i=0}^{7} \epsilon_{i,p}$.

References

[1] J. Baker, *A general functional equation and its stability*, Proc. Natl. Acad. Sci. **133(6)** (2005), 1657–1664.

[2] B. Bouikhalene, A. Charifi, and E. Elqorachi, *Hyers-Ulam-Rassias stability of a generalized Pexider functional equation*, Banach J. Math. Anal. **1(2)**(2007), 176–185

[3] I. S. Chang, Y. H. Lee, and J. Roh, *On the stability of the general sextic functional equation*, J. Chungcheong Math. Soc. **34(3)**(2021), 295–306.

[4] I. S. Chang, Y. H. Lee, and J. Roh, *Nearly general septic functional equation*, J. Function spaces. **2021**(2021), https://doi.org/10.1155/2021/5643145.

[5] Z. Gajda, *On stability of additive mappings*, Int. J. Math. & Math. Sci. **14(3)** (1991), 431–434.

[6] P. Găvruta, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl. **184** (1994), 431–436.

[7] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci. USA **27** (1941), 222–224.

[8] G. Isac and T. M. Rassias, *On the Hyers-Ulam stability of ψ-additive mappings*, J. Approx. Theory **72** (1993), 131–137.

[9] S. S. Jin and Y. H. Lee, *Hyers-Ulam-Rassias stability of a functional equation related to general quadratic mappings*, Honam Math. J. **39(3)** (2017), 417–430.

[10] S. S. Jin and Y. H. Lee, *Stability of the General Quintic Functional Equation*, Int. J. Math. Anal.(Ruse) **15(6)**(2021), 271–282.

[11] K.-W. Jun and H.-M. Kim, *On the Hyers-Ulam-Rassias stability of a general cubic functional equation*, Math. Inequal. Appl. **6** (2003), 289—302.

[12] Y.-H. Lee, *On the generalized Hyers-Ulam stability of the generalized polynomial function of degree 3*, Tamsui Oxf. J. Math. Sci. **24(4)** (2008), 429–444.

[13] Y.-H. Lee, *On the Hyers-Ulam-Rassias stability of the generalized polynomial function of degree 2*, J. Chungcheong Math. Soc. **22(2)** (2009), 201–209.

[14] Y.-H. Lee, *Stability of a monomial functional equation on a restricted domain*, Mathematics **5** (2017), 53.

[15] Y.-H. Lee, *On the Hyers-Ulam-Rassias stability of a general quartic functional equation*, East Asian Math. J. **35(3)** (2019), 351–356.

[16] Y.-H. Lee, *On the Hyers-Ulam-Rassias stability of a general quintic functional equation and a general sextic functional equation*, Mathematics **7** (2019), 510.

[17] Y.-H. Lee, *On the Hyers-Ulam-Rassias stability of an additive-quadratic-cubic functional equation*, J. Chungcheong Math. Soc. **32(3)** (2019), 295–307.

[18] Y.-H. Lee and K. W. Jun, *A generalization of the Hyers–Ulam–Rassias stability of Jensen's equation*, J. Math. Anal. Appl. **238(1)** (1999),305–315.

[19] Y.-H. Lee, and S.-M. Jung, *A fixed point approach to the stability of an additive-quadratic-cubic-quartic type functional equation*, J. Funct. Spaces **2016** (2016).

[20] Y.-H. Lee, and S.-M. Jung, *A general theorem on the stability of a class of functional equations including quadratic-additive functional equations*, SpringerPlus **5(1)** (2016), 1–16.

[21] Y.-H. Lee, and S.-M. Jung, *General uniqueness theorem concerning the stability of additive, quadratic, and cubic functional equations*, Adv. Differ. Equ. (2016)

[22] Y.-H. Lee, and S.-M. Jung, *A general theorem on the stability of a class of functional equations including quartic-cubic-quadratic-additive equations*, Mathematics **6(12)** (2018), 282.

[23] Y.-H. Lee, and S.-M. Jung, *A general uniqueness theorem concerning the stability of AQCQ type functional equations*, Kyungpook Math. J. **58(2)** (2018), 291–305.

[24] Y.-H. Lee, and S.-M. Jung, *A fixed point approach to the stability of a general quartic functional equation*, J. Math. Comput. Sci. **20** (2020), 207–215.

[25] Y.-H. Lee, and S.-M. Jung, *Generalized Hyers-Ulam stability of some cubic-quadratic-additive type functional equations*, Kyungpook Math. J. **60(1)** (2020), 133–144.

[26] Th. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), 297–300.

[27] J. Roh,Y.-H. Lee, and S.-M. Jung, *The stability of a general sextic functional equation by fixed point theory*, J. Funct. Spaces **2020** (2020).

[28] S.M. Ulam, *A Collection of Mathematical Problems*, Interscience, New York, 1960.