Original Research Article

Hyers-Ulam-Rassias stability of a general septic functional equation

Abstract. In this paper, we investigate the stability of a general septic functional equation

$$\sum_{i=0}^{8} {}_8C_i(-1)^{8-i}f(x+(i-4)y)=0$$

in the sense of Rassias.

AMS Subject Classification: 39B82; 39B52.

Key Words: stability of a functional equation; general septic functional equation; general septic mapping.

1 Introduction

It is well known that the study on the stability of functional equations began as an attempt to solve Ulam's question in [26] about the stability of the group homomorphisms. As a partial answer to this question, Hyers [5] solved the stability of the Cauchy functional equation in the following year. Since then, many mathematicians have generalized Hyers' results by showing the stability of various kind of functional equations, see [3, 4, 6, 12, 24]. Today the term 'Hyers-Ulam-Rassias stability' refers to the generalization introduced by Rassias [24].

Throughout this paper, let V, X, and Y be a real vector space, a real normed space, and a real Banach space, respectively. For a mapping f from V to Y, we consider the functional equation

$$\sum_{i=0}^{k} {}_kC_i(-1)^{k-i}f(x+iy)=0, \tag{1.1}$$

where $k = 2, 3, 4, 5, 6, 7, 8$. A solution mapping $f : V \to Y$ of (1.1) is a "generalized polynomial mapping of degree at most 7" in the sense of J. Baker in [1]. So we call the functional equation (1.1) a Jensen, a general quadratic, a general cubic, a general quartic, a general quintic, a general sextic, and a general septic functional equation, for

$k = 2, 3, 4, 5, 6, 7, 8$, respectively. Also we call each solution mapping of (1.1), for $k = 2, 3, 4, 5, 6, 7, 8$, a Jensen, a general quadratic, a general cubic, a general quartic, a general quintic, a general sextic, and a general septic mapping, respectively. Prior this paper, the stability problems for the functional equation (1.1) were studied in many ways. In the case of a Jensen functional equation, K.-W. Jun et al.[16] showed the stability result. The stability of a general quadratic function equation were obtained by Y. H. Lee [11], Y. H. Lee et al. [18], and S. S. Jin et al.[7]. The stability of a general cubic function equation was studied by Y. H. Lee [10, 15], S. M. Jun et al. [9], and Y. H. Lee et al. [23, 19], and we can see the stability of the general quartic function equation in Y. H. Lee [13] and Y. H. Lee et al. [17, 19, 20, 21, 22]. Moreover, the stability of a general quintic functional equation has been studied by S. S. Jin et al. [8], and the stability of the general sextic function equation has been obtained by Y. H. Lee [14], I. S, Chang ey al. [2], and J. Roh et al. [25].

In this article, we investigate the stability of a general septic functional equation

$$Df(x,y) := \sum_{i=0}^{8} {}_8C_i(-1)^{8-i} f(x+(i-4)y) = 0 \tag{1.2}$$

in the sense of Rassias. Precisely, for a nonnegative real number $p \neq 1, 2, 3, 4, 5, 6, 7$ and $\theta > 0$, let $f : X \to Y$ satisfy

$$\|Df(x,y)\| \le \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$, then we will show that there exists a unique septic mapping F, i.e., $DF(x,y) = 0$ for all $x, y \in X$, such that $F(0) = 0$ and

$$\|\tilde{f}(x) - F(x)\| \le \epsilon_p \theta \|x\|^p$$

for all $x \in X$, where the constant ϵ_p depends only on p, see (2.14).

2 Stability of a general septic functional equation

Definition 2.1 *For a given mapping $f : V \to Y$, we define the mappings $Df : V^2 \to Y$, $\tilde{f}$, f_o, f_e, Γf, and $\Delta f : V \to Y$ as*

$$\begin{aligned}
Df(x,y) :=& \sum_{i=0}^{8} {}_8C_i(-1)^{8-i} f(x+(i-4)y),\\
\tilde{f}(x) :=& f(x) - f(0), \quad f_o(x) := \frac{f(x)-f(-x)}{2}, \quad f_e(x) := \frac{f(x)+f(-x)}{2},\\
\Gamma f(x) :=& Df_o(8x,2x) + 8Df_o(6x,2x) + 36Df_o(4x,2x) + 120Df_o(2x,2x)\\
&+ 160Df_o(4x,x) + 1280Df_o(3x,x) + 4032Df_o(2x,x) + 5376Df_o(x,x),\\
\Delta f(x) :=& Df_e(4x,x) + 8Df_e(3x,x) + 36Df_e(2x,x) + 120Df_e(x,x) + 123Df_e(0,x)
\end{aligned}$$

for all $x, y \in V$.

Theorem 2.2 *For $f : V \to Y$, let us define f_1, f_2, f_3, f_4, f_5, f_6, and $f_7 : V \to Y$ by followings;*

$$f_1(x) := \frac{1}{M}\begin{vmatrix} f_o(x) & 1 & 1 & 1 \\ f_o(2x) & 8 & 32 & 128 \\ f_o(4x) & 8^2 & 32^2 & 128^2 \\ f_o(8x) & 8^3 & 32^3 & 128^3 \end{vmatrix}, \quad f_2(x) := \frac{1}{M'}\begin{vmatrix} f_e(x) & 1 & 1 \\ f_e(2x) & 16 & 64 \\ f_e(4x) & 16^2 & 64^2 \end{vmatrix},$$

$$f_3(x) := \frac{1}{M}\begin{vmatrix} 1 & f_o(x) & 1 & 1 \\ 2 & f_o(2x) & 32 & 128 \\ 2^2 & f_o(4x) & 32^2 & 128^2 \\ 2^3 & f_o(8x) & 32^3 & 128^3 \end{vmatrix}, \quad f_4(x) := \frac{1}{M'}\begin{vmatrix} 1 & f_e(x) & 1 \\ 4 & f_e(2x) & 64 \\ 4^2 & f_e(4x) & 64^2 \end{vmatrix},$$

$$f_5(x) := \frac{1}{M}\begin{vmatrix} 1 & 1 & f_o(x) & 1 \\ 2 & 8 & f_o(2x) & 128 \\ 2^2 & 8^2 & f_o(4x) & 128^2 \\ 2^3 & 8^3 & f_o(8x) & 128^3 \end{vmatrix}, \quad f_6(x) := \frac{1}{M'}\begin{vmatrix} 1 & 1 & f_e(x) \\ 4 & 16 & f_e(2x) \\ 4^2 & 16^2 & f_e(4x) \end{vmatrix},$$

$$f_7(x) := \frac{1}{M}\begin{vmatrix} 1 & 1 & 1 & f_o(x) \\ 2 & 8 & 32 & f_o(2x) \\ 2^2 & 8^2 & 32^2 & f_o(4x) \\ 2^3 & 8^3 & 32^3 & f_o(8x) \end{vmatrix}$$

for all $x \in V$, where

$$M := \begin{vmatrix} 1 & 1 & 1 & 1 \\ 2 & 8 & 32 & 128 \\ 2^2 & 8^2 & 32^2 & 128^2 \\ 2^3 & 8^3 & 32^3 & 128^3 \end{vmatrix} \quad \text{and} \quad M' := \begin{vmatrix} 1 & 1 & 1 \\ 4 & 16 & 64 \\ 4^2 & 16^2 & 64^2 \end{vmatrix}.$$

Then ~~we have that~~

$$f(x) = f_o(x) + f_e(x) = \sum_{i=1}^{7} f_i(x) \tag{2.1}$$

for all $x \in V$.

Proof. We note that $M \neq 0$ and $M' \neq 0$. The uniqueness of solution (stated in Cramer's rule) implies that the family $\{f_1(x), f_3(x), f_5(x), f_7(x)\}$ is the only solution to the system of nonhomogeneous linear equations

$$\begin{cases} f_1(x) + f_3(x) + f_5(x) + f_7(x) = f_o(x) \\ 2f_1(x) + 8f_3(x) + 32f_5(x) + 128f_7(x) = f_o(2x) \\ 2^2 f_1(x) + 8^2 f_3(x) + 32^2 f_5(x) + 128^2 f_7(x) = f_o(4x) \\ 2^3 f_1(x) + 8^3 f_3(x) + 32^3 f_5(x) + 128^3 f_7(x) = f_o(8x) \end{cases}$$

for all $x \in V$. Similarly, we have $f_e(x) = f_2(x) + f_4(x) + f_6(x)$ for all $x \in V$.

□

By laborious computation we can get the following equalities;

$$\Gamma \tilde{f}(x) = f_o(16x) - 170 f_o(8x) + 5712 f_o(4x) - 43520 f_o(2x) + 65536 f_o(x),$$
$$\Delta \tilde{f}(x) = \tilde{f}_e(8x) - 84 \tilde{f}_e(4x) + 1344 \tilde{f}_e(2x) - 4096 \tilde{f}_e(x),$$

$$f_1(x) = \frac{32768 f_o(x) - 5376 f_o(2x) + 168 f_o(4x) - f_o(8x)}{22680}, \tag{2.2}$$
$$f_2(x) = \frac{1024 f_e(x) - 80 f_e(2x) + f_e(4x)}{720}, \tag{2.3}$$
$$f_3(x) = -\frac{8192 f_o(x) - 4416 f_o(2x) + 162 f_o(4x) - f_o(8x)}{17280}, \tag{2.4}$$
$$f_4(x) = -\frac{256 f_e(x) - 68 f_e(2x) + f_e(4x)}{576}, \tag{2.5}$$
$$f_5(x) = \frac{2048 f_o(x) - 1296 f_o(2x) + 138 f_o(4x) - f_o(8x)}{69120}, \tag{2.6}$$
$$f_6(x) = \frac{64 f_e(x) - 20 f_e(2x) + f_e(4x)}{2880}, \tag{2.7}$$
$$f_7(x) = -\frac{512 f_o(x) - 336 f_o(2x) + 42 f_o(4x) - f_o(8x)}{1451520}, \tag{2.8}$$

as well as

$$\tilde{f}_1(x) - \frac{\tilde{f}_1(2x)}{2} = \frac{\Gamma \tilde{f}_o(x)}{45360}, \qquad \tilde{f}_2(x) - \frac{\tilde{f}_2(2x)}{4} = -\frac{\Delta \tilde{f}_e(x)}{2880}, \tag{2.9}$$
$$\tilde{f}_3(x) - \frac{\tilde{f}_3(2x)}{8} = -\frac{\Gamma \tilde{f}_o(x)}{138240}, \qquad \tilde{f}_4(x) - \frac{\tilde{f}_4(2x)}{16} = \frac{\Delta \tilde{f}_e(x)}{9216}, \tag{2.10}$$
$$\tilde{f}_5(x) - \frac{\tilde{f}_5(2x)}{32} = \frac{\Gamma \tilde{f}_o(x)}{2211840}, \qquad \tilde{f}_6(x) - \frac{\tilde{f}_6(2x)}{64} = -\frac{\Delta \tilde{f}_e(x)}{184320}, \tag{2.11}$$
$$\tilde{f}_7(x) - \frac{\tilde{f}_7(2x)}{128} = -\frac{\Gamma \tilde{f}_o(x)}{185794560} \tag{2.12}$$

for all $x \in V$.

Now, ~~we show~~ the stability of the general septic functional equation (1.2);

Theorem 2.3 *Let $p \neq 1, 2, 3, 4, 5, 6, 7$ be a nonnegative real number, and let $\theta > 0$. Suppose that $f : X \to Y$ satisfies*

$$\|Df(x, y)\| \leq \theta(\|x\|^p + \|y\|^p) \tag{2.13}$$

for all $x, y \in X$, then there exists a unique mapping F such that $F(0) = 0$, $DF(x, y) = 0$ for all $x, y \in X$, and

$$\begin{aligned}\|\tilde{f}(x) - F(x)\| \leq & \frac{K'\theta\|x\|^p}{22680 \cdot |2 - 2^p|} + \frac{K\theta\|x\|^p}{720 \cdot |4 - 2^p|} + \frac{K'\theta\|x\|^p}{17280 \cdot |8 - 2^p|} + \frac{K\theta\|x\|^p}{576 \cdot |16 - 2^p|} \\ & + \frac{K'\theta\|x\|^p}{69120 \cdot |32 - 2^p|} + \frac{K\theta\|x\|^p}{2880 \cdot |64 - 2^p|} + \frac{K'\theta\|x\|^p}{1451520 \cdot |2^p - 128|}\end{aligned} \tag{2.14}$$

for all $x \in X$, where K and K' are constants given by $K := \left(4^p + 8 \cdot 3^p + 36 \cdot 2^p + 408\right)$ and $K' := \left(8^p + 8 \cdot 6^p + 196 \cdot 4^p + 1280 \cdot 3^p + 4317 \cdot 2^p + 16224\right)$.

Proof. Notice that $\tilde{f}(0) = 0$, $D\tilde{f}(x, y) = Df(x, y)$, and

$$\|Df_o(x, y)\|, \ \|D\tilde{f}_e(x, y)\| \leq \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$, by (2.13). Then, together with the definitions of $\Gamma\tilde{f}$ and $\Delta\tilde{f}$, we get

$$\begin{aligned}\|\Gamma\tilde{f}_o(x)\| =&\big\|Df_o(8x,2x)+8Df_o(6x,2x)+36Df_o(4x,2x)+120Df_o(2x,2x)\\&+160Df_o(4x,x)+1280Df_o(3x,x)+4032Df_o(2x,x)+5376Df_o(x,x)\|\\ \leq&\big(8^p+2^p+8\cdot 6^p+8\cdot 2^p+36\cdot 4^p+36\cdot 2^p+240\cdot 2^p+160\cdot 4^p\\&+160+1280\cdot 3^p+1280+4032\cdot 2^p+4032+10752\big)\theta\|x\|^p\\ \leq&\ K'\theta\|x\|^p, \qquad (2.15)\\ \|\Delta\tilde{f}_e(x)\| =&\big\|D\tilde{f}_e(4x,x)+8D\tilde{f}_e(3x,x)+36D\tilde{f}_e(2x,x)+120D\tilde{f}_e(x,x)+123D\tilde{f}_e(0,x)\big\|\\ \leq&\ K\theta\|x\|^p \qquad (2.16)\end{aligned}$$

for all $x \in X$. ~~We will prove~~ the theorem by seven steps in the following way;

Step 1. For $p \neq 1$, there exists a mapping $F^{(1)} : X \to Y$ satisfying $F^{(1)}(0) = 0$, $DF^{(1)}(x, y) = 0$ for all $x, y \in X$, and

$$\|\tilde{f}_1(x) - F^{(1)}(x)\| \leq \frac{K'\theta\|x\|^p}{22680\cdot|2-2^p|} \qquad (2.17)$$

for all $x \in X$.

(1) If $0 \leq p < 1$, then it follows from (2.9) and (2.15) that we obtain that

$$\begin{aligned}\left\|\frac{\tilde{f}_1(2^n x)}{2^n}-\frac{\tilde{f}_1(2^{n+m}x)}{2^{n+m}}\right\| &= \left\|\sum_{i=n}^{n+m-1}\left(\frac{\tilde{f}_1(2^i x)}{2^i}-\frac{\tilde{f}_1(2^{i+1}x)}{2^{i+1}}\right)\right\|\\ &\leq \sum_{i=n}^{n+m-1}\left\|\frac{\Gamma\tilde{f}_o(2^i x)}{45360\cdot 2^i}\right\| \leq \sum_{i=n}^{n+m-1}\frac{K'\theta 2^{ip}\|x\|^p}{45360\cdot 2^i} \qquad (2.18)\end{aligned}$$

for all $x \in X$ and $n, m \in \mathbb{N}\cup\{0\}$, so $\left\{\frac{\tilde{f}_1(2^n x)}{2^n}\right\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, it converges and we can define a mapping $F^{(1)} : X \to Y$ by

$$F^{(1)}(x) := \lim_{n\to\infty}\frac{\tilde{f}_1(2^n x)}{2^n}$$

for all $x \in X$. Moreover, letting $n = 0$ and passing the limit $m \to \infty$ in (2.18) we get the inequality

$$\|\tilde{f}_1(x) - F^{(1)}(x)\| \leq \frac{K'\theta\|x\|^p}{22680(2-2^p)}$$

for all $x \in X$, and, together with (2.2), it holds that

$$\begin{aligned}\|DF^{(1)}(x,y)\| &= \lim_{n\to\infty}\left\|\frac{D\tilde{f}_1(2^n x,2^n y)}{2^n}\right\| \leq \lim_{n\to\infty}\left(\left\|\frac{32768D\tilde{f}_o(2^n x,2^n y)}{22680\cdot 2^n}\right\|\right.\\ &\left.+\left\|\frac{5376D\tilde{f}_o(2^{n+1}x,2^{n+1}y)}{22680\cdot 2^n}\right\|+\left\|\frac{168D\tilde{f}_o(2^{n+2}x,2^{n+2}y)}{22680\cdot 2^n}\right\|+\left\|\frac{D\tilde{f}_o(2^{n+3}x,2^{n+3}y)}{22680\cdot 2^n}\right\|\right)\\ &\leq \lim_{n\to\infty}\left(32768\cdot 2^{np}+5376\cdot 2^{(n+1)p}+168\cdot 2^{(n+2)p}+2^{(n+3)p}\right)\frac{\theta(\|x\|^p+\|y\|^p)}{22680\cdot 2^n}\\ &= 0\end{aligned}$$

for all $x, y \in X$.

(2) If $p > 1$, then it follows from (2.9) and (2.15) that ~~we obtain that~~

$$\left\|2^n \tilde{f}_1(2^{-n}x) - 2^{n+m}\tilde{f}_1(2^{-n-m}x)\right\| = \left\|\sum_{i=n}^{n+m-1}\left(2^i \tilde{f}_1(2^{-i}x) - 2^{i+1}\tilde{f}_1(2^{-i-1}x)\right)\right\|$$
$$\leq \sum_{i=n}^{n+m-1}\left\|\frac{2^{i+1}\Gamma\tilde{f}_o(2^{-i-1}x)}{45360}\right\| \leq \sum_{i=n}^{n+m-1}\frac{2^i K'\theta\|x\|^p}{22680\cdot 2^{(i+1)p}} \tag{2.19}$$

for all $x \in X$ and $n, m \in \mathbb{N} \cup \{0\}$, so $\{2^n \tilde{f}_1(2^{-n}x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence converges and we can define a mapping $F^{(1)} : X \to Y$ by

$$F^{(1)}(x) := \lim_{n\to\infty} 2^n \tilde{f}_1(2^{-n}x)$$

for all $x \in X$. Moreover, letting $n = 0$ and passing the limit $m \to \infty$ in (2.19) we get the inequality

$$\|\tilde{f}_1(x) - F^{(1)}(x)\| \leq \frac{K'\theta\|x\|^p}{22680(2^p - 2)}$$

for all $x \in X$, and, using (2.2), we have

$$\|DF^{(1)}(x,y)\| = \lim_{n\to\infty} 2^n\left\|D\tilde{f}_1(2^{-n}x, 2^{-n}y)\right\| \leq \lim_{n\to\infty} 2^n\left(\left\|\frac{32768 D\tilde{f}_o(2^{-n}x, 2^{-n}y)}{22680}\right\|\right.$$
$$+\left\|\frac{5376 D\tilde{f}_o(2^{-n+1}x, 2^{-n+1}y)}{22680}\right\| + \left\|\frac{168 D\tilde{f}_o(2^{-n+2}x, 2^{-n+2}y)}{22680}\right\| + \left.\left\|\frac{D\tilde{f}_o(2^{-n+3}x, 2^{-n+3}y)}{22680}\right\|\right)$$
$$\leq \lim_{n\to\infty}\left(32768\cdot 2^{-np} + 5376\cdot 2^{-np+p} + 168\cdot 2^{-np+2p} + 2^{-np+3p}\right)\frac{2^n\theta(\|x\|^p + \|y\|^p)}{22680}$$
$$= 0$$

for all $x, y \in X$.

Step 2. For $p \neq 2$, there exists a mapping $F^{(2)} : X \to Y$ satisfying $F^{(2)}(0) = 0$, $DF^{(2)}(x,y) = 0$ for all $x, y \in X$, and

$$\|\tilde{f}_2(x) - F^{(2)}(x)\| \leq \frac{K\theta\|x\|^p}{720\cdot|4 - 2^p|} \tag{2.20}$$

for all $x \in X$.

(1) If $p < 2$, then it follows from (2.9) and (2.16) that ~~we obtain that~~

$$\left\|\frac{\tilde{f}_2(2^n x)}{4^n} - \frac{\tilde{f}_2(2^{n+m}x)}{4^{n+m}}\right\| = \left\|\sum_{i=n}^{n+m-1}\left(\frac{\tilde{f}_2(2^i x)}{4^i} - \frac{\tilde{f}_2(2^{i+1}x)}{4^{i+1}}\right)\right\|$$
$$\leq \sum_{i=n}^{n+m-1}\left\|\frac{\Delta\tilde{f}_e(2^i x)}{2880\cdot 4^i}\right\| \leq \sum_{i=n}^{n+m-1}\frac{K\theta 2^{ip}\|x\|^p}{2880\cdot 4^i} \tag{2.21}$$

for all $x \in X$ and $n, m \in \mathbb{N} \cup \{0\}$, so $\left\{\frac{\tilde{f}_2(2^n x)}{4^n}\right\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, it converges and we can define a mapping $F^{(2)} : X \to Y$ by

$$F^{(2)}(x) := \lim_{n\to\infty} \frac{\tilde{f}_2(2^n x)}{4^n}$$

for all $x \in X$. Moreover, letting $n = 0$ and passing the limit $m \to \infty$ in (2.21) we get the inequality

$$\|\tilde{f}_2(x) - F^{(2)}(x)\| \leq \frac{K\theta\|x\|^p}{720(4 - 2^p)}$$

for all $x \in X$, and by (2.3) it holds that

$$\begin{aligned}
\|DF^{(2)}(x,y)\| &= \lim_{n\to\infty} \left\| \frac{D\tilde{f}_2(2^n x, 2^n y)}{4^n} \right\| \\
&= \lim_{n\to\infty} \left\| \frac{1024 D\tilde{f}_e(2^n x, 2^n y)}{720 \cdot 4^n} - \frac{80 D\tilde{f}_e(2^{n+1}x, 2^{n+1}y)}{720 \cdot 4^n} + \frac{D\tilde{f}_e(2^{n+2}x, 2^{n+2}y)}{720 \cdot 4^n} \right\| \\
&\leq \lim_{n\to\infty} \left(1024 \cdot 2^{np} + 80 \cdot 2^{(n+1)p} + 2^{(n+2)p}\right) \frac{\theta(\|x\|^p + \|y\|^p)}{720 \cdot 4^n} \\
&= 0
\end{aligned}$$

for all $x, y \in X$.

(2) If $p > 2$, then it follows from (2.9) and (2.16) that ~~we obtain that~~

$$\begin{aligned}
\left\|4^n \tilde{f}_2(2^{-n}x) - 4^{n+m}\tilde{f}_2(2^{-n-m}x)\right\| &= \left\| \sum_{i=n}^{n+m-1} \left(4^i \tilde{f}_2(2^{-i}x) - 4^{i+1}\tilde{f}_2(2^{-i-1}x)\right) \right\| \\
&\leq \sum_{i=n}^{n+m-1} \left\| \frac{4^{i+1}\Delta \tilde{f}_e(2^{-i-1}x)}{2880} \right\| \leq \sum_{i=n}^{n+m-1} \frac{4^i K\theta\|x\|^p}{720 \cdot 2^{(i+1)p}} \qquad (2.22)
\end{aligned}$$

for all $x \in X$ and $n, m \in \mathbb{N} \cup \{0\}$, so $\{4^n \tilde{f}_2(2^{-n}x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, it converges and we can define a mapping $F^{(2)} : X \to Y$ by

$$F^{(2)}(x) := \lim_{n\to\infty} 4^n \tilde{f}_2(2^{-n}x)$$

for all $x \in X$. Moreover, letting $n = 0$ and passing the limit $m \to \infty$ in (2.22) we get the inequality

$$\|\tilde{f}_2(x) - F^{(2)}(x)\| \leq \frac{K\theta\|x\|^p}{720(2^p - 4)}$$

and

$$\begin{aligned}
\|DF^{(2)}(x,y)\| &= \lim_{n\to\infty} 4^n \left\|D\tilde{f}_2(2^{-n}x, 2^{-n}y)\right\| \\
&= \lim_{n\to\infty} 4^n \left\| \frac{1024 D\tilde{f}_e(2^{-n}x, 2^{-n}y)}{720} - \frac{80 D\tilde{f}_e(2^{-n+1}x, 2^{-n+1}y)}{720} + \frac{D\tilde{f}_e(2^{-n+2}x, 2^{-n+2}y)}{720} \right\| \\
&\leq \lim_{n\to\infty} \left(1024 \cdot 2^{-np} + 80 \cdot 2^{-np+p} + 2^{-np+2p}\right) \frac{4^n\theta(\|x\|^p + \|y\|^p)}{720} \\
&= 0
\end{aligned}$$

for all $x, y \in X$.

Step 3. For $p \neq 3$, there exists a mapping $F^{(3)} : X \to Y$ satisfying $F^{(3)}(0) = 0$, $DF^{(3)}(x, y) = 0$ for all $x, y \in X$, and

$$\|\tilde{f}_3(x) - F^{(3)}(x)\| \leq \frac{K'\theta\|x\|^p}{17280 \cdot |8 - 2^p|} \tag{2.23}$$

for all $x \in X$.

(1) If $p < 3$, then it follows from (2.10) and (2.15) that

$$\begin{aligned} \left\|\frac{\tilde{f}_3(2^n x)}{8^n} - \frac{\tilde{f}_3(2^{n+m}x)}{8^{n+m}}\right\| &= \left\|\sum_{i=n}^{n+m-1}\left(\frac{\tilde{f}_3(2^i x)}{8^i} - \frac{\tilde{f}_3(2^{i+1}x)}{8^{i+1}}\right)\right\| \\ &\leq \sum_{i=n}^{n+m-1}\left\|\frac{\Gamma\tilde{f}_o(2^i x)}{138240 \cdot 8^i}\right\| \leq \sum_{i=n}^{n+m-1}\frac{K'\theta 2^{ip}\|x\|^p}{138240 \cdot 8^i} \end{aligned} \tag{2.24}$$

for all $x \in X$ and $n, m \in \mathbb{N} \cup \{0\}$. Then $\left\{\frac{\tilde{f}_3(2^n x)}{8^n}\right\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, it converges and we can define a mapping $F^{(3)} : X \to Y$ by

$$F^{(3)}(x) := \lim_{n\to\infty}\frac{\tilde{f}_3(2^n x)}{8^n}$$

for all $x \in X$. Moreover, letting $n = 0$ and passing the limit $m \to \infty$ in (2.24) we get the inequality

$$\|\tilde{f}_3(x) - F^{(3)}(x)\| \leq \frac{K'\theta\|x\|^p}{17280(8 - 2^p)}$$

for all $x \in X$, and by (2.4) it holds that

$$\begin{aligned} \|DF^{(3)}(x,y)\| &= \lim_{n\to\infty}\left\|\frac{D\tilde{f}_3(2^n x, 2^n y)}{8^n}\right\| = \lim_{n\to\infty}\left(\left\|\frac{8192 D\tilde{f}_o(2^n x, 2^n y)}{17280 \cdot 8^n}\right\|\right. \\ &\quad \left. + \left\|\frac{4416 D\tilde{f}_o(2^{n+1}x, 2^{n+1}y)}{17280 \cdot 8^n}\right\| + \left\|\frac{162 D\tilde{f}_o(2^{n+2}x, 2^{n+2}y)}{17280 \cdot 8^n}\right\| + \left\|\frac{D\tilde{f}_o(2^{n+3}x, 2^{n+3}y)}{17280 \cdot 8^n}\right\|\right) \\ &\leq \lim_{n\to\infty}\left(8192 \cdot 2^{np} + 4416 \cdot 2^{(n+1)p} + 162 \cdot 2^{(n+2)p} + 2^{(n+3)p}\right)\frac{\theta(\|x\|^p + \|y\|^p)}{17280 \cdot 8^n} \\ &= 0 \end{aligned}$$

for all $x, y \in X$.

(2) If $p > 3$, then it follows from (2.10) and (2.15) that

$$\begin{aligned} \left\|8^n\tilde{f}_3(2^{-n}x) - 8^{n+m}\tilde{f}_3(2^{-n-m}x)\right\| &= \left\|\sum_{i=n}^{n+m-1}\left(8^i\tilde{f}_3(2^{-i}x) - 8^{i+1}\tilde{f}_3(2^{-i-1}x)\right)\right\| \\ &\leq \sum_{i=n}^{n+m-1}\left\|\frac{8^{i+1}\Gamma\tilde{f}_o(2^{-i-1}x)}{138240}\right\| \leq \sum_{i=n}^{n+m-1}\frac{8^i K'\theta\|x\|^p}{17280 \cdot 2^{(i+1)p}} \end{aligned} \tag{2.25}$$

for all $x \in X$ and $n, m \in \mathbb{N} \cup \{0\}$, so $\{8^n \tilde{f}_3(2^{-n}x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence converges and we can define a mapping $F^{(3)} : X \to Y$ by

$$F^{(3)}(x) := \lim_{n\to\infty} 8^n \tilde{f}_3(2^{-n}x)$$

for all $x \in X$. Moreover, letting $n = 0$ and passing the limit $m \to \infty$ in (2.25) we get the inequality

$$\|\tilde{f}_3(x) - F^{(3)}(x)\| \le \frac{K'\theta\|x\|^p}{17280(2^p - 8)}$$

for all $x \in X$, and by (2.4) it holds that

$$\begin{aligned}
\|DF^{(3)}(x,y)\| &= \lim_{n\to\infty} 8^n \left\|D\tilde{f}_3(2^{-n}x, 2^{-n}y)\right\| = \lim_{n\to\infty} 8^n \left(\left\| \frac{8192 D\tilde{f}_o(2^{-n}x, 2^{-n}y)}{17280} \right\| \right. \\
&+ \left. \left\| \frac{4416 D\tilde{f}_o(2^{-n+1}x, 2^{-n+1}y)}{17280} \right\| + \left\| \frac{162 D\tilde{f}_o(2^{-n+2}x, 2^{-n+2}y)}{17280} \right\| + \left\| \frac{D\tilde{f}_o(2^{-n+3}x, 2^{-n+3}y)}{17280} \right\| \right) \\
&\le \lim_{n\to\infty} \left(8192 \cdot 2^{-np} + 4416 \cdot 2^{-np+p} + 162 \cdot 2^{-np+2p} + 2^{-np+3p}\right) \frac{8^n\theta(\|x\|^p + \|y\|^p)}{17280} \\
&= 0
\end{aligned}$$

for all $x, y \in X$.

Step 4. For $p \neq 4$, there exists a mapping $F^{(4)} : X \to Y$ satisfying $F^{(4)}(0) = 0$, $DF^{(4)}(x,y) = 0$ for all $x, y \in X$, and

$$\|\tilde{f}_4(x) - F^{(4)}(x)\| \le \frac{K\theta\|x\|^p}{576 \cdot |16 - 2^p|} \tag{2.26}$$

for all $x \in X$.

(1) If $p < 4$, then it follows from (2.10) and (2.16) that

$$\begin{aligned}
\left\| \frac{\tilde{f}_4(2^n x)}{16^n} - \frac{\tilde{f}_4(2^{n+m}x)}{16^{n+m}} \right\| &= \left\| \sum_{i=n}^{n+m-1} \left(\frac{\tilde{f}_4(2^i x)}{16^i} - \frac{\tilde{f}_4(2^{i+1}x)}{16^{i+1}} \right) \right\| \\
&\le \sum_{i=n}^{n+m-1} \left\| \frac{\Delta\tilde{f}_e(2^i x)}{9216 \cdot 16^i} \right\| \le \sum_{i=n}^{n+m-1} \frac{K\theta 2^{ip}\|x\|^p}{9216 \cdot 16^i}
\end{aligned} \tag{2.27}$$

for all $x \in X$ and $n, m \in \mathbb{N} \cup \{0\}$, so $\left\{\frac{\tilde{f}_4(2^n x)}{16^n}\right\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, it converges and we can define a mapping $F^{(4)} : X \to Y$ by

$$F^{(4)}(x) := \lim_{n\to\infty} \frac{\tilde{f}_4(2^n x)}{16^n}$$

for all $x \in X$. Moreover, letting $n = 0$ and passing the limit $m \to \infty$ in (2.27) we get the inequality

$$\|\tilde{f}_4(x) - F^{(4)}(x)\| \le \frac{K\theta\|x\|^p}{576(16 - 2^p)}$$

for all $x \in X$, and by (2.5) it holds that

$$\begin{aligned}
\|DF^{(4)}(x,y)\| &= \lim_{n\to\infty} \left\| \frac{D\tilde{f}_4(2^n x, 2^n y)}{16^n} \right\| \\
&= \lim_{n\to\infty} \left\| -\frac{256 D\tilde{f}_e(2^n x, 2^n y)}{576 \cdot 16^n} + \frac{68 D\tilde{f}_e(2^{n+1}x, 2^{n+1}y)}{576 \cdot 16^n} - \frac{D\tilde{f}_e(2^{n+2}x, 2^{n+2}y)}{576 \cdot 16^n} \right\| \\
&\le \lim_{n\to\infty} \left(256 \cdot 2^{np} + 68 \cdot 2^{(n+1)p} + 2^{(n+2)p}\right) \frac{\theta(\|x\|^p + \|y\|^p)}{576 \cdot 16^n} \\
&= 0
\end{aligned}$$

for all $x, y \in X$.

(2) If $p > 4$, then it follows from (2.10) and (2.16) that

$$\begin{aligned}
&\left\|16^n \tilde{f}_4(2^{-n}x) - 16^{n+m}\tilde{f}_4(2^{-n-m}x)\right\| = \left\| \sum_{i=n}^{n+m-1} \left(16^i \tilde{f}_4(2^{-i}x) - 16^{i+1}\tilde{f}_4(2^{-i-1}x)\right) \right\| \\
&\quad \le \sum_{i=n}^{n+m-1} \left\| \frac{16^{i+1}\Delta \tilde{f}_e(2^{-i-1}x)}{9216} \right\| \le \sum_{i=n}^{n+m-1} \frac{K\theta 16^i \|x\|^p}{576 \cdot 2^{(i+1)p}}
\end{aligned} \tag{2.28}$$

for all $x \in X$ and $n, m \in \mathbb{N} \cup \{0\}$, so $\{16^n \tilde{f}_4(2^{-n}x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, it converges and we can define a mapping $F^{(4)} : X \to Y$ by

$$F^{(4)}(x) := \lim_{n\to\infty} 16^n \tilde{f}_4(2^{-n}x)$$

for all $x \in X$. Moreover, letting $n = 0$ and passing the limit $m \to \infty$ in (2.28) we get

$$\|\tilde{f}_4(x) - F^{(4)}(x)\| \le \frac{K\theta\|x\|^p}{576(2^p - 16)}$$

for all $x \in X$, and by (2.5) it holds that

$$\begin{aligned}
\|DF^{(4)}(x,y)\| &= \lim_{n\to\infty} 16^n \left\| D\tilde{f}_4(2^{-n}x, 2^{-n}y) \right\| \\
&= \lim_{n\to\infty} 16^n \left\| -\frac{256 D\tilde{f}_e(2^{-n}x, 2^{-n}y)}{576} + \frac{68 D\tilde{f}_e(2^{-n+1}x, 2^{-n+1}y)}{576} - \frac{D\tilde{f}_e(2^{-n+2}x, 2^{-n+2}y)}{576} \right\| \\
&\le \lim_{n\to\infty} \left(256 \cdot 2^{-np} + 68 \cdot 2^{-np+p} + 2^{-np+2p}\right) \frac{16^n \theta(\|x\|^p + \|y\|^p)}{576} \\
&= 0
\end{aligned}$$

for all $x, y \in X$.

Step 5. For $p \neq 5$, there exists a mapping $F^{(5)} : X \to Y$ satisfying $F^{(5)}(0) = 0$, $DF^{(5)}(x,y) = 0$ for all $x, y \in X$, and

$$\|\tilde{f}_5(x) - F^{(5)}(x)\| \le \frac{K'\theta\|x\|^p}{69120 \cdot |32 - 2^p|} \tag{2.29}$$

for all $x, y \in X$.

(1) If $p < 5$, then it follows from (2.11) and (2.15) that ~~we obtain that~~

$$\left\| \frac{\tilde{f}_5(2^n x)}{32^n} - \frac{\tilde{f}_5(2^{n+m}x)}{32^{n+m}} \right\| = \left\| \sum_{i=n}^{n+m-1} \left(\frac{\tilde{f}_5(2^i x)}{32^i} - \frac{\tilde{f}_5(2^{i+1}x)}{32^{i+1}} \right) \right\|$$
$$\leq \sum_{i=n}^{n+m-1} \left\| \frac{\Gamma \tilde{f}_o(2^i x)}{2211840 \cdot 32^i} \right\| \leq \sum_{i=n}^{n+m-1} \frac{K'\theta 2^{ip} \|x\|^p}{2211840 \cdot 32^i} \tag{2.30}$$

for all $x \in X$ and $n, m \in \mathbb{N} \cup \{0\}$. Then $\left\{ \frac{\tilde{f}_5(2^n x)}{32^n} \right\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, it converges and we can define a mapping $F^{(5)} : X \to Y$ by

$$F^{(5)}(x) := \lim_{n\to\infty} \frac{\tilde{f}_5(2^n x)}{32^n}$$

for all $x \in X$. Moreover, letting $n = 0$ and passing the limit $m \to \infty$ in (2.30) we get

$$\|\tilde{f}_5(x) - F^{(5)}(x)\| \leq \frac{K'\theta\|x\|^p}{69120(32 - 2^p)}$$

for all $x \in X$, and by (2.6) it holds that

$$\|DF^{(5)}(x,y)\| = \lim_{n\to\infty} \left\| \frac{D\tilde{f}_5(2^n x, 2^n y)}{32^n} \right\| \leq \lim_{n\to\infty} \left(\left\| \frac{2048 D\tilde{f}_o(2^n x, 2^n y)}{69120 \cdot 32^n} \right\| \right.$$
$$+ \left\| \frac{1296 D\tilde{f}_o(2^{n+1}x, 2^{n+1}y)}{69120 \cdot 32^n} \right\| + \left\| \frac{138 D\tilde{f}_o(2^{n+2}x, 2^{n+2}y)}{69120 \cdot 32^n} \right\| + \left. \left\| \frac{D\tilde{f}_o(2^{n+3}x, 2^{n+3}y)}{69120 \cdot 32^n} \right\| \right)$$
$$\leq \lim_{n\to\infty} \left(2048 \cdot 2^{np} + 1296 \cdot 2^{(n+1)p} + 138 \cdot 2^{(n+2)p} + 2^{(n+3)p} \right) \frac{\theta(\|x\|^p + \|y\|^p)}{69120 \cdot 32^n}$$
$$= 0$$

for all $x, y \in X$.

(2) If $p > 5$, then it follows from (2.11) and (2.15) that ~~we obtain that~~

$$\left\| 32^n \tilde{f}_5(2^{-n}x) - 32^{n+m} \tilde{f}_5(2^{-n-m}x) \right\| = \left\| \sum_{i=n}^{n+m-1} \left(32^i \tilde{f}_5(2^{-i}x) - 32^{i+1} \tilde{f}_5(2^{-i-1}x) \right) \right\|$$
$$\leq \sum_{i=n}^{n+m-1} \left\| \frac{32^{i+1} \Gamma \tilde{f}_o(2^{-i-1}x)}{2211840} \right\| \leq \sum_{i=n}^{n+m-1} \frac{32^i K'\theta\|x\|^p}{69120 \cdot 2^{(i+1)p}} \tag{2.31}$$

for all $x \in X$ and $n, m \in \mathbb{N} \cup \{0\}$, so $\{32^n \tilde{f}_5(2^{-n}x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, it converges and we can define a mapping $F^{(5)} : X \to Y$ by

$$F^{(5)}(x) := \lim_{n\to\infty} 32^n \tilde{f}_5(2^{-n}x)$$

for all $x \in X$. Moreover, letting $n = 0$ and passing the limit $m \to \infty$ in (2.31), we get

$$\|\tilde{f}_5(x) - F^{(5)}(x)\| \leq \frac{K'\theta\|x\|^p}{69120(2^p - 32)}$$

for all $x \in X$, and, using (2.6), we have

$$\|DF^{(5)}(x,y)\| = \lim_{n\to\infty} 32^n \left\|D\tilde{f}_5(2^{-n}x, 2^{-n}y)\right\| \le \lim_{n\to\infty} 32^n \left(\left\|\frac{2048D\tilde{f}_o(2^{-n}x, 2^{-n}y)}{69120}\right\| + \left\|\frac{1296D\tilde{f}_o(2^{-n+1}x, 2^{-n+1}y)}{69120}\right\| + \left\|\frac{138D\tilde{f}_o(2^{-n+2}x, 2^{-n+2}y)}{69120}\right\| + \left\|\frac{D\tilde{f}_o(2^{-n+3}x, 2^{-n+3}y)}{69120}\right\|\right)$$
$$\le \lim_{n\to\infty} \left(2048\cdot 2^{-np} + 1296\cdot 2^{-np+p} + 138\cdot 2^{-np+2p} + 2^{-np+3p}\right) \frac{32^n\theta(\|x\|^p + \|y\|^p))}{69120}$$
$$= 0$$

for all $x, y \in X$.

Step 6. For $p \neq 6$, there exists a mapping $F^{(6)} : X \to Y$ satisfying $F^{(6)}(0) = 0$, $DF^{(6)}(x,y) = 0$ for all $x, y \in X$, and

$$\|\tilde{f}_6(x) - F^{(6)}(x)\| \le \frac{K\theta\|x\|^p}{2880\cdot|64-2^p|} \tag{2.32}$$

for all $x, y \in X$.

(1) If $p < 6$, then it follows from (2.11) and (2.16) that ~~we obtain that~~

$$\left\|\frac{\tilde{f}_6(2^n x)}{64^n} - \frac{\tilde{f}_6(2^{n+m}x)}{64^{n+m}}\right\| = \left\|\sum_{i=n}^{n+m-1}\left(\frac{\tilde{f}_6(2^i x)}{64^i} - \frac{\tilde{f}_6(2^{i+1}x)}{64^{i+1}}\right)\right\|$$
$$\le \sum_{i=n}^{n+m-1}\left\|\frac{\Delta\tilde{f}_e(2^i x)}{184320\cdot 64^i}\right\| \le \sum_{i=n}^{n+m-1}\frac{K2^{ip}\theta\|x\|^p}{184320\cdot 64^i} \tag{2.33}$$

for all $x \in X$ and $n, m \in \mathbb{N}\cup\{0\}$, so $\left\{\frac{\tilde{f}_6(2^n x)}{64^n}\right\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, it converges and we can define a mapping $F^{(6)} : X \to Y$ by

$$F^{(6)}(x) := \lim_{n\to\infty}\frac{\tilde{f}_6(2^n x)}{64^n}$$

for all $x \in X$. Moreover, letting $n = 0$ and passing the limit $m \to \infty$ in (2.33) we get

$$\|\tilde{f}_6(x) - F^{(6)}(x)\| \le \frac{K\theta\|x\|^p}{2880(64-2^p)}$$

for all $x \in X$, and by (2.7) it holds that

$$\|DF^{(6)}(x,y)\| = \lim_{n\to\infty}\left\|\frac{D\tilde{f}_6(2^n x, 2^n y)}{64^n}\right\|$$
$$= \lim_{n\to\infty}\left\|\frac{64D\tilde{f}_e(2^n x, 2^n y)}{2880\cdot 64^n} - \frac{5376D\tilde{f}_o(2^{n+1}x, 2^{n+1}y)}{2880\cdot 64^n} + \frac{168D\tilde{f}_o(2^{n+2}x, 2^{n+2}y)}{2880\cdot 64^n}\right\|$$
$$\le \lim_{n\to\infty}\left(64\cdot 2^{np} + 20\cdot 2^{(n+1)p} + 2^{(n+2)p}\right)\frac{\theta(\|x\|^p + \|y\|^p)}{2880\cdot 64^n}$$
$$= 0$$

for all $x, y \in X$.

(2) If $p > 6$, then it follows from (2.11) and (2.16) that

$$\left\|64^n \tilde{f}_6(2^{-n}x) - 64^{n+m}\tilde{f}_6(2^{-n-m}x)\right\| = \left\|\sum_{i=n}^{n+m-1}\left(64^i \tilde{f}_6(2^{-i}x) - 64^{i+1}\tilde{f}_6(2^{-i-1}x)\right)\right\|$$
$$\le \sum_{i=n}^{n+m-1}\left\|\frac{64^{i+1}\Delta\tilde{f}_e(2^{-i-1}x)}{184320}\right\| \le \sum_{i=n}^{n+m-1}\frac{64^i K\theta\|x\|^p}{2880\cdot 2^{(i+1)p}} \tag{2.34}$$

for all $x \in X$ and $n, m \in \mathbb{N}\cup\{0\}$, so $\{64^n \tilde{f}_6(2^{-n}x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, it converges and we can define a mapping $F^{(6)} : X \to Y$ by

$$F^{(6)}(x) := \lim_{n\to\infty} 64^n \tilde{f}_6(2^{-n}x)$$

for all $x \in X$. Moreover, letting $n = 0$ and passing the limit $m \to \infty$ in (2.34) we get

$$\|\tilde{f}_6(x) - F^{(6)}(x)\| \le \frac{K\theta\|x\|^p}{2880(2^p - 64)}$$

for all $x \in X$, and by (2.7) it holds that

$$\begin{aligned}
&\|DF^{(6)}(x,y)\| = \lim_{n\to\infty} 64^n \left\|D\tilde{f}_6(2^{-n}x, 2^{-n}y)\right\| \\
&= \lim_{n\to\infty}\frac{64^n}{2880}\left\|64D\tilde{f}_e(2^{-n}x, 2^{-n}y) - 20D\tilde{f}_e(2^{-n+1}x, 2^{-n+1}y) + D\tilde{f}_e(2^{-n+2}x, 2^{-n+2}y)\right\| \\
&\le \lim_{n\to\infty}\left(64\cdot 2^{-np} + 20\cdot 2^{-np+p} + 2^{-np+2p}\right)\frac{64^n\theta(\|x\|^p + \|y\|^p)}{2880} \\
&= 0
\end{aligned}$$

for all $x, y \in X$.

Step 7. For $p \ne 7$, there exists a mapping $F^{(7)} : X \to Y$ satisfying $F^{(7)}(0) = 0$, $DF^{(7)}(x,y) = 0$ for all $x, y \in X$, and

$$\|\tilde{f}_7(x) - F^{(7)}(x)\| \le \frac{K'\theta\|x\|^p}{1451520\cdot|128 - 2^p|} \tag{2.35}$$

for all $x \in X$.

(1) If $p < 7$, then it follows from (2.12) and (2.15) that

$$\left\|\frac{\tilde{f}_7(2^n x)}{128^n} - \frac{\tilde{f}_7(2^{n+m}x)}{128^{n+m}}\right\| = \left\|\sum_{i=n}^{n+m-1}\left(\frac{\tilde{f}_7(2^i x)}{128^i} - \frac{\tilde{f}_7(2^{i+1}x)}{128^{i+1}}\right)\right\|$$
$$\le \sum_{i=n}^{n+m-1}\left\|\frac{\Gamma\tilde{f}_o(2^i x)}{185794560\cdot 128^i}\right\| \le \sum_{i=n}^{n+m-1}\frac{K'\theta 2^{ip}\|x\|^p}{185794560\cdot 128^i} \tag{2.36}$$

for all $x \in X$ and $n, m \in \mathbb{N} \cup \{0\}$, so $\left\{ \frac{\tilde{f}_7(2^n x)}{128^n} \right\}$ becomes a Cauchy sequence for all $x \in X$. Since Y is complete, it converges, and hence we can define a mapping $F^{(7)} : X \to Y$ by

$$F^{(7)}(x) := \lim_{n\to\infty} \frac{\tilde{f}_7(2^n x)}{128^n}$$

for all $x \in X$. Moreover, letting $n = 0$ and passing the limit $m \to \infty$ in (2.36) we get

$$\|\tilde{f}_7(x) - F^{(7)}(x)\| \leq \frac{K'\theta\|x\|^p}{1451520(128 - 2^p)}$$

for all $x \in X$, and, using (2.8), we have

$$\begin{aligned}
\|DF^{(7)}(x,y)\| &= \lim_{n\to\infty} \left\|\frac{D\tilde{f}_7(2^n x, 2^n y)}{128^n}\right\| = \lim_{n\to\infty} \left\| -\frac{512 D\tilde{f}_o(2^n x, 2^n y)}{65536 \cdot 128^n} \right. \\
&\quad \left. + \frac{336 D\tilde{f}_o(2^{n+1}x, 2^{n+1}y)}{65536 \cdot 128^n} - \frac{42 D\tilde{f}_o(2^{n+2}x, 2^{n+2}y)}{65536 \cdot 128^n} + \frac{D\tilde{f}_o(2^{n+3}x, 2^{n+3}y)}{65536 \cdot 128^n} \right\| \\
&\leq \lim_{n\to\infty} \left(512 \cdot 2^{np} + 336 \cdot 2^{(n+1)p} + 42 \cdot 2^{(n+2)p} + 2^{(n+3)p}\right) \frac{\theta(\|x\|^p + \|y\|^p)}{65536 \cdot 128^n} \\
&= 0
\end{aligned}$$

for all $x, y \in X$.

(2) If $p > 7$, then it follows from (2.12) and (2.15) that

$$\begin{aligned}
\left\|128^n \tilde{f}_7(2^{-n}x) - 128^{n+m}\tilde{f}_7(2^{-n-m}x)\right\| &= \left\| \sum_{i=n}^{n+m-1} \left(128^i \tilde{f}_7(2^{-i}x) - 128^{i+1}\tilde{f}_7(2^{-i-1}x)\right) \right\| \\
&\leq \sum_{i=n}^{n+m-1} \left\| -\frac{128^{i+1}\Gamma\tilde{f}_o(2^{-i-1}x)}{185794560} \right\| \leq \sum_{i=n}^{n+m-1} \frac{128^i K'\theta\|x\|^p}{1451520 \cdot 2^{(i+1)p}} \qquad (2.37)
\end{aligned}$$

for all $x \in X$ and $n, m \in \mathbb{N} \cup \{0\}$, so $\{128^n \tilde{f}_7(2^{-n}x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, it converges and we can define a mapping $F^{(7)} : X \to Y$ by

$$F^{(7)}(x) := \lim_{n\to\infty} 128^n \tilde{f}_7(2^{-n}x)$$

for all $x \in X$. Moreover, letting $n = 0$ and passing the limit $m \to \infty$ in (2.37), we get

$$\|\tilde{f}_7(x) - F^{(7)}(x)\| \leq \frac{K'\theta\|x\|^p}{1451520(2^p - 128)}$$

for all $x \in X$, and, using (2.8), we have

$$\begin{aligned}
\|DF^{(7)}(x,y)\| &= \lim_{n\to\infty} 128^n \left\|D\tilde{f}_7(2^{-n}x, 2^{-n}y)\right\| = \lim_{n\to\infty} 128^n \left\| \frac{-512 D\tilde{f}_o(2^{-n}x, 2^{-n}y)}{2835 \cdot 512} \right. \\
&\quad \left. + \frac{336 D\tilde{f}_o(2^{-n+1}x, 2^{-n+1}y)}{2835 \cdot 512} - \frac{42 D\tilde{f}_o(2^{-n+2}x, 2^{-n+2}y)}{2835 \cdot 8} + \frac{D\tilde{f}_o(2^{-n+3}x, 2^{-n+3}y)}{2835 \cdot 512} \right\| \\
&\leq \lim_{n\to\infty} \left(512 \cdot 2^{-np} + 336 \cdot 2^{-np+p} + 42 \cdot 2^{-np+2p} + 2^{-np+3p}\right) \frac{128^n\theta(\|x\|^p + \|y\|^p))}{1451520} \\
&= 0
\end{aligned}$$

for all $x, y \in X$.

Now, using the functions $F^{(1)}, F^{(2)}, \cdots, F^{(7)} : X \to Y$ of Step1, Step 2, ..., and Step 7, respectively, we put

$$F(x) := \sum_{i=1}^{7} F^{(i)}(x)$$

for all $x \in X$. Since $\|\tilde{f}(x) - F(x)\| \le \sum_{i=1}^{7} \|\tilde{f}_i(x) - F^{(i)}(x)\|$ for all $x \in X$, together with (2.17), (2.20), (2.23), (2.26), (2.29), (2.32), (2.35), we obtain the property (2.14). And it is clear that

$$DF(x, y) = \sum_{i=1}^{7} DF^{(i)}(x, y) = 0$$

for all $x, y \in X$. Finally, to prove the uniqueness of F, let $G : X \to Y$ be another mapping, which satisfies the property (2.14), $G(0) = 0$, and $DG(x, y) = 0$ for all $x, y \in X$. And let $G_1, G_2, \cdots, G_7 : X \to Y$ be defined as in Definition 2.1. Since $DG_o(x, y) = DG_e(x, y) = 0$ for all $x, y \in X$,

$$DG_i(x, y) = 0, \quad i = 1, 2, \cdots, 7$$

for all $x, y \in X$, Additionally, by (2.14) with the definitions of the odd function and the even function of Definition 2.1, we get

$$\|\tilde{f}_e(x) - G_e(x)\|, \quad \|\tilde{f}_e(x) - G_e(x)\| \le N\theta\|x\|^p \tag{2.38}$$

for all $x \in X$, where

$$\begin{aligned} N := & \frac{K'}{22680 \cdot |2 - 2^p|} + \frac{K}{720 \cdot |4 - 2^p|} + \frac{K'}{17280 \cdot |8 - 2^p|} + \frac{K}{576 \cdot |16 - 2^p|} \\ & + \frac{K'}{69120 \cdot |32 - 2^p|} + \frac{K}{2880 \cdot |64 - 2^p|} + \frac{K'}{1451520 \cdot |2^p - 128|}. \end{aligned}$$

Notice that $\Gamma G(x) = \Delta G(x) = 0$ for all $x \in X$ by (2.9)-(2.12), we can prove that $G_i(2x) = 2^i G_i(x)$, $i = 1, 2, \cdots, 7$, for all $x \in X$. Particularly, they hold that

$$G_2(x) = 2^2 G_2\left(\frac{x}{2}\right) = \cdots\cdots = 4^n G_2\left(\frac{x}{2^n}\right) \tag{2.39}$$

$$G_3(2^n x) = 2^3 G_3(2^{n-1}x) = \cdots\cdots = 8^n G_3(x) \tag{2.40}$$

for all $x \in X$ and $n \in \mathbb{N}$. Now, observe that, in the case of $2 < p < 3$, we have

$$\begin{aligned} \left\| 4^n \tilde{f}_2\left(\frac{x}{2^n}\right) - G_2(x) \right\| &= \left\| 4^n \tilde{f}_2\left(\frac{x}{2^n}\right) - 4^n G_2\left(\frac{x}{2^n}\right) \right\| \le \frac{1024 \cdot 4^n}{720} \left\| \tilde{f}_e\left(\frac{x}{2^n}\right) - G_e\left(\frac{x}{2^n}\right) \right\| \\ & + \frac{80 \cdot 4^n}{720} \left\| \tilde{f}_e\left(\frac{2x}{2^n}\right) - G_e\left(\frac{2x}{2^n}\right) \right\| + \frac{4^n}{720} \left\| \tilde{f}_e\left(\frac{4x}{2^n}\right) - G_e\left(\frac{4x}{2^n}\right) \right\| \\ & \le (1024 + 80 \cdot 2^p + 4^p) \frac{4^n}{720 \cdot 2^{np}} N\theta\|x\|^p, \end{aligned}$$

and

$$\begin{aligned}\left\|\frac{\tilde{f}_3(2^n x)}{8^n} - G_3(x)\right\| &= \left\|\frac{\tilde{f}_3(2^n x)}{8^n} - \frac{G_3(2^n x)}{8^n}\right\| \\ &\leq \left\|\frac{8192(\tilde{f}_o(2^n x) - G_o(2^n x))}{17280 \cdot 8^n}\right\| + \left\|\frac{4416(\tilde{f}_o(2^{n+1} x) - G_o(2^{n+1} x))}{17280 \cdot 8^n}\right\| \\ &\quad + \left\|\frac{162(\tilde{f}_o(2^{n+2} x) - G_o(2^{n+2} x))}{17280 \cdot 8^n}\right\| + \left\|\frac{(\tilde{f}_o(2^{n+3} x) - G_o(2^{n+3} x))}{17280 \cdot 8^n}\right\| \\ &\leq \left(8192 + 4416 \cdot 2^p + 162 \cdot 2^{2p} + 2^{3p}\right) \frac{2^{np} N\theta \|x\|^p}{172808^n}\end{aligned}$$

for all $x \in X$ and all positive integers n. Taking the limit in the above inequalities as $n \to \infty$, we obtain

$$G_2(x) = \lim_{n\to\infty} \tilde{4}^n \tilde{f}_2\left(\frac{x}{2^n}\right) = F^{(2)}(x), \qquad G_3(x) = \lim_{n\to\infty} \frac{\tilde{f}_3(2^n x)}{8^n} = F^{(3)}(x)$$

for all $x \in X$. Also, in the same way, we can show that $G_i(x) = F^{(i)}(x)$, $i = 1, 4, 5, 6, 7$, for all $x \in X$. Therefore, we have shown that $G(x) = F(x)$ for all $x \in X$ in the case of $2 < p < 3$. Similarly, ~~we can prove~~ the uniqueness of F in other cases of p. ~~It has been finished the proof of the theorem.~~ □

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