

# Hyers-Ulam-Rassias stability of a general septic functional equation

**Abstract.** In this paper, we investigate the stability of a general septic functional equation

$$\sum_{i=0}^8 {}_8C_i (-1)^{8-i} f(x + (i-4)y) = 0$$

in the sense of Rassias.

**AMS Subject Classification:** 39B82; 39B52.

**Key Words:** stability of a functional equation; general septic functional equation; general septic mapping.

## 1 Introduction

It is well known that the study on the stability of functional equations began as an attempt to solve Ulam's question in [26] about the stability of the group homomorphisms. As a partial answer to this question, Hyers [5] solved the stability of the Cauchy functional equation in the following year. Since then, many mathematicians have generalized Hyers' results by showing the stability of various kind of functional equations, see [3, 4, 6, 12, 24]. Today the term 'Hyers-Ulam-Rassias stability' refers to the generalization introduced by Rassias [24].

Throughout this paper, let  $V$ ,  $X$ , and  $Y$  be a real vector space, a real normed space, and a real Banach space, respectively. For a mapping  $f$  from  $V$  to  $Y$ , we consider the functional equation

$$\sum_{i=0}^k {}_kC_i (-1)^{k-i} f(x + iy) = 0, \tag{1.1}$$

where  $k = 2, 3, 4, 5, 6, 7, 8$ . A solution mapping  $f : V \rightarrow Y$  of (1.1) is a "generalized polynomial mapping of degree at most 7" in the sense of J. Baker in [1]. So we call the functional equation (1.1) a Jensen, a general quadratic, a general cubic, a general quartic, a general quintic, a general sextic, and a general septic functional equation, for

$k = 2, 3, 4, 5, 6, 7, 8$ , respectively. Also we call each solution mapping of (1.1), for  $k = 2, 3, 4, 5, 6, 7, 8$ , a Jensen, a general quadratic, a general cubic, a general quartic, a general quintic, a general sextic, and a general septic mapping, respectively. Prior this paper, the stability problems for the functional equation (1.1) were studied in many ways. In the case of a Jensen functional equation, K.-W. Jun et al.[16] showed the stability result. The stability of a general quadratic function equation were obtained by Y. H. Lee [11], Y. H. Lee et al. [18], and S. S. Jin et al.[7]. The stability of a general cubic function equation was studied by Y. H. Lee [10, 15], S. M. Jun et al. [9], and Y. H. Lee et al. [23, 19], and we can see the stability of the general quartic function equation in Y. H. Lee [13] and Y. H. Lee et al. [17, 19, 20, 21, 22]. Moreover, the stability of a general quintic functional equation has been studied by S. S. Jin et al. [8], and the stability of the general sextic function equation has been obtained by Y. H. Lee [14], I. S, Chang ey al. [2], and J. Roh et al. [25].

In this article, we investigate the stability of a general septic functional equation

$$Df(x, y) := \sum_{i=0}^8 {}_8C_i(-1)^{8-i} f(x + (i - 4)y) = 0 \tag{1.2}$$

in the sense of Rassias. Precisely, for a nonnegative real number  $p \neq 1, 2, 3, 4, 5, 6, 7$  and  $\theta > 0$ , let  $f : X \rightarrow Y$  satisfy

$$\|Df(x, y)\| \leq \theta(\|x\|^p + \|y\|^p)$$

for all  $x, y \in X$ , then we will show that there exists a unique septic mapping  $F$ , i.e.,  $DF(x, y) = 0$  for all  $x, y \in X$ , such that  $F(0) = 0$  and

$$\|\tilde{f}(x) - F(x)\| \leq \epsilon_p \theta \|x\|^p$$

for all  $x \in X$ , where the constant  $\epsilon_p$  depends only on  $p$ , see (2.14).

## 2 Stability of a general septic functional equation

**Definition 2.1** For a given mapping  $f : V \rightarrow Y$ , we define the mappings  $Df : V^2 \rightarrow Y$ ,  $\tilde{f}$ ,  $f_o$ ,  $f_e$ ,  $\Gamma f$ , and  $\Delta f : V \rightarrow Y$  as

$$\begin{aligned} Df(x, y) &:= \sum_{i=0}^8 {}_8C_i(-1)^{8-i} f(x + (i - 4)y), \\ \tilde{f}(x) &:= f(x) - f(0), \quad f_o(x) := \frac{f(x) - f(-x)}{2}, \quad f_e(x) := \frac{f(x) + f(-x)}{2}, \\ \Gamma f(x) &:= Df_o(8x, 2x) + 8Df_o(6x, 2x) + 36Df_o(4x, 2x) + 120Df_o(2x, 2x) \\ &\quad + 160Df_o(4x, x) + 1280Df_o(3x, x) + 4032Df_o(2x, x) + 5376Df_o(x, x), \\ \Delta f(x) &:= Df_e(4x, x) + 8Df_e(3x, x) + 36Df_e(2x, x) + 120Df_e(x, x) + 123Df_e(0, x) \end{aligned}$$

for all  $x, y \in V$ .

**Theorem 2.2** For  $f : V \rightarrow Y$ , let us define  $f_1, f_2, f_3, f_4, f_5, f_6$ , and  $f_7 : V \rightarrow Y$  by followings;

$$\begin{aligned}
 f_1(x) &:= \frac{1}{M} \begin{vmatrix} f_o(x) & 1 & 1 & 1 \\ f_o(2x) & 8 & 32 & 128 \\ f_o(4x) & 8^2 & 32^2 & 128^2 \\ f_o(8x) & 8^3 & 32^3 & 128^3 \end{vmatrix}, & f_2(x) &:= \frac{1}{M'} \begin{vmatrix} f_e(x) & 1 & 1 \\ f_e(2x) & 16 & 64 \\ f_e(4x) & 16^2 & 64^2 \end{vmatrix}, \\
 f_3(x) &:= \frac{1}{M} \begin{vmatrix} 1 & f_o(x) & 1 & 1 \\ 2 & f_o(2x) & 32 & 128 \\ 2^2 & f_o(4x) & 32^2 & 128^2 \\ 2^3 & f_o(8x) & 32^3 & 128^3 \end{vmatrix}, & f_4(x) &:= \frac{1}{M'} \begin{vmatrix} 1 & f_e(x) & 1 \\ 4 & f_e(2x) & 64 \\ 4^2 & f_e(4x) & 64^2 \end{vmatrix}, \\
 f_5(x) &:= \frac{1}{M} \begin{vmatrix} 1 & 1 & f_o(x) & 1 \\ 2 & 8 & f_o(2x) & 128 \\ 2^2 & 8^2 & f_o(4x) & 128^2 \\ 2^3 & 8^3 & f_o(8x) & 128^3 \end{vmatrix}, & f_6(x) &:= \frac{1}{M'} \begin{vmatrix} 1 & 1 & f_e(x) \\ 4 & 16 & f_e(2x) \\ 4^2 & 16^2 & f_e(4x) \end{vmatrix}, \\
 f_7(x) &:= \frac{1}{M} \begin{vmatrix} 1 & 1 & 1 & f_o(x) \\ 2 & 8 & 32 & f_o(2x) \\ 2^2 & 8^2 & 32^2 & f_o(4x) \\ 2^3 & 8^3 & 32^3 & f_o(8x) \end{vmatrix}
 \end{aligned}$$

for all  $x \in V$ , where

$$M := \begin{vmatrix} 1 & 1 & 1 & 1 \\ 2 & 8 & 32 & 128 \\ 2^2 & 8^2 & 32^2 & 128^2 \\ 2^3 & 8^3 & 32^3 & 128^3 \end{vmatrix} \quad \text{and} \quad M' := \begin{vmatrix} 1 & 1 & 1 \\ 4 & 16 & 64 \\ 4^2 & 16^2 & 64^2 \end{vmatrix}.$$

Then we have that

$$f(x) = f_o(x) + f_e(x) = \sum_{i=1}^7 f_i(x) \tag{2.1}$$

for all  $x \in V$ .

**Proof.** We note that  $M \neq 0$  and  $M' \neq 0$ . The uniqueness of solution (stated in Cramer's rule) implies that the family  $\{f_1(x), f_3(x), f_5(x), f_7(x)\}$  is the only solution to the system of nonhomogeneous linear equations

$$\begin{cases} f_1(x) + f_3(x) + f_5(x) + f_7(x) = f_o(x) \\ 2f_1(x) + 8f_3(x) + 32f_5(x) + 128f_7(x) = f_o(2x) \\ 2^2f_1(x) + 8^2f_3(x) + 32^2f_5(x) + 128^2f_7(x) = f_o(4x) \\ 2^3f_1(x) + 8^3f_3(x) + 32^3f_5(x) + 128^3f_7(x) = f_o(8x) \end{cases}$$

for all  $x \in V$ . Similarly, we have  $f_e(x) = f_2(x) + f_4(x) + f_6(x)$  for all  $x \in V$ . □

By laborious computation we can get the following equalities;

$$\begin{aligned}
 \Gamma \tilde{f}(x) &= f_o(16x) - 170f_o(8x) + 5712f_o(4x) - 43520f_o(2x) + 65536f_o(x), \\
 \Delta \tilde{f}(x) &= \tilde{f}_e(8x) - 84\tilde{f}_e(4x) + 1344\tilde{f}_e(2x) - 4096\tilde{f}_e(x),
 \end{aligned}$$

$$f_1(x) = \frac{32768f_o(x) - 5376f_o(2x) + 168f_o(4x) - f_o(8x)}{22680}, \quad (2.2)$$

$$f_2(x) = \frac{1024f_e(x) - 80f_e(2x) + f_e(4x)}{720}, \quad (2.3)$$

$$f_3(x) = -\frac{8192f_o(x) - 4416f_o(2x) + 162f_o(4x) - f_o(8x)}{17280}, \quad (2.4)$$

$$f_4(x) = -\frac{256f_e(x) - 68f_e(2x) + f_e(4x)}{576}, \quad (2.5)$$

$$f_5(x) = \frac{2048f_o(x) - 1296f_o(2x) + 138f_o(4x) - f_o(8x)}{69120}, \quad (2.6)$$

$$f_6(x) = \frac{64f_e(x) - 20f_e(2x) + f_e(4x)}{2880}, \quad (2.7)$$

$$f_7(x) = -\frac{512f_o(x) - 336f_o(2x) + 42f_o(4x) - f_o(8x)}{1451520}, \quad (2.8)$$

as well as

$$\tilde{f}_1(x) - \frac{\tilde{f}_1(2x)}{2} = \frac{\Gamma \tilde{f}_o(x)}{45360}, \quad \tilde{f}_2(x) - \frac{\tilde{f}_2(2x)}{4} = -\frac{\Delta \tilde{f}_e(x)}{2880}, \quad (2.9)$$

$$\tilde{f}_3(x) - \frac{\tilde{f}_3(2x)}{8} = -\frac{\Gamma \tilde{f}_o(x)}{138240}, \quad \tilde{f}_4(x) - \frac{\tilde{f}_4(2x)}{16} = \frac{\Delta \tilde{f}_e(x)}{9216}, \quad (2.10)$$

$$\tilde{f}_5(x) - \frac{\tilde{f}_5(2x)}{32} = \frac{\Gamma \tilde{f}_o(x)}{2211840}, \quad \tilde{f}_6(x) - \frac{\tilde{f}_6(2x)}{64} = -\frac{\Delta \tilde{f}_e(x)}{184320}, \quad (2.11)$$

$$\tilde{f}_7(x) - \frac{\tilde{f}_7(2x)}{128} = -\frac{\Gamma \tilde{f}_o(x)}{185794560} \quad (2.12)$$

for all  $x \in V$ .

Now, we show the stability of the general septic functional equation (1.2);

**Theorem 2.3** *Let  $p \neq 1, 2, 3, 4, 5, 6, 7$  be a nonnegative real number, and let  $\theta > 0$ . Suppose that  $f : X \rightarrow Y$  satisfies*

$$\|Df(x, y)\| \leq \theta(\|x\|^p + \|y\|^p) \quad (2.13)$$

for all  $x, y \in X$ , then there exists a unique mapping  $F$  such that  $F(0) = 0$ ,  $DF(x, y) = 0$  for all  $x, y \in X$ , and

$$\begin{aligned} \|\tilde{f}(x) - F(x)\| \leq & \frac{K'\theta\|x\|^p}{22680 \cdot |2 - 2^p|} + \frac{K\theta\|x\|^p}{720 \cdot |4 - 2^p|} + \frac{K'\theta\|x\|^p}{17280 \cdot |8 - 2^p|} + \frac{K\theta\|x\|^p}{576 \cdot |16 - 2^p|} \\ & + \frac{K'\theta\|x\|^p}{69120 \cdot |32 - 2^p|} + \frac{K\theta\|x\|^p}{2880 \cdot |64 - 2^p|} + \frac{K'\theta\|x\|^p}{1451520 \cdot |2^p - 128|} \end{aligned} \quad (2.14)$$

for all  $x \in X$ , where  $K$  and  $K'$  are constants given by  $K := (4^p + 8 \cdot 3^p + 36 \cdot 2^p + 408)$  and  $K' := (8^p + 8 \cdot 6^p + 196 \cdot 4^p + 1280 \cdot 3^p + 4317 \cdot 2^p + 16224)$ .

**Proof.** Notice that  $\tilde{f}(0) = 0$ ,  $D\tilde{f}(x, y) = Df(x, y)$ , and

$$\|Df_o(x, y)\|, \|D\tilde{f}_e(x, y)\| \leq \theta(\|x\|^p + \|y\|^p)$$

for all  $x, y \in X$ , by (2.13). Then, together with the definitions of  $\Gamma\tilde{f}$  and  $\Delta\tilde{f}$ , we get

$$\begin{aligned} \|\Gamma\tilde{f}_o(x)\| &= \|Df_o(8x, 2x) + 8Df_o(6x, 2x) + 36Df_o(4x, 2x) + 120Df_o(2x, 2x) \\ &\quad + 160Df_o(4x, x) + 1280Df_o(3x, x) + 4032Df_o(2x, x) + 5376Df_o(x, x)\| \\ &\leq (8^p + 2^p + 8 \cdot 6^p + 8 \cdot 2^p + 36 \cdot 4^p + 36 \cdot 2^p + 240 \cdot 2^p + 160 \cdot 4^p \\ &\quad + 160 + 1280 \cdot 3^p + 1280 + 4032 \cdot 2^p + 4032 + 10752)\theta\|x\|^p \\ &\leq K'\theta\|x\|^p, \end{aligned} \tag{2.15}$$

$$\begin{aligned} \|\Delta\tilde{f}_e(x)\| &= \|D\tilde{f}_e(4x, x) + 8D\tilde{f}_e(3x, x) + 36D\tilde{f}_e(2x, x) + 120D\tilde{f}_e(x, x) + 123D\tilde{f}_e(0, x)\| \\ &\leq K\theta\|x\|^p \end{aligned} \tag{2.16}$$

for all  $x \in X$ . We will prove the theorem by seven steps in the following way;

*Step 1.* For  $p \neq 1$ , there exists a mapping  $F^{(1)} : X \rightarrow Y$  satisfying  $F^{(1)}(0) = 0$ ,  $DF^{(1)}(x, y) = 0$  for all  $x, y \in X$ , and

$$\|\tilde{f}_1(x) - F^{(1)}(x)\| \leq \frac{K'\theta\|x\|^p}{22680 \cdot |2 - 2^p|} \tag{2.17}$$

for all  $x \in X$ .

(1) If  $0 \leq p < 1$ , then it follows from (2.9) and (2.15) that we obtain that

$$\begin{aligned} \left\| \frac{\tilde{f}_1(2^n x)}{2^n} - \frac{\tilde{f}_1(2^{n+m} x)}{2^{n+m}} \right\| &= \left\| \sum_{i=n}^{n+m-1} \left( \frac{\tilde{f}_1(2^i x)}{2^i} - \frac{\tilde{f}_1(2^{i+1} x)}{2^{i+1}} \right) \right\| \\ &\leq \sum_{i=n}^{n+m-1} \left\| \frac{\Gamma\tilde{f}_o(2^i x)}{45360 \cdot 2^i} \right\| \leq \sum_{i=n}^{n+m-1} \frac{K'\theta 2^{ip} \|x\|^p}{45360 \cdot 2^i} \end{aligned} \tag{2.18}$$

for all  $x \in X$  and  $n, m \in \mathbb{N} \cup \{0\}$ , so  $\left\{ \frac{\tilde{f}_1(2^n x)}{2^n} \right\}$  is a Cauchy sequence for all  $x \in X$ . Since  $Y$  is complete, it converges and we can define a mapping  $F^{(1)} : X \rightarrow Y$  by

$$F^{(1)}(x) := \lim_{n \rightarrow \infty} \frac{\tilde{f}_1(2^n x)}{2^n}$$

for all  $x \in X$ . Moreover, letting  $n = 0$  and passing the limit  $m \rightarrow \infty$  in (2.18) we get the inequality

$$\|\tilde{f}_1(x) - F^{(1)}(x)\| \leq \frac{K'\theta\|x\|^p}{22680(2 - 2^p)}$$

for all  $x \in X$ , and, together with (2.2), it holds that

$$\begin{aligned} \|DF^{(1)}(x, y)\| &= \lim_{n \rightarrow \infty} \left\| \frac{D\tilde{f}_1(2^n x, 2^n y)}{2^n} \right\| \leq \lim_{n \rightarrow \infty} \left( \left\| \frac{32768D\tilde{f}_o(2^n x, 2^n y)}{22680 \cdot 2^n} \right\| \right. \\ &\quad \left. + \left\| \frac{5376D\tilde{f}_o(2^{n+1} x, 2^{n+1} y)}{22680 \cdot 2^n} \right\| + \left\| \frac{168D\tilde{f}_o(2^{n+2} x, 2^{n+2} y)}{22680 \cdot 2^n} \right\| + \left\| \frac{D\tilde{f}_o(2^{n+3} x, 2^{n+3} y)}{22680 \cdot 2^n} \right\| \right) \\ &\leq \lim_{n \rightarrow \infty} \left( 32768 \cdot 2^{np} + 5376 \cdot 2^{(n+1)p} + 168 \cdot 2^{(n+2)p} + 2^{(n+3)p} \right) \frac{\theta(\|x\|^p + \|y\|^p)}{22680 \cdot 2^n} \\ &= 0 \end{aligned}$$

6

for all  $x, y \in X$ .

(2) If  $p > 1$ , then it follows from (2.9) and (2.15) that we obtain that

$$\begin{aligned} \left\| 2^n \tilde{f}_1(2^{-n}x) - 2^{n+m} \tilde{f}_1(2^{-n-m}x) \right\| &= \left\| \sum_{i=n}^{n+m-1} \left( 2^i \tilde{f}_1(2^{-i}x) - 2^{i+1} \tilde{f}_1(2^{-i-1}x) \right) \right\| \\ &\leq \sum_{i=n}^{n+m-1} \left\| \frac{2^{i+1} \Gamma \tilde{f}_o(2^{-i-1}x)}{45360} \right\| \leq \sum_{i=n}^{n+m-1} \frac{2^i K' \theta \|x\|^p}{22680 \cdot 2^{(i+1)p}} \end{aligned} \quad (2.19)$$

for all  $x \in X$  and  $n, m \in \mathbb{N} \cup \{0\}$ , so  $\{2^n \tilde{f}_1(2^{-n}x)\}$  is a Cauchy sequence for all  $x \in X$ . Since  $Y$  is complete, the sequence converges and we can define a mapping  $F^{(1)} : X \rightarrow Y$  by

$$F^{(1)}(x) := \lim_{n \rightarrow \infty} 2^n \tilde{f}_1(2^{-n}x)$$

for all  $x \in X$ . Moreover, letting  $n = 0$  and passing the limit  $m \rightarrow \infty$  in (2.19) we get the inequality

$$\|\tilde{f}_1(x) - F^{(1)}(x)\| \leq \frac{K' \theta \|x\|^p}{22680(2^p - 2)}$$

for all  $x \in X$ , and, using (2.2), we have

$$\begin{aligned} \|DF^{(1)}(x, y)\| &= \lim_{n \rightarrow \infty} 2^n \left\| D\tilde{f}_1(2^{-n}x, 2^{-n}y) \right\| \leq \lim_{n \rightarrow \infty} 2^n \left( \left\| \frac{32768 D\tilde{f}_o(2^{-n}x, 2^{-n}y)}{22680} \right\| \right. \\ &+ \left\| \frac{5376 D\tilde{f}_o(2^{-n+1}x, 2^{-n+1}y)}{22680} \right\| + \left\| \frac{168 D\tilde{f}_o(2^{-n+2}x, 2^{-n+2}y)}{22680} \right\| + \left. \left\| \frac{D\tilde{f}_o(2^{-n+3}x, 2^{-n+3}y)}{22680} \right\| \right) \\ &\leq \lim_{n \rightarrow \infty} (32768 \cdot 2^{-np} + 5376 \cdot 2^{-np+p} + 168 \cdot 2^{-np+2p} + 2^{-np+3p}) \frac{2^n \theta (\|x\|^p + \|y\|^p)}{22680} \\ &= 0 \end{aligned}$$

for all  $x, y \in X$ .

*Step 2.* For  $p \neq 2$ , there exists a mapping  $F^{(2)} : X \rightarrow Y$  satisfying  $F^{(2)}(0) = 0$ ,  $DF^{(2)}(x, y) = 0$  for all  $x, y \in X$ , and

$$\|\tilde{f}_2(x) - F^{(2)}(x)\| \leq \frac{K\theta \|x\|^p}{720 \cdot |4 - 2^p|} \quad (2.20)$$

for all  $x \in X$ .

(1) If  $p < 2$ , then it follows from (2.9) and (2.16) that we obtain that

$$\begin{aligned} \left\| \frac{\tilde{f}_2(2^n x)}{4^n} - \frac{\tilde{f}_2(2^{n+m} x)}{4^{n+m}} \right\| &= \left\| \sum_{i=n}^{n+m-1} \left( \frac{\tilde{f}_2(2^i x)}{4^i} - \frac{\tilde{f}_2(2^{i+1} x)}{4^{i+1}} \right) \right\| \\ &\leq \sum_{i=n}^{n+m-1} \left\| \frac{\Delta \tilde{f}_e(2^i x)}{2880 \cdot 4^i} \right\| \leq \sum_{i=n}^{n+m-1} \frac{K\theta 2^{ip} \|x\|^p}{2880 \cdot 4^i} \end{aligned} \quad (2.21)$$

for all  $x \in X$  and  $n, m \in \mathbb{N} \cup \{0\}$ , so  $\left\{ \frac{\tilde{f}_2(2^n x)}{4^n} \right\}$  is a Cauchy sequence for all  $x \in X$ . Since  $Y$  is complete, it converges and we can define a mapping  $F^{(2)} : X \rightarrow Y$  by

$$F^{(2)}(x) := \lim_{n \rightarrow \infty} \frac{\tilde{f}_2(2^n x)}{4^n}$$

for all  $x \in X$ . Moreover, letting  $n = 0$  and passing the limit  $m \rightarrow \infty$  in (2.21) we get the inequality

$$\|\tilde{f}_2(x) - F^{(2)}(x)\| \leq \frac{K\theta\|x\|^p}{720(4 - 2^p)}$$

for all  $x \in X$ , and by (2.3) it holds that

$$\begin{aligned} \|DF^{(2)}(x, y)\| &= \lim_{n \rightarrow \infty} \left\| \frac{D\tilde{f}_2(2^n x, 2^n y)}{4^n} \right\| \\ &= \lim_{n \rightarrow \infty} \left\| \frac{1024D\tilde{f}_e(2^n x, 2^n y)}{720 \cdot 4^n} - \frac{80D\tilde{f}_e(2^{n+1}x, 2^{n+1}y)}{720 \cdot 4^n} + \frac{D\tilde{f}_e(2^{n+2}x, 2^{n+2}y)}{720 \cdot 4^n} \right\| \\ &\leq \lim_{n \rightarrow \infty} \left( 1024 \cdot 2^{np} + 80 \cdot 2^{(n+1)p} + 2^{(n+2)p} \right) \frac{\theta(\|x\|^p + \|y\|^p)}{720 \cdot 4^n} \\ &= 0 \end{aligned}$$

for all  $x, y \in X$ .

(2) If  $p > 2$ , then it follows from (2.9) and (2.16) that we obtain that

$$\begin{aligned} \left\| 4^n \tilde{f}_2(2^{-n}x) - 4^{n+m} \tilde{f}_2(2^{-n-m}x) \right\| &= \left\| \sum_{i=n}^{n+m-1} \left( 4^i \tilde{f}_2(2^{-i}x) - 4^{i+1} \tilde{f}_2(2^{-(i+1)}x) \right) \right\| \\ &\leq \sum_{i=n}^{n+m-1} \left\| \frac{4^{i+1} \Delta \tilde{f}_e(2^{-i-1}x)}{2880} \right\| \leq \sum_{i=n}^{n+m-1} \frac{4^i K\theta\|x\|^p}{720 \cdot 2^{(i+1)p}} \quad (2.22) \end{aligned}$$

for all  $x \in X$  and  $n, m \in \mathbb{N} \cup \{0\}$ , so  $\{4^n \tilde{f}_2(2^{-n}x)\}$  is a Cauchy sequence for all  $x \in X$ . Since  $Y$  is complete, it converges and we can define a mapping  $F^{(2)} : X \rightarrow Y$  by

$$F^{(2)}(x) := \lim_{n \rightarrow \infty} 4^n \tilde{f}_2(2^{-n}x)$$

for all  $x \in X$ . Moreover, letting  $n = 0$  and passing the limit  $m \rightarrow \infty$  in (2.22) we get the inequality

$$\|\tilde{f}_2(x) - F^{(2)}(x)\| \leq \frac{K\theta\|x\|^p}{720(2^p - 4)}$$

and

$$\begin{aligned} \|DF^{(2)}(x, y)\| &= \lim_{n \rightarrow \infty} 4^n \left\| D\tilde{f}_2(2^{-n}x, 2^{-n}y) \right\| \\ &= \lim_{n \rightarrow \infty} 4^n \left\| \frac{1024D\tilde{f}_e(2^{-n}x, 2^{-n}y)}{720} - \frac{80D\tilde{f}_e(2^{-n+1}x, 2^{-n+1}y)}{720} + \frac{D\tilde{f}_e(2^{-n+2}x, 2^{-n+2}y)}{720} \right\| \\ &\leq \lim_{n \rightarrow \infty} \left( 1024 \cdot 2^{-np} + 80 \cdot 2^{-np+p} + 2^{-np+2p} \right) \frac{4^n \theta(\|x\|^p + \|y\|^p)}{720} \\ &= 0 \end{aligned}$$

8

for all  $x, y \in X$ .

*Step 3.* For  $p \neq 3$ , there exists a mapping  $F^{(3)} : X \rightarrow Y$  satisfying  $F^{(3)}(0) = 0$ ,  $DF^{(3)}(x, y) = 0$  for all  $x, y \in X$ , and

$$\|\tilde{f}_3(x) - F^{(3)}(x)\| \leq \frac{K'\theta\|x\|^p}{17280 \cdot |8 - 2^p|} \quad (2.23)$$

for all  $x \in X$ .

(1) If  $p < 3$ , then it follows from (2.10) and (2.15) that

$$\begin{aligned} \left\| \frac{\tilde{f}_3(2^n x)}{8^n} - \frac{\tilde{f}_3(2^{n+m} x)}{8^{n+m}} \right\| &= \left\| \sum_{i=n}^{n+m-1} \left( \frac{\tilde{f}_3(2^i x)}{8^i} - \frac{\tilde{f}_3(2^{i+1} x)}{8^{i+1}} \right) \right\| \\ &\leq \sum_{i=n}^{n+m-1} \left\| \frac{\Gamma \tilde{f}_o(2^i x)}{138240 \cdot 8^i} \right\| \leq \sum_{i=n}^{n+m-1} \frac{K'\theta 2^{ip} \|x\|^p}{138240 \cdot 8^i} \end{aligned} \quad (2.24)$$

for all  $x \in X$  and  $n, m \in \mathbb{N} \cup \{0\}$ . Then  $\left\{ \frac{\tilde{f}_3(2^n x)}{8^n} \right\}$  is a Cauchy sequence for all  $x \in X$ . Since  $Y$  is complete, it converges and we can define a mapping  $F^{(3)} : X \rightarrow Y$  by

$$F^{(3)}(x) := \lim_{n \rightarrow \infty} \frac{\tilde{f}_3(2^n x)}{8^n}$$

for all  $x \in X$ . Moreover, letting  $n = 0$  and passing the limit  $m \rightarrow \infty$  in (2.24) we get the inequality

$$\|\tilde{f}_3(x) - F^{(3)}(x)\| \leq \frac{K'\theta\|x\|^p}{17280(8 - 2^p)}$$

for all  $x \in X$ , and by (2.4) it holds that

$$\begin{aligned} \|DF^{(3)}(x, y)\| &= \lim_{n \rightarrow \infty} \left\| \frac{D\tilde{f}_3(2^n x, 2^n y)}{8^n} \right\| = \lim_{n \rightarrow \infty} \left( \left\| \frac{8192D\tilde{f}_o(2^n x, 2^n y)}{17280 \cdot 8^n} \right\| \right. \\ &\quad \left. + \left\| \frac{4416D\tilde{f}_o(2^{n+1} x, 2^{n+1} y)}{17280 \cdot 8^n} \right\| + \left\| \frac{162D\tilde{f}_o(2^{n+2} x, 2^{n+2} y)}{17280 \cdot 8^n} \right\| + \left\| \frac{D\tilde{f}_o(2^{n+3} x, 2^{n+3} y)}{17280 \cdot 8^n} \right\| \right) \\ &\leq \lim_{n \rightarrow \infty} \left( 8192 \cdot 2^{np} + 4416 \cdot 2^{(n+1)p} + 162 \cdot 2^{(n+2)p} + 2^{(n+3)p} \right) \frac{\theta(\|x\|^p + \|y\|^p)}{17280 \cdot 8^n} \\ &= 0 \end{aligned}$$

for all  $x, y \in X$ .

(2) If  $p > 3$ , then it follows from (2.10) and (2.15) that

$$\begin{aligned} \left\| 8^n \tilde{f}_3(2^{-n} x) - 8^{n+m} \tilde{f}_3(2^{-n-m} x) \right\| &= \left\| \sum_{i=n}^{n+m-1} \left( 8^i \tilde{f}_3(2^{-i} x) - 8^{i+1} \tilde{f}_3(2^{-i-1} x) \right) \right\| \\ &\leq \sum_{i=n}^{n+m-1} \left\| \frac{8^{i+1} \Gamma \tilde{f}_o(2^{-i-1} x)}{138240} \right\| \leq \sum_{i=n}^{n+m-1} \frac{8^i K'\theta\|x\|^p}{17280 \cdot 2^{(i+1)p}} \end{aligned} \quad (2.25)$$

for all  $x \in X$  and  $n, m \in \mathbb{N} \cup \{0\}$ , so  $\{8^n \tilde{f}_3(2^{-n}x)\}$  is a Cauchy sequence for all  $x \in X$ . Since  $Y$  is complete, the sequence converges and we can define a mapping  $F^{(3)} : X \rightarrow Y$  by

$$F^{(3)}(x) := \lim_{n \rightarrow \infty} 8^n \tilde{f}_3(2^{-n}x)$$

for all  $x \in X$ . Moreover, letting  $n = 0$  and passing the limit  $m \rightarrow \infty$  in (2.25) we get the inequality

$$\|\tilde{f}_3(x) - F^{(3)}(x)\| \leq \frac{K'\theta\|x\|^p}{17280(2^p - 8)}$$

for all  $x \in X$ , and by (2.4) it holds that

$$\begin{aligned} \|DF^{(3)}(x, y)\| &= \lim_{n \rightarrow \infty} 8^n \left\| D\tilde{f}_3(2^{-n}x, 2^{-n}y) \right\| = \lim_{n \rightarrow \infty} 8^n \left( \left\| \frac{8192D\tilde{f}_o(2^{-n}x, 2^{-n}y)}{17280} \right\| \right. \\ &+ \left\| \frac{4416D\tilde{f}_o(2^{-n+1}x, 2^{-n+1}y)}{17280} \right\| + \left\| \frac{162D\tilde{f}_o(2^{-n+2}x, 2^{-n+2}y)}{17280} \right\| + \left. \left\| \frac{D\tilde{f}_o(2^{-n+3}x, 2^{-n+3}y)}{17280} \right\| \right) \\ &\leq \lim_{n \rightarrow \infty} (8192 \cdot 2^{-np} + 4416 \cdot 2^{-np+p} + 162 \cdot 2^{-np+2p} + 2^{-np+3p}) \frac{8^n \theta (\|x\|^p + \|y\|^p)}{17280} \\ &= 0 \end{aligned}$$

for all  $x, y \in X$ .

*Step 4.* For  $p \neq 4$ , there exists a mapping  $F^{(4)} : X \rightarrow Y$  satisfying  $F^{(4)}(0) = 0$ ,  $DF^{(4)}(x, y) = 0$  for all  $x, y \in X$ , and

$$\|\tilde{f}_4(x) - F^{(4)}(x)\| \leq \frac{K\theta\|x\|^p}{576 \cdot |16 - 2^p|} \tag{2.26}$$

for all  $x \in X$ .

(1) If  $p < 4$ , then it follows from (2.10) and (2.16) that

$$\begin{aligned} \left\| \frac{\tilde{f}_4(2^n x)}{16^n} - \frac{\tilde{f}_4(2^{n+m} x)}{16^{n+m}} \right\| &= \left\| \sum_{i=n}^{n+m-1} \left( \frac{\tilde{f}_4(2^i x)}{16^i} - \frac{\tilde{f}_4(2^{i+1} x)}{16^{i+1}} \right) \right\| \\ &\leq \sum_{i=n}^{n+m-1} \left\| \frac{\Delta \tilde{f}_e(2^i x)}{9216 \cdot 16^i} \right\| \leq \sum_{i=n}^{n+m-1} \frac{K\theta 2^{ip} \|x\|^p}{9216 \cdot 16^i} \end{aligned} \tag{2.27}$$

for all  $x \in X$  and  $n, m \in \mathbb{N} \cup \{0\}$ , so  $\left\{ \frac{\tilde{f}_4(2^n x)}{16^n} \right\}$  is a Cauchy sequence for all  $x \in X$ . Since  $Y$  is complete, it converges and we can define a mapping  $F^{(4)} : X \rightarrow Y$  by

$$F^{(4)}(x) := \lim_{n \rightarrow \infty} \frac{\tilde{f}_4(2^n x)}{16^n}$$

for all  $x \in X$ . Moreover, letting  $n = 0$  and passing the limit  $m \rightarrow \infty$  in (2.27) we get the inequality

$$\|\tilde{f}_4(x) - F^{(4)}(x)\| \leq \frac{K\theta\|x\|^p}{576(16 - 2^p)}$$

10

for all  $x \in X$ , and by (2.5) it holds that

$$\begin{aligned} \|DF^{(4)}(x, y)\| &= \lim_{n \rightarrow \infty} \left\| \frac{D\tilde{f}_4(2^n x, 2^n y)}{16^n} \right\| \\ &= \lim_{n \rightarrow \infty} \left\| -\frac{256D\tilde{f}_e(2^n x, 2^n y)}{576 \cdot 16^n} + \frac{68D\tilde{f}_e(2^{n+1}x, 2^{n+1}y)}{576 \cdot 16^n} - \frac{D\tilde{f}_e(2^{n+2}x, 2^{n+2}y)}{576 \cdot 16^n} \right\| \\ &\leq \lim_{n \rightarrow \infty} \left( 256 \cdot 2^{np} + 68 \cdot 2^{(n+1)p} + 2^{(n+2)p} \right) \frac{\theta(\|x\|^p + \|y\|^p)}{576 \cdot 16^n} \\ &= 0 \end{aligned}$$

for all  $x, y \in X$ .

(2) If  $p > 4$ , then it follows from (2.10) and (2.16) that

$$\begin{aligned} \left\| 16^n \tilde{f}_4(2^{-n}x) - 16^{n+m} \tilde{f}_4(2^{-n-m}x) \right\| &= \left\| \sum_{i=n}^{n+m-1} \left( 16^i \tilde{f}_4(2^{-i}x) - 16^{i+1} \tilde{f}_4(2^{-i-1}x) \right) \right\| \\ &\leq \sum_{i=n}^{n+m-1} \left\| \frac{16^{i+1} \Delta \tilde{f}_e(2^{-i-1}x)}{9216} \right\| \leq \sum_{i=n}^{n+m-1} \frac{K\theta 16^i \|x\|^p}{576 \cdot 2^{(i+1)p}} \end{aligned} \quad (2.28)$$

for all  $x \in X$  and  $n, m \in \mathbb{N} \cup \{0\}$ , so  $\{16^n \tilde{f}_4(2^{-n}x)\}$  is a Cauchy sequence for all  $x \in X$ . Since  $Y$  is complete, it converges and we can define a mapping  $F^{(4)} : X \rightarrow Y$  by

$$F^{(4)}(x) := \lim_{n \rightarrow \infty} 16^n \tilde{f}_4(2^{-n}x)$$

for all  $x \in X$ . Moreover, letting  $n = 0$  and passing the limit  $m \rightarrow \infty$  in (2.28) we get

$$\|\tilde{f}_4(x) - F^{(4)}(x)\| \leq \frac{K\theta \|x\|^p}{576(2^p - 16)}$$

for all  $x \in X$ , and by (2.5) it holds that

$$\begin{aligned} \|DF^{(4)}(x, y)\| &= \lim_{n \rightarrow \infty} 16^n \left\| D\tilde{f}_4(2^{-n}x, 2^{-n}y) \right\| \\ &= \lim_{n \rightarrow \infty} 16^n \left\| -\frac{256D\tilde{f}_e(2^{-n}x, 2^{-n}y)}{576} + \frac{68D\tilde{f}_e(2^{-n+1}x, 2^{-n+1}y)}{576} - \frac{D\tilde{f}_e(2^{-n+2}x, 2^{-n+2}y)}{576} \right\| \\ &\leq \lim_{n \rightarrow \infty} \left( 256 \cdot 2^{-np} + 68 \cdot 2^{-np+p} + 2^{-np+2p} \right) \frac{16^n \theta (\|x\|^p + \|y\|^p)}{576} \\ &= 0 \end{aligned}$$

for all  $x, y \in X$ .

*Step 5.* For  $p \neq 5$ , there exists a mapping  $F^{(5)} : X \rightarrow Y$  satisfying  $F^{(5)}(0) = 0$ ,  $DF^{(5)}(x, y) = 0$  for all  $x, y \in X$ , and

$$\|\tilde{f}_5(x) - F^{(5)}(x)\| \leq \frac{K'\theta \|x\|^p}{69120 \cdot |32 - 2^p|} \quad (2.29)$$

for all  $x, y \in X$ .

(1) If  $p < 5$ , then it follows from (2.11) and (2.15) that we obtain that

$$\begin{aligned} \left\| \frac{\tilde{f}_5(2^n x)}{32^n} - \frac{\tilde{f}_5(2^{n+m} x)}{32^{n+m}} \right\| &= \left\| \sum_{i=n}^{n+m-1} \left( \frac{\tilde{f}_5(2^i x)}{32^i} - \frac{\tilde{f}_5(2^{i+1} x)}{32^{i+1}} \right) \right\| \\ &\leq \sum_{i=n}^{n+m-1} \left\| \frac{\Gamma \tilde{f}_o(2^i x)}{2211840 \cdot 32^i} \right\| \leq \sum_{i=n}^{n+m-1} \frac{K' \theta 2^{ip} \|x\|^p}{2211840 \cdot 32^i} \end{aligned} \quad (2.30)$$

for all  $x \in X$  and  $n, m \in \mathbb{N} \cup \{0\}$ . Then  $\left\{ \frac{\tilde{f}_5(2^n x)}{32^n} \right\}$  is a Cauchy sequence for all  $x \in X$ . Since  $Y$  is complete, it converges and we can define a mapping  $F^{(5)} : X \rightarrow Y$  by

$$F^{(5)}(x) := \lim_{n \rightarrow \infty} \frac{\tilde{f}_5(2^n x)}{32^n}$$

for all  $x \in X$ . Moreover, letting  $n = 0$  and passing the limit  $m \rightarrow \infty$  in (2.30) we get

$$\|\tilde{f}_5(x) - F^{(5)}(x)\| \leq \frac{K' \theta \|x\|^p}{69120(32 - 2^p)}$$

for all  $x \in X$ , and by (2.6) it holds that

$$\begin{aligned} \|DF^{(5)}(x, y)\| &= \lim_{n \rightarrow \infty} \left\| \frac{D\tilde{f}_5(2^n x, 2^n y)}{32^n} \right\| \leq \lim_{n \rightarrow \infty} \left( \left\| \frac{2048 D\tilde{f}_o(2^n x, 2^n y)}{69120 \cdot 32^n} \right\| \right. \\ &+ \left\| \frac{1296 D\tilde{f}_o(2^{n+1} x, 2^{n+1} y)}{69120 \cdot 32^n} \right\| + \left\| \frac{138 D\tilde{f}_o(2^{n+2} x, 2^{n+2} y)}{69120 \cdot 32^n} \right\| + \left. \left\| \frac{D\tilde{f}_o(2^{n+3} x, 2^{n+3} y)}{69120 \cdot 32^n} \right\| \right) \\ &\leq \lim_{n \rightarrow \infty} \left( 2048 \cdot 2^{np} + 1296 \cdot 2^{(n+1)p} + 138 \cdot 2^{(n+2)p} + 2^{(n+3)p} \right) \frac{\theta(\|x\|^p + \|y\|^p)}{69120 \cdot 32^n} \\ &= 0 \end{aligned}$$

for all  $x, y \in X$ .

(2) If  $p > 5$ , then it follows from (2.11) and (2.15) that we obtain that

$$\begin{aligned} \left\| 32^n \tilde{f}_5(2^{-n} x) - 32^{n+m} \tilde{f}_5(2^{-n-m} x) \right\| &= \left\| \sum_{i=n}^{n+m-1} \left( 32^i \tilde{f}_5(2^{-i} x) - 32^{i+1} \tilde{f}_5(2^{-(i+1)} x) \right) \right\| \\ &\leq \sum_{i=n}^{n+m-1} \left\| \frac{32^{i+1} \Gamma \tilde{f}_o(2^{-i-1} x)}{2211840} \right\| \leq \sum_{i=n}^{n+m-1} \frac{32^i K' \theta \|x\|^p}{69120 \cdot 2^{(i+1)p}} \end{aligned} \quad (2.31)$$

for all  $x \in X$  and  $n, m \in \mathbb{N} \cup \{0\}$ , so  $\{32^n \tilde{f}_5(2^{-n} x)\}$  is a Cauchy sequence for all  $x \in X$ . Since  $Y$  is complete, it converges and we can define a mapping  $F^{(5)} : X \rightarrow Y$  by

$$F^{(5)}(x) := \lim_{n \rightarrow \infty} 32^n \tilde{f}_5(2^{-n} x)$$

for all  $x \in X$ . Moreover, letting  $n = 0$  and passing the limit  $m \rightarrow \infty$  in (2.31), we get

$$\|\tilde{f}_5(x) - F^{(5)}(x)\| \leq \frac{K' \theta \|x\|^p}{69120(2^p - 32)}$$

12

for all  $x \in X$ , and, using (2.6), we have

$$\begin{aligned} \|DF^{(5)}(x, y)\| &= \lim_{n \rightarrow \infty} 32^n \left\| D\tilde{f}_5(2^{-n}x, 2^{-n}y) \right\| \leq \lim_{n \rightarrow \infty} 32^n \left( \left\| \frac{2048D\tilde{f}_o(2^{-n}x, 2^{-n}y)}{69120} \right\| \right. \\ &+ \left\| \frac{1296D\tilde{f}_o(2^{-n+1}x, 2^{-n+1}y)}{69120} \right\| + \left\| \frac{138D\tilde{f}_o(2^{-n+2}x, 2^{-n+2}y)}{69120} \right\| + \left. \left\| \frac{D\tilde{f}_o(2^{-n+3}x, 2^{-n+3}y)}{69120} \right\| \right) \\ &\leq \lim_{n \rightarrow \infty} (2048 \cdot 2^{-np} + 1296 \cdot 2^{-np+p} + 138 \cdot 2^{-np+2p} + 2^{-np+3p}) \frac{32^n \theta(\|x\|^p + \|y\|^p)}{69120} \\ &= 0 \end{aligned}$$

for all  $x, y \in X$ .

*Step 6.* For  $p \neq 6$ , there exists a mapping  $F^{(6)} : X \rightarrow Y$  satisfying  $F^{(6)}(0) = 0$ ,  $DF^{(6)}(x, y) = 0$  for all  $x, y \in X$ , and

$$\|\tilde{f}_6(x) - F^{(6)}(x)\| \leq \frac{K\theta\|x\|^p}{2880 \cdot |64 - 2^p|} \quad (2.32)$$

for all  $x, y \in X$ .

(1) If  $p < 6$ , then it follows from (2.11) and (2.16) that we obtain that

$$\begin{aligned} \left\| \frac{\tilde{f}_6(2^n x)}{64^n} - \frac{\tilde{f}_6(2^{n+m} x)}{64^{n+m}} \right\| &= \left\| \sum_{i=n}^{n+m-1} \left( \frac{\tilde{f}_6(2^i x)}{64^i} - \frac{\tilde{f}_6(2^{i+1} x)}{64^{i+1}} \right) \right\| \\ &\leq \sum_{i=n}^{n+m-1} \left\| \frac{\Delta \tilde{f}_e(2^i x)}{184320 \cdot 64^i} \right\| \leq \sum_{i=n}^{n+m-1} \frac{K2^{ip}\theta\|x\|^p}{184320 \cdot 64^i} \end{aligned} \quad (2.33)$$

for all  $x \in X$  and  $n, m \in \mathbb{N} \cup \{0\}$ , so  $\left\{ \frac{\tilde{f}_6(2^n x)}{64^n} \right\}$  is a Cauchy sequence for all  $x \in X$ . Since  $Y$  is complete, it converges and we can define a mapping  $F^{(6)} : X \rightarrow Y$  by

$$F^{(6)}(x) := \lim_{n \rightarrow \infty} \frac{\tilde{f}_6(2^n x)}{64^n}$$

for all  $x \in X$ . Moreover, letting  $n = 0$  and passing the limit  $m \rightarrow \infty$  in (2.33) we get

$$\|\tilde{f}_6(x) - F^{(6)}(x)\| \leq \frac{K\theta\|x\|^p}{2880(64 - 2^p)}$$

for all  $x \in X$ , and by (2.7) it holds that

$$\begin{aligned} \|DF^{(6)}(x, y)\| &= \lim_{n \rightarrow \infty} \left\| \frac{D\tilde{f}_6(2^n x, 2^n y)}{64^n} \right\| \\ &= \lim_{n \rightarrow \infty} \left\| \frac{64D\tilde{f}_e(2^n x, 2^n y)}{2880 \cdot 64^n} - \frac{5376D\tilde{f}_o(2^{n+1}x, 2^{n+1}y)}{2880 \cdot 64^n} + \frac{168D\tilde{f}_o(2^{n+2}x, 2^{n+2}y)}{2880 \cdot 64^n} \right\| \\ &\leq \lim_{n \rightarrow \infty} \left( 64 \cdot 2^{np} + 20 \cdot 2^{(n+1)p} + 2^{(n+2)p} \right) \frac{\theta(\|x\|^p + \|y\|^p)}{2880 \cdot 64^n} \\ &= 0 \end{aligned}$$

for all  $x, y \in X$ .

(2) If  $p > 6$ , then it follows from (2.11) and (2.16) that

$$\begin{aligned} \left\| 64^n \tilde{f}_6(2^{-n}x) - 64^{n+m} \tilde{f}_6(2^{-n-m}x) \right\| &= \left\| \sum_{i=n}^{n+m-1} \left( 64^i \tilde{f}_6(2^{-i}x) - 64^{i+1} \tilde{f}_6(2^{-i-1}x) \right) \right\| \\ &\leq \sum_{i=n}^{n+m-1} \left\| \frac{64^{i+1} \Delta \tilde{f}_e(2^{-i-1}x)}{184320} \right\| \leq \sum_{i=n}^{n+m-1} \frac{64^i K \theta \|x\|^p}{2880 \cdot 2^{(i+1)p}} \end{aligned} \quad (2.34)$$

for all  $x \in X$  and  $n, m \in \mathbb{N} \cup \{0\}$ , so  $\{64^n \tilde{f}_6(2^{-n}x)\}$  is a Cauchy sequence for all  $x \in X$ . Since  $Y$  is complete, it converges and we can define a mapping  $F^{(6)} : X \rightarrow Y$  by

$$F^{(6)}(x) := \lim_{n \rightarrow \infty} 64^n \tilde{f}_6(2^{-n}x)$$

for all  $x \in X$ . Moreover, letting  $n = 0$  and passing the limit  $m \rightarrow \infty$  in (2.34) we get

$$\|\tilde{f}_6(x) - F^{(6)}(x)\| \leq \frac{K\theta \|x\|^p}{2880(2^p - 64)}$$

for all  $x \in X$ , and by (2.7) it holds that

$$\begin{aligned} \|DF^{(6)}(x, y)\| &= \lim_{n \rightarrow \infty} 64^n \left\| D\tilde{f}_6(2^{-n}x, 2^{-n}y) \right\| \\ &= \lim_{n \rightarrow \infty} \frac{64^n}{2880} \left\| 64D\tilde{f}_e(2^{-n}x, 2^{-n}y) - 20D\tilde{f}_e(2^{-n+1}x, 2^{-n+1}y) + D\tilde{f}_e(2^{-n+2}x, 2^{-n+2}y) \right\| \\ &\leq \lim_{n \rightarrow \infty} (64 \cdot 2^{-np} + 20 \cdot 2^{-np+p} + 2^{-np+2p}) \frac{64^n \theta (\|x\|^p + \|y\|^p)}{2880} \\ &= 0 \end{aligned}$$

for all  $x, y \in X$ .

*Step 7.* For  $p \neq 7$ , there exists a mapping  $F^{(7)} : X \rightarrow Y$  satisfying  $F^{(7)}(0) = 0$ ,  $DF^{(7)}(x, y) = 0$  for all  $x, y \in X$ , and

$$\|\tilde{f}_7(x) - F^{(7)}(x)\| \leq \frac{K'\theta \|x\|^p}{1451520 \cdot |128 - 2^p|} \quad (2.35)$$

for all  $x \in X$ .

(1) If  $p < 7$ , then it follows from (2.12) and (2.15) that

$$\begin{aligned} \left\| \frac{\tilde{f}_7(2^n x)}{128^n} - \frac{\tilde{f}_7(2^{n+m} x)}{128^{n+m}} \right\| &= \left\| \sum_{i=n}^{n+m-1} \left( \frac{\tilde{f}_7(2^i x)}{128^i} - \frac{\tilde{f}_7(2^{i+1} x)}{128^{i+1}} \right) \right\| \\ &\leq \sum_{i=n}^{n+m-1} \left\| \frac{\Gamma \tilde{f}_o(2^i x)}{185794560 \cdot 128^i} \right\| \leq \sum_{i=n}^{n+m-1} \frac{K'\theta 2^{ip} \|x\|^p}{185794560 \cdot 128^i} \end{aligned} \quad (2.36)$$

14

for all  $x \in X$  and  $n, m \in \mathbb{N} \cup \{0\}$ , so  $\left\{ \frac{\tilde{f}_7(2^n x)}{128^n} \right\}$  becomes a Cauchy sequence for all  $x \in X$ . Since  $Y$  is complete, it converges, and hence we can define a mapping  $F^{(7)} : X \rightarrow Y$  by

$$F^{(7)}(x) := \lim_{n \rightarrow \infty} \frac{\tilde{f}_7(2^n x)}{128^n}$$

for all  $x \in X$ . Moreover, letting  $n = 0$  and passing the limit  $m \rightarrow \infty$  in (2.36) we get

$$\|\tilde{f}_7(x) - F^{(7)}(x)\| \leq \frac{K'\theta\|x\|^p}{1451520(128 - 2^p)}$$

for all  $x \in X$ , and, using (2.8), we have

$$\begin{aligned} \|DF^{(7)}(x, y)\| &= \lim_{n \rightarrow \infty} \left\| \frac{D\tilde{f}_7(2^n x, 2^n y)}{128^n} \right\| = \lim_{n \rightarrow \infty} \left\| -\frac{512D\tilde{f}_o(2^n x, 2^n y)}{65536 \cdot 128^n} \right. \\ &\quad \left. + \frac{336D\tilde{f}_o(2^{n+1}x, 2^{n+1}y)}{65536 \cdot 128^n} - \frac{42D\tilde{f}_o(2^{n+2}x, 2^{n+2}y)}{65536 \cdot 128^n} + \frac{D\tilde{f}_o(2^{n+3}x, 2^{n+3}y)}{65536 \cdot 128^n} \right\| \\ &\leq \lim_{n \rightarrow \infty} \left( 512 \cdot 2^{np} + 336 \cdot 2^{(n+1)p} + 42 \cdot 2^{(n+2)p} + 2^{(n+3)p} \right) \frac{\theta(\|x\|^p + \|y\|^p)}{65536 \cdot 128^n} \\ &= 0 \end{aligned}$$

for all  $x, y \in X$ .

(2) If  $p > 7$ , then it follows from (2.12) and (2.15) that

$$\begin{aligned} \left\| 128^n \tilde{f}_7(2^{-n}x) - 128^{n+m} \tilde{f}_7(2^{-n-m}x) \right\| &= \left\| \sum_{i=n}^{n+m-1} \left( 128^i \tilde{f}_7(2^{-i}x) - 128^{i+1} \tilde{f}_7(2^{-i-1}x) \right) \right\| \\ &\leq \sum_{i=n}^{n+m-1} \left\| -\frac{128^{i+1} \Gamma \tilde{f}_o(2^{-i-1}x)}{185794560} \right\| \leq \sum_{i=n}^{n+m-1} \frac{128^i K'\theta\|x\|^p}{1451520 \cdot 2^{(i+1)p}} \end{aligned} \quad (2.37)$$

for all  $x \in X$  and  $n, m \in \mathbb{N} \cup \{0\}$ , so  $\{128^n \tilde{f}_7(2^{-n}x)\}$  is a Cauchy sequence for all  $x \in X$ . Since  $Y$  is complete, it converges and we can define a mapping  $F^{(7)} : X \rightarrow Y$  by

$$F^{(7)}(x) := \lim_{n \rightarrow \infty} 128^n \tilde{f}_7(2^{-n}x)$$

for all  $x \in X$ . Moreover, letting  $n = 0$  and passing the limit  $m \rightarrow \infty$  in (2.37), we get

$$\|\tilde{f}_7(x) - F^{(7)}(x)\| \leq \frac{K'\theta\|x\|^p}{1451520(2^p - 128)}$$

for all  $x \in X$ , and, using (2.8), we have

$$\begin{aligned} \|DF^{(7)}(x, y)\| &= \lim_{n \rightarrow \infty} 128^n \left\| D\tilde{f}_7(2^{-n}x, 2^{-n}y) \right\| = \lim_{n \rightarrow \infty} 128^n \left\| \frac{-512D\tilde{f}_o(2^{-n}x, 2^{-n}y)}{2835 \cdot 512} \right. \\ &\quad \left. + \frac{336D\tilde{f}_o(2^{-n+1}x, 2^{-n+1}y)}{2835 \cdot 512} - \frac{42D\tilde{f}_o(2^{-n+2}x, 2^{-n+2}y)}{2835 \cdot 8} + \frac{D\tilde{f}_o(2^{-n+3}x, 2^{-n+3}y)}{2835 \cdot 512} \right\| \\ &\leq \lim_{n \rightarrow \infty} \left( 512 \cdot 2^{-np} + 336 \cdot 2^{-np+p} + 42 \cdot 2^{-np+2p} + 2^{-np+3p} \right) \frac{128^n \theta(\|x\|^p + \|y\|^p)}{1451520} \\ &= 0 \end{aligned}$$

for all  $x, y \in X$ .

Now, using the functions  $F^{(1)}, F^{(2)}, \dots, F^{(7)} : X \rightarrow Y$  of Step 1, Step 2, ..., and Step 7, respectively, we put

$$F(x) := \sum_{i=1}^7 F^{(i)}(x)$$

for all  $x \in X$ . Since  $\|\tilde{f}(x) - F(x)\| \leq \sum_{i=1}^7 \|\tilde{f}_i(x) - F^{(i)}(x)\|$  for all  $x \in X$ , together with (2.17), (2.20), (2.23), (2.26), (2.29), (2.32), (2.35), we obtain the property (2.14). And it is clear that

$$DF(x, y) = \sum_{i=1}^7 DF^{(i)}(x, y) = 0$$

for all  $x, y \in X$ . Finally, to prove the uniqueness of  $F$ , let  $G : X \rightarrow Y$  be another mapping, which satisfies the property (2.14),  $G(0) = 0$ , and  $DG(x, y) = 0$  for all  $x, y \in X$ . And let  $G_1, G_2, \dots, G_7 : X \rightarrow Y$  be defined as in Definition 2.1. Since  $DG_o(x, y) = DG_e(x, y) = 0$  for all  $x, y \in X$ ,

$$DG_i(x, y) = 0, \quad i = 1, 2, \dots, 7$$

for all  $x, y \in X$ . Additionally, by (2.14) with the definitions of the odd function and the even function of Definition 2.1, we get

$$\|\tilde{f}_e(x) - G_e(x)\|, \quad \|\tilde{f}_o(x) - G_o(x)\| \leq N\theta\|x\|^p \tag{2.38}$$

for all  $x \in X$ , where

$$N := \frac{K'}{22680 \cdot |2 - 2^p|} + \frac{K}{720 \cdot |4 - 2^p|} + \frac{K'}{17280 \cdot |8 - 2^p|} + \frac{K}{576 \cdot |16 - 2^p|} \\ + \frac{K'}{69120 \cdot |32 - 2^p|} + \frac{K}{2880 \cdot |64 - 2^p|} + \frac{K'}{1451520 \cdot |2^p - 128|}.$$

Notice that  $\Gamma G(x) = \Delta G(x) = 0$  for all  $x \in X$  by (2.9)-(2.12), we can prove that  $G_i(2x) = 2^i G_i(x)$ ,  $i = 1, 2, \dots, 7$ , for all  $x \in X$ . Particularly, they hold that

$$G_2(x) = 2^2 G_2\left(\frac{x}{2}\right) = \dots = 4^n G_2\left(\frac{x}{2^n}\right) \tag{2.39}$$

$$G_3(2^n x) = 2^3 G_3(2^{n-1} x) = \dots = 8^n G_3(x) \tag{2.40}$$

for all  $x \in X$  and  $n \in \mathbb{N}$ . Now, observe that, in the case of  $2 < p < 3$ , we have

$$\left\| 4^n \tilde{f}_2\left(\frac{x}{2^n}\right) - G_2(x) \right\| = \left\| 4^n \tilde{f}_2\left(\frac{x}{2^n}\right) - 4^n G_2\left(\frac{x}{2^n}\right) \right\| \leq \frac{1024 \cdot 4^n}{720} \left\| \tilde{f}_e\left(\frac{x}{2^n}\right) - G_e\left(\frac{x}{2^n}\right) \right\| \\ + \frac{80 \cdot 4^n}{720} \left\| \tilde{f}_e\left(\frac{2x}{2^n}\right) - G_e\left(\frac{2x}{2^n}\right) \right\| + \frac{4^n}{720} \left\| \tilde{f}_e\left(\frac{4x}{2^n}\right) - G_e\left(\frac{4x}{2^n}\right) \right\| \\ \leq (1024 + 80 \cdot 2^p + 4^p) \frac{4^n}{720 \cdot 2^{np}} N\theta\|x\|^p,$$

16

and

$$\begin{aligned} \left\| \frac{\tilde{f}_3(2^n x)}{8^n} - G_3(x) \right\| &= \left\| \frac{\tilde{f}_3(2^n x)}{8^n} - \frac{G_3(2^n x)}{8^n} \right\| \\ &\leq \left\| \frac{8192(\tilde{f}_o(2^n x) - G_o(2^n x))}{17280 \cdot 8^n} \right\| + \left\| \frac{4416(\tilde{f}_o(2^{n+1} x) - G_o(2^{n+1} x))}{17280 \cdot 8^n} \right\| \\ &\quad + \left\| \frac{162(\tilde{f}_o(2^{n+2} x) - G_o(2^{n+2} x))}{17280 \cdot 8^n} \right\| + \left\| \frac{(\tilde{f}_o(2^{n+3} x) - G_o(2^{n+3} x))}{17280 \cdot 8^n} \right\| \\ &\leq \left( 8192 + 4416 \cdot 2^p + 162 \cdot 2^{2p} + 2^{3p} \right) \frac{2^{np} N \theta \|x\|^p}{17280 8^n} \end{aligned}$$

for all  $x \in X$  and all positive integers  $n$ . Taking the limit in the above inequalities as  $n \rightarrow \infty$ , we obtain

$$G_2(x) = \lim_{n \rightarrow \infty} \tilde{4}^n \tilde{f}_2 \left( \frac{x}{2^n} \right) = F^{(2)}(x), \quad G_3(x) = \lim_{n \rightarrow \infty} \frac{\tilde{f}_3(2^n x)}{8^n} = F^{(3)}(x)$$

for all  $x \in X$ . Also, in the same way, we can show that  $G_i(x) = F^{(i)}(x)$ ,  $i = 1, 4, 5, 6, 7$ , for all  $x \in X$ . Therefore, we have shown that  $G(x) = F(x)$  for all  $x \in X$  in the case of  $2 < p < 3$ . Similarly, we can prove the uniqueness of  $F$  in other cases of  $p$ . It has been finished the proof of the theorem.  $\square$

## References

- [1] J. Baker, *A general functional equation and its stability*, Proc. Natl. Acad. Sci. **133(6)** (2005), 1657–1664.
- [2] I. S. Chang, Y. H. Lee, and J. Roh, *On the stability of the general sextic functional equation*, J. Chungcheong Math. Soc. **34(3)**(2021), 295–306.
- [3] Z. Gajda, *On stability of additive mappings*, Int. J. Math. & Math. Sci. **14(3)** (1991), 431–434.
- [4] P. Găvruta, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl. **184** (1994), 431–436.
- [5] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci. USA **27** (1941), 222–224.
- [6] G. Isac and T. M. Rassias, *On the Hyers-Ulam stability of  $\psi$ -additive mappings*, J. Approx. Theory **72** (1993), 131–137.
- [7] S. S. Jin and Y. H. Lee, *Hyers-Ulam-Rassias stability of a functional equation related to general quadratic mappings*, Honam Math. J. **39(3)** (2017), 417–430.
- [8] S. S. Jin and Y. H. Lee, *Stability of the General Quintic Functional Equation*, Int. J. Math. Anal.(Ruse) **15(6)**(2021), 271–282.

- [9] K.-W. Jun and H.-M. Kim, *On the Hyers-Ulam-Rassias stability of a general cubic functional equation*, Math. Inequal. Appl. **6** (2003), 289–302.
- [10] Y.-H. Lee, *On the generalized Hyers-Ulam stability of the generalized polynomial function of degree 3*, Tamsui Oxf. J. Math. Sci. **24**(4) (2008), 429–444.
- [11] Y.-H. Lee, *On the Hyers-Ulam-Rassias stability of the generalized polynomial function of degree 2*, J. Chungcheong Math. Soc. **22**(2) (2009), 201–209.
- [12] Y.-H. Lee, *Stability of a monomial functional equation on a restricted domain*, Mathematics **5** (2017), 53.
- [13] Y.-H. Lee, *On the Hyers-Ulam-Rassias stability of a general quartic functional equation*, East Asian Math. J. **35**(3) (2019), 351–356.
- [14] Y.-H. Lee, *On the Hyers-Ulam-Rassias stability of a general quintic functional equation and a general sextic functional equation*, Mathematics **7** (2019), 510.
- [15] Y.-H. Lee, *On the Hyers-Ulam-Rassias stability of an additive-quadratic-cubic functional equation*, J. Chungcheong Math. Soc. **32**(3) (2019), 295–307.
- [16] Y.-H. Lee and K. W. Jun, *A generalization of the Hyers-Ulam-Rassias stability of Jensen's equation*, J. Math. Anal. Appl. **238**(1) (1999), 305–315.
- [17] Y.-H. Lee, and S.-M. Jung, *A fixed point approach to the stability of an additive-quadratic-cubic-quartic type functional equation*, J. Funct. Spaces **2016** (2016).
- [18] Y.-H. Lee, and S.-M. Jung, *A general theorem on the stability of a class of functional equations including quadratic-additive functional equations*, SpringerPlus **5**(1) (2016), 1–16.
- [19] Y.-H. Lee, and S.-M. Jung, *General uniqueness theorem concerning the stability of additive, quadratic, and cubic functional equations*, Adv. Differ. Equ. (2016)
- [20] Y.-H. Lee, and S.-M. Jung, *A general theorem on the stability of a class of functional equations including quartic-cubic-quadratic-additive equations*, Mathematics **6**(12) (2018), 282.
- [21] Y.-H. Lee, and S.-M. Jung, *A general uniqueness theorem concerning the stability of AQCQ type functional equations*, Kyungpook Math. J. **58**(2) (2018), 291–305.
- [22] Y.-H. Lee, and S.-M. Jung, *A fixed point approach to the stability of a general quartic functional equation*, J. Math. Comput. Sci. **20** (2020), 207–215.
- [23] Y.-H. Lee, and S.-M. Jung, *Generalized Hyers-Ulam stability of some cubic-quadratic-additive type functional equations*, Kyungpook Math. J. **60**(1) (2020), 133–144.
- [24] Th. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), 297–300.
- [25] J. Roh, Y.-H. Lee, and S.-M. Jung, *The stability of a general sextic functional equation by fixed point theory*, J. Funct. Spaces **2020** (2020).
- [26] S.M. Ulam, *A Collection of Mathematical Problems*, Interscience, New York, 1960.