

On Products of Composition and Differentiation Operators

Abstract

In this paper we consider products of composition and differentiation operators on the Hardy space. We provide a complete characterization of boundedness and compactness of these operators. M. Moradi and M. Fatehi [5] obtain the explicit condition for these operators to be Hilbert-Schmidt operators. We have a theoretical application on the composition operators with series of perfect symbols.

Keywords. Boundedness, composition operator, compactness, differentiation operators.

1. Preliminaries

For \mathbb{D} be the open unit disk in the complex plane \mathbb{C} . The Hardy space H^2 is the Hilbert space of all analytic functions f_j on \mathbb{D} such that

$$\|f_j\|^2 = \lim_{r \rightarrow 1} \frac{1}{2\pi} \int_0^{2\pi} \sum_j |f_j(re^{i\theta})|^2 d\theta < \infty. \quad (1.1)$$

It is well known that the Hardy space H^2 is a reproducing kernel Hilbert space, with the inner product

$$\langle f_j, g_j \rangle = \frac{1}{2\pi} \int_0^{2\pi} \sum_j f_j(e^{i\theta}) \overline{g_j(e^{i\theta})} d\theta, \quad (1.2)$$

and with kernel functions $K_{w_j}^{(n)}(z) = \sum_j \frac{n!z^n}{(1-\bar{w}_j z)^{n+1}}$, where n is a non-negative integer and $z, w_j \in \mathbb{D}$. These kernel functions satisfy $\langle f_j, K_{w_j}^{(n)} \rangle = f_j^{(n)}(w_j)$ for each $f_j \in H^2$. To simplify notation we write K_{w_j} in case $n = 0$. In particular note that $\sum_j \|K_{w_j}\|^2 = \sum_j K_{w_j}(w_j) = \sum_j \frac{1}{1-|w_j|^2}$. Let $\hat{f}_j(n)$ be the n th coefficient of f_j in its Maclaurin series. Moreover, we have another representation for the norm of f_j on H^2 as follows

$$\|f_j\|^2 = \sum_{n=0}^{\infty} \sum_j |\hat{f}_j(n)|^2 < \infty. \quad (1.3)$$

The space H^∞ is the Banach space of bounded analytic functions f_j on \mathbb{D} with $\|f_j\|_\infty = \sup \sum \{|f_j(z)| : z \in \mathbb{D}\}$.

For φ_j an analytic self-map of \mathbb{D} , the composition operator $C_{\Sigma\varphi_j}$ is defined for analytic functions f_j on \mathbb{D} by $\Sigma C_{\varphi_j}(f_j) = \Sigma f_j \circ \varphi_j$. It is well known that every composition operator $C_{\Sigma\varphi_j}$ is bounded on H^2 (see [1. Corollary 3.7]). For each positive integer k , the operator $D^{(k)}$ for any $f_j \in H^2$ is defined by the rule $D^{(k)} \Sigma(f_j) = \Sigma f_j^{(k)}$. This operator is called the differentiation operator of order k .

For convenience, we use the notation D when $k = 1$. The differentiation operators $D^{(k)}$ are unbounded on H^2 , whereas Ohno [9] found a characterization for $C_{\Sigma\varphi_j}D$ and $DC_{\Sigma\varphi_j}$ to be bounded and compact on H^2 . The study of operators $C_{\Sigma\varphi_j}D$ and $DC_{\Sigma\varphi_j}$ was initially addressed by Hibscheiler, Portnoy, and Ohno (see [8] and [9]) and has been noticed by many researchers ([3],[4], and [10]). [5] be considering a slightly broader class of these operators.

M. Moradi and M. Fatehi [5] showed explicit improvement raised condition for the composition and differential operators.

Massimo Sorella, Riccardo Tione [6] showed the four-state problem and convex integration for linear differential operators is flexible.

Naoki Hamada, Naoto Shida, and Naohito Tomita [7] showed the bilinear pseudo-differential operators are determined in the framework of Besov spaces that improve boundedness of the operators

For each positive integer n , we write $D_{\Sigma\varphi_j,n}$ to denote the operator on H^2 given by the rule $\sum D_{\varphi_j,n}(f_j) = \sum C_{\varphi_j}D^{(n)}(f_j) = \sum f_j^{(n)} \circ \varphi_j$. The main results provide complete characterizations of the boundedness and compactness of operators $D_{\Sigma\varphi_j,n}$ on H^2 (Theorems 2.1 and 2.2). In addition, we characterize the Hilbert-Schmidt operators $D_{\Sigma\varphi_j,n}$ on H^2 (Theorem 3.3). [5] use some ideas which are found in [9].

For φ_j be an analytic self-map of \mathbb{D} . The Nevanlinna counting function $N_{\Sigma\varphi_j}$ of φ_j is defined by

$$\sum N_{\varphi_j}(w_j) = \sum_{\varphi_j(z)=w_j} \sum \log\left(\frac{1}{|z|}\right) \quad w_j \in \mathbb{D} \setminus \{\varphi_j(0)\} \quad (1.4)$$

and $\sum N_{\varphi_j}(\varphi_j(0)) = \infty$. Note that $\sum N_{\varphi_j}(w_j) = 0$ when w_j is not in $\varphi_j(\mathbb{D})$. For each $f_j \in H^2$, by using change of variables formula and Littlewood-Paley Identity, the norm of $\sum C_{\varphi_j}f_j$ is determined as follows:

$$\sum \|f_j \circ \varphi_j\|^2 = \sum |f_j(\varphi_j(0))|^2 + 2 \int_{\mathbb{D}} \sum |f_j'(w_j)|^2 N_{\varphi_j}(w_j) dA(w_j), \quad (1.5)$$

where dA is the normalized area measure on \mathbb{D} (see [1. Theorem 2.31]). Moreover, to obtain the lower bound estimate on $\|D_{\Sigma\varphi_j,n}\|$ we need the following well known lemma as follows (see [1. p. 137]):

Suppose that φ_j is an analytic self-map of \mathbb{D} and f_j is analytic in \mathbb{D} . Assume that Δ is any disk not containing $\{f_j^{-1}(\varphi_j(0))\}$ and centered at a . Then

$$\sum N_{\varphi_j}(f_j(a)) \leq \frac{1}{|\Delta|} \int_{\Delta} \sum N_{\varphi_j}(f_j(w_j)) dA(w_j), \quad (1.6)$$

where $|\Delta|$ is the normalized area measure of Δ .

2. Boundedness and Compactness of $D_{\Sigma\varphi_j,n}$

Here we show under some specific conditions well mentioned useable ways of finding bounded operators.

We determine which of these operators $D_{\Sigma\varphi_j,n}$ are bounded and compact (see [5]).

Theorem 2.1. Let φ_j be an analytic self-map of \mathbb{D} and n be a positive integer. The operator $D_{\Sigma\varphi_j,n}$ is bounded on H^2 if and only if

$$\sum N_{\varphi_j}(w_j) = O\left(\left[\sum \log(1/|w_j|)\right]^{2n+1}\right) \quad (|w_j| \rightarrow 1). \quad (2.1)'$$

Proof. Suppose that $D_{\Sigma\varphi_j,n}$ is bounded on H^2 . Let $f_j(z) = \frac{K_\lambda(z)}{\|K_\lambda\|} = \frac{\sqrt{1-|\lambda|^2}}{1-\bar{\lambda}z}$ for $\lambda \in \mathbb{D}$. By (1.5), we see that

$$\begin{aligned} & \sum \|D_{\varphi_j,n}\|^2 \\ & \geq \sum \|D_{\varphi_j,n}f_j\|^2 \\ & = \sum \left\| C_{\varphi_j} \left(\frac{n! \bar{\lambda}^n \sqrt{1-|\lambda|^2}}{(1-\bar{\lambda}z)^{n+1}} \right) \right\|^2 \\ & = \sum \left| \frac{n! \bar{\lambda}^n \sqrt{1-|\lambda|^2}}{(1-\bar{\lambda}\varphi_j(0))^{n+1}} \right|^2 + 2 \int_{\mathbb{D}} \sum \left| \frac{(n+1)! \bar{\lambda}^{n+1} \sqrt{1-|\lambda|^2}}{(1-\bar{\lambda}w_j)^{n+2}} \right|^2 N_{\varphi_j}(w_j) dA(w_j) \\ & \geq \int_{\mathbb{D}} \sum \frac{2((n+1)!)^2 |\lambda|^{2n+2} (1-|\lambda|^2)}{|1-\bar{\lambda}w_j|^{2n+4}} N_{\varphi_j}(w_j) dA(w_j). \end{aligned} \quad (2.1)$$

Substitute $w_j = \sum \alpha_\lambda(u_j) = \sum \frac{\lambda - u_j}{1 - \bar{\lambda}u_j}$ back into (2.1) and using [11, Theorem 7.26] to obtain

$$\begin{aligned} & \sum \|D_{\varphi_j,n}\|^2 \\ & \geq \int_{\mathbb{D}} \sum \frac{2((n+1)!)^2 |\lambda|^{2n+2} (1-|\lambda|^2)}{|1-\bar{\lambda}\alpha_\lambda(u_j)|^{2n+4}} N_{\varphi_j}(\alpha_\lambda(u_j)) |\alpha'_\lambda(u_j)|^2 dA(u_j). \end{aligned} \quad (2.2)$$

Since $1 - \sum \bar{\lambda}\alpha_\lambda(u_j) = \sum \frac{1-|\lambda|^2}{1-\bar{\lambda}u_j}$ and $\sum \alpha'_\lambda(u_j) = \sum \frac{|\lambda|^2-1}{(1-\bar{\lambda}u_j)^2}$, by substituting α'_λ and $1 - \bar{\lambda}\alpha_\lambda$ back into (2.2), we see that

$$\begin{aligned} & \sum \|D_{\varphi_j,n}\|^2 \\ & \geq \int_{\mathbb{D}} \sum \frac{2((n+1)!)^2 |\lambda|^{2n+2} |1-\bar{\lambda}u_j|^{2n}}{(1-|\lambda|^2)^{2n+1}} N_{\varphi_j}(\alpha_\lambda(u_j)) dA(u_j). \end{aligned} \quad (2.3)$$

Because $\sum |1-\bar{\lambda}u_j| \geq \frac{1}{2}$ for any $u_j \in \mathbb{D}/2$, we get from (2.3) that

$$\sum \|D_{\varphi_j,n}\|^2 \geq \int_{\mathbb{D}/2} \sum \frac{2((n+1)!)^2 |\lambda|^{2n+2}}{2^{2n} (1-|\lambda|^2)^{2n+1}} N_{\varphi_j}(\alpha_\lambda(u_j)) dA(u_j). \quad (2.4)$$

There exists $r < 1$ such that for each λ with $r < |\lambda| < 1$, $\sum \alpha_\lambda^{-1}(\varphi_j(0)) \notin \mathbb{D}/2$ because $\sum |\alpha_\lambda^{-1}(\varphi_j(0))| = \sum |\alpha_{\varphi_j(0)}(\lambda)|$ and $\sum \alpha_{\varphi_j(0)}$ is an automorphism of \mathbb{D} . By (1.6) and (2.4), we have

$$\begin{aligned}
\sum \|D_{\varphi_j, n}\|^2 &\geq \frac{2((n+1)!)^2 |\lambda|^{2n+2}}{2^{2n}(1-|\lambda|^2)^{2n+1}} \int_{\frac{\mathbb{D}}{2}} \sum N_{\varphi_j}(\alpha_\lambda(u_j)) dA(u_j) \\
&\geq \frac{2((n+1)!)^2 |\lambda|^{2n+2}}{2^{2n}(1-|\lambda|^2)^{2n+1}} \cdot \sum \frac{N_{\varphi_j}(\alpha_\lambda(0))}{4} \\
&= \frac{((n+1)!)^2 |\lambda|^{2n+2}}{2^{2n+1}(1-|\lambda|^2)^{2n+1}} \sum N_{\varphi_j}(\lambda)
\end{aligned} \tag{2.5}$$

for each λ with $r < |\lambda| < 1$. Since $D_{\Sigma \varphi_j, n}$ is bounded, there exists a constant number M so that

$$\lim_{|\lambda| \rightarrow 1} \sum \frac{((n+1)!)^2 |\lambda|^{2n+2}}{2^{2n+1}(1-|\lambda|^2)^{2n+1}} N_{\varphi_j}(\lambda) \leq M. \tag{2.6}$$

We know that $\log(1/|\lambda|)$ is comparable to $1-|\lambda|$ as $|\lambda| \rightarrow 1^-$. Note that

$$\begin{aligned}
&\lim_{|\lambda| \rightarrow 1} \sum \frac{((n+1)!)^2 |\lambda|^{2n+2}}{2^{2n+1}(1-|\lambda|^2)^{2n+1}} N_{\varphi_j}(\lambda) \\
&= \lim_{|\lambda| \rightarrow 1} \sum \frac{((n+1)!)^2 |\lambda|^{2n+2}}{2^{2n+1}(1+|\lambda|)^{2n+1}} \left(\frac{\log(1/|\lambda|)}{1-|\lambda|} \right)^{2n+1} \frac{N_{\varphi_j}(\lambda)}{(\log(1/|\lambda|))^{2n+1}} \\
&\geq \frac{((n+1)!)^2}{2^{6n+4}} \lim_{|\lambda| \rightarrow 1} \sum \frac{N_{\varphi_j}(\lambda)}{(\log(1/|\lambda|))^{2n+1}}.
\end{aligned} \tag{2.7}$$

By (2.6) and (2.7), we can see that

$$N_{\Sigma \varphi_j}(\lambda) = O([\log(1/|\lambda|)]^{2n+1}) \quad (|\lambda| \rightarrow 1). \tag{2.7}'$$

Conversely, suppose that for some R with $0 < R < 1$, there exists a constant M satisfying

$$\sup_{R < |w_j| < 1} \sum N_{\varphi_j}(w_j) / [\log(1/|w_j|)]^{2n+1} \leq M.$$

Let f_j be an arbitrary function in H^2 . It follows from (1.5) that

$$\begin{aligned}
&\sum \|D_{\varphi_j, n} f_j\|^2 \\
&= \sum |f_j^{(n)}(\varphi_j(0))|^2 + 2 \int_{\mathbb{D}} \sum |f_j^{(n+1)}(w)_j|^2 N_{\varphi_j}(w_j) dA(w_j) \\
&= |f_j^{(n)}(\varphi_j(0))|^2 \\
&+ 2 \sum \left(\int_{R\mathbb{D}} |f_j^{(n+1)}(w_j)|^2 N_{\varphi_j}(w_j) dA(w_j) \right. \\
&\left. + \int_{\mathbb{D} \setminus R\mathbb{D}} |f_j^{(n+1)}(w_j)|^2 N_{\varphi_j}(w_j) dA(w_j) \right).
\end{aligned} \tag{2.8}$$

First we estimate the first and the second terms in the right-hand of (2.8). Observe that

$$\sum f_j^{(n)}(z) = \sum \langle f_j, K_z^{(n)} \rangle = \int_0^{2\pi} \sum \frac{n! e^{-in\theta} f_j(e^{i\theta}) d\theta}{(1 - e^{-i\theta} z)^{n+1} 2\pi}$$

and hence

$$\sum |f_j^{(n)}(z)| \leq \frac{n!}{(1-|z|)^{n+1}} \int_0^{2\pi} \sum |f_j(e^{i\theta})| \frac{d\theta}{2\pi} \leq \frac{n!}{(1-|z|)^{n+1}} \sum \|f_j\| \quad (2.9)$$

for any $z \in \mathbb{D}$. It follows from (2.9) that

$$\sum |f_j^{(n)}(\varphi_j(0))| \leq \sum \frac{n! \|f_j\|}{(1-|\varphi_j(0)|)^{n+1}}. \quad (2.10)$$

Moreover, we can see that

$$\sum |f_j^{(n+1)}(z)| = \sum |\langle f_j, K_z^{(n+1)} \rangle| = \frac{(n+1)!}{(1-|z|)^{n+2}} \sum \|f_j\| \quad (2.11)$$

for any $z \in \mathbb{D}$. Therefore by (2.11), we see that

$$\begin{aligned} \int_{R\mathbb{D}} \sum |f_j^{(n+1)}(w_j)|^2 N_{\varphi_j}(w_j) dA(w_j) \\ \leq \left(\frac{(n+1)!}{(1-R)^{n+2}} \right)^2 \sum \|f_j\|^2 \int_{R\mathbb{D}} N_{\varphi_j}(w_j) dA(w_j). \end{aligned}$$

Since $\sum \| \varphi_j \|^2 = |\sum \varphi_j(0)|^2 + 2 \int_{\mathbb{D}} N_{\varphi_j}(w_j) dA(w_j)$ by (1.5), we obtain

$$\int_{\mathbb{D}} \sum N_{\varphi_j}(w_j) dA(w_j) = \frac{1}{2} \sum (\| \varphi_j \|^2 - |\varphi_j(0)|^2) < 1. \quad (2.12)$$

From (2.11) and (2.12), we see that

$$\int_{R\mathbb{D}} \sum |f_j^{(n+1)}(w_j)|^2 N_{\varphi_j}(w_j) dA(w_j) \leq \left(\frac{(n+1)!}{(1-R)^{n+2}} \right)^2 \sum \|f_j\|^2. \quad (2.13)$$

Now we estimate the third term in the right-hand of (2.8). We have

$$\begin{aligned} \int_{\mathbb{D} \setminus R\mathbb{D}} \sum |f_j^{(n+1)}(w_j)|^2 N_{\varphi_j}(w_j) dA(w_j) \\ = \int_{\mathbb{D} \setminus R\mathbb{D}} \sum |f_j^{(n+1)}(w_j)|^2 (\log(1/|w_j|))^{2n+1} \frac{N_{\varphi_j}(w_j)}{(\log(1/|w_j|))^{2n+1}} dA(w_j) \\ \leq \sup_{R < |w_j| < 1} \sum \frac{N_{\varphi_j}(w_j)}{(\log(1/|w_j|))^{2n+1}} \int_{\mathbb{D} \setminus R\mathbb{D}} |f_j^{(n+1)}(w_j)|^2 (\log(1/|w_j|))^{2n+1} dA(w_j) \\ \leq M \int_{\mathbb{D} \setminus R\mathbb{D}} \sum |f_j^{(n+1)}(w_j)|^2 (\log(1/|w_j|))^{2n+1} dA(w_j). \quad (2.14) \end{aligned}$$

Let $f_j(z) = \sum_{m=0}^{\infty} a_m^j z^m$. We get

$$\int_{\mathbb{D} \setminus R\mathbb{D}} \sum |f_j^{(n+1)}(w_j)|^2 (\log(1/|w_j|))^{2n+1} dA(w_j)$$

$$\begin{aligned}
&\leq \sum_{m=n+1}^{\infty} \sum m^2(m-1)^2 \dots (m-n)^2 |a_m^j|^2 \int_{\mathbb{D} \setminus R\mathbb{D}} |(w_j)^{m-(n+1)}|^2 (\log(1/|w_j|))^{2n+1} dA(w_j) \\
&\leq \sum_{m=n+1}^{\infty} \sum m^2(m-1)^2 \dots (m-n)^2 |a_m^j|^2 \int_{\mathbb{D}} |(w_j)^{m-(n+1)}|^2 (\log(1/|w_j|))^{2n+1} dA(w_j) \\
&= \sum_{m=n+1}^{\infty} \sum m^2(m-1)^2 \dots (m-n)^2 |a_m^j|^2 \int_0^1 \int_0^{2\pi} |r_j e^{i\theta}|^{2(m-(n+1))} (\log(1/r_j))^{2n+1} r_j dr_j \frac{d\theta}{\pi} \\
&\leq \sum_{m=n+1}^{\infty} \sum m^2(m-1)^2 \dots (m-n)^2 |a_m^j|^2 \\
&\quad \int_0^1 (r_j)^{2(m-(n+1))} (\log(1/r_j))^{2n+1} 2r_j dr_j. \tag{2.15}
\end{aligned}$$

Now substitute $t_j = r_j^2$ and $u_j = \log(1/t_j)$ to obtain

$$\begin{aligned}
&\int_0^1 \sum (r_j)^{2(m-(n+1))} (\log(1/r_j))^{2n+1} 2r dr_j \\
&= \int_0^1 \sum t_j^{(m-(n+1))} \left(\frac{1}{2} \log(1/t_j)\right)^{2n+1} dt_j \\
&= (1/2)^{2n+1} \int_0^{\infty} \sum e^{-u_j(m-n)} u_j^{2n+1} du_j. \tag{2.16}
\end{aligned}$$

By substituting $x_j = (m-n)u_j$ back into (2.16), we have

$$\begin{aligned}
&(1/2)^{2n+1} \int_0^{\infty} \sum e^{-u_j(m-n)} u_j^{2n+1} du_j \\
&= \frac{1}{2^{2n+1}(m-n)^{2n+2}} \int_0^{\infty} \sum e^{-x} x_j^{2n+1} dx_j \\
&= \frac{\Gamma(2n+2)}{2^{2n+1}(m-n)^{2n+2}}. \tag{2.17}
\end{aligned}$$

By (2.14), (2.15), (2.16) and (2.17), we can see that

$$\begin{aligned}
&\int_{\mathbb{D} \setminus R\mathbb{D}} \sum |f_j^{(n+1)}(w_j)|^2 N_{\varphi_j}(w_j) dA(w_j) \\
&\leq M \sum_{m=n+1}^{\infty} \sum m^2(m-1)^2 \dots (m-n)^2 |a_m^j|^2 \frac{\Gamma(2n+2)}{2^{2n+1}(m-n)^{2n+2}} \\
&= M \frac{(2n+1)!}{2^{2n+1}} \sum_{m=n+1}^{\infty} \sum \frac{m^2(m-1)^2 \dots (m-n+1)^2}{(m-n)^{2n}} |a_m^j|^2 \\
&\leq M\lambda \frac{(2n+1)!}{2^{2n+1}} \sum_{m=n+1}^{\infty} \sum |a_m^j|^2
\end{aligned}$$

$$\leq M\lambda \frac{(2n+1)!}{2^{2n+1}} \sum \|f_j\|^2, \quad (2.18)$$

where λ is a constant so that $\frac{m^2(m-1)^2 \dots (m-n+1)^2}{(m-n)^{2n}} \leq \lambda$ for each $m \geq n+1$ (note that the function $f_j(x_j) = \frac{x_j^2(x-1)^2 \dots (x_j-n+1)^2}{(x-n)^{2n}}$ is bounded on $[n+1, +\infty)$). Then (2.8), (2.10), (2.13) and (2.18) show that $D_{\Sigma\varphi_j, n}$ is bounded. We have the following (see [5])

Theorem 2.2. Let φ_j be an analytic self-map of \mathbb{D} and n be a positive integer. The operator $D_{\Sigma\varphi_j, n}$ is compact on H^2 if and only if

$$\sum N_{\varphi_j}(w_j) = o \sum ([\log(1/|w_j|)]^{2n+1}) \quad (|w_j| \rightarrow 1). \quad (2.19)$$

Proof. Let $h_m(z) = \frac{\sqrt{1-|\lambda_m|^2}}{1-\bar{\lambda}_m z}$ for a sequence $\{\lambda_m\}$ in \mathbb{D} so that $|\lambda_m| \rightarrow 1$ as $m \rightarrow \infty$. Then $h_m \rightarrow 0$ weakly as $m \rightarrow \infty$ by [1 Theorem 2.17]. First suppose that $D_{\Sigma\varphi_j, n}$ is compact. Hence $\sum \|D_{\varphi_j, n} h_m\| \rightarrow 0$ as $m \rightarrow \infty$. Therefore (2.5) shows that

$$\lim_{m \rightarrow \infty} \sum \frac{((n+1)!)^2 |\lambda_m|^{2n+2}}{2^{2n+1} (1-|\lambda_m|^2)^{2n+1}} N_{\varphi_j}(\lambda_m) = 0.$$

Since $\log(1/|\lambda_m|)$ is comparable to $1-|\lambda_m|$ as $m \rightarrow \infty$, the result follows.

Conversely, suppose that (2.19) holds. Let $\epsilon > 0$. Then there exists $R, 0 < R < 1$, such that

$$\sup_{R < |w_j| < 1} \sum N_{\varphi_j}(w_j) / [\log(1/|w_j|)]^{2n+1} < \epsilon. \quad (2.20)$$

Let $\{(f_j)_m\}$ be any bounded sequence in H^2 . By using the idea which was stated in the proof of [1. Proposition 3.11], we can see that $\{(f_j)_m\}$ is a normal family and there exists a subsequence $\{(f_j)_{m_k}\}$ which converges to some function $f_j \in H^2$ uniformly on all compact subsets of \mathbb{D} . Let $(g_j)_{m_k} = (f_j)_{m_k} - f_j$ for each positive integer k . Note that $\{(g_j)_{m_k}\}$ is a bounded sequence in H^2 which converges to 0 uniformly on all compact subsets of \mathbb{D} . By (2.8), we obtain

$$\begin{aligned} & \sum \|D_{\varphi_j, n}(g_j)_{m_k}\|^2 \\ &= \sum |(g_j)_{m_k}^{(n)}(\varphi_j(0))|^2 + 2 \int_{R\mathbb{D}} \sum |(g_j)_{m_k}^{(n+1)}(w_j)|^2 N_{\varphi_j}(w_j) dA(w_j) \\ &+ 2 \int_{\mathbb{D} \setminus R\mathbb{D}} \sum |(g_j)_{m_k}^{(n+1)}(w_j)|^2 N_{\varphi_j}(w_j) dA(w_j). \end{aligned} \quad (2.21)$$

By [2. Theorem 2.1, p. 151], we can choose k_ϵ so that

$$\sum |(g_j)_{m_k}^{(n)}(\varphi_j(0))| < \sqrt{\epsilon} \quad (2.22)$$

and $\sum |(g_j)_{m_k}^{(n+1)}| < \sqrt{\epsilon}$ on $R\mathbb{D}$ whenever $k > k_\epsilon$. Substituting $f_j(z) = z$ into (1.5), we see that

$$\begin{aligned} \int_{R\mathbb{D}} \sum |(g_j)_{m_k}^{(n+1)}(w_j)|^2 N_{\varphi_j}(w_j) dA(w_j) &\leq \epsilon \int_{R\mathbb{D}} \sum N_{\varphi_j}(w_j) dA(w_j) \\ &\leq \frac{\epsilon}{2} \sum (\|\varphi_j\|^2 - |\varphi_j(0)|^2) \end{aligned} \quad (2.23)$$

for $k > k_\epsilon$. On the other hand by (2.20) and the same idea as stated in the proof of (2.14) and (2.18), we see that

$$\begin{aligned} \int_{\mathbb{D} \setminus R\mathbb{D}} \sum |(g_j)(w)|^2 N_{\varphi_j}(w_j) dA(w_j) \\ \leq \sup_{R < |w_j| < 1} \sum \frac{N_{\varphi_j}(w_j)}{[\log(1/|w_j|)]^{2n+1}} \\ \int_{\mathbb{D} \setminus R\mathbb{D}} |(g_j)_{m_k}^{(n+1)}(w_j)|^2 [\log(1/|w_j|)]^{2n+1} dA(w_j) \\ \leq C\epsilon \sum \|(g_j)_{m_k}\|, \end{aligned} \quad (2.24)$$

where C is a constant. Hence we conclude that $\sum \|D_{\varphi_j, n}(g_j)_{m_k}\|$ converges to zero as $k \rightarrow \infty$ by (2.21), (2.22), (2.23) and (2.24) and so $D_{\varphi, n}$ is compact.

The preceding theorems lead to characterizations of all bounded and compact operators $D_{\varphi_j, n}$ when φ_j is a univalent self-map (see [5]).

Corollary 2.3. Let φ_j be a univalent self-map of \mathbb{D} and n be a positive integer. Then the following hold.

(i) $D_{\Sigma \varphi_j, n}$ is bounded on H^2 if and only if

$$\sup_{w_j \in \mathbb{D}} \sum \frac{1 - |w_j|}{(1 - |\varphi_j(w_j)|)^{2n+1}} < \infty \quad (2.25)$$

(ii) $D_{\Sigma \varphi_j, n}$ is compact on H^2 if and only if

$$\lim_{|w_j| \rightarrow 1} \sum \sum \frac{1 - |w_j|}{(1 - |\varphi_j(w_j)|)^{2n+1}} = 0 \quad (2.26)$$

Proof. Since $\Sigma \varphi_j$ is univalent, we can see that $N_{\varphi_j}(w_j) = \log(1/|z|)$, where $\varphi_j(z) = w_j$. We observe that

$$\sum \frac{N_{\varphi_j}(w_j)}{[\log(1/|w_j|)]^{2n+1}} = \sum \frac{-\log(|z|)}{(-\log(|\varphi_j(z)|))^{2n+1}}. \quad (2.27)$$

Moreover, we know that $\log(1/|z|)$ is comparable to $1 - |z|$ as $|z| \rightarrow 1^-$. Furthermore $|z| \rightarrow 1$ as $|\varphi_j(z)| \rightarrow 1$. Therefore the results follow immediately from Theorems 2.1 and 2.2

3. Hilbert-Schmidt Operator $D_{\Sigma \varphi_j, n}$

We begin with a few easy observations that help us in the proof of Theorem 3.3.

By showing some needable tools of Hilbert-Schmidt operator, its norm and bound, by considering some Lemmas.

In the proof of the following lemma, we assume that $0^0 = 1$ (see [5]).

Lemma 3.1. Let n be a positive integer and $\alpha_k > 0$ for each $0 \leq k \leq n$. Then for $0 \leq x < 1$, the following statements hold.

$$(a) \sum_{k=0}^n \sum \frac{\alpha_k^j x^k}{(1-x)^{n+k+1}} \leq \frac{\sum_{k=0}^n \sum \alpha_k^j}{(1-x)^{2n+1}}.$$

$$(b) \text{ There exists a positive number } \beta^j \text{ such that } \sum_{k=0}^n \sum \frac{\alpha_k^j x^k}{(1-x)^{n+k+1}} \geq \sum \frac{\beta^j}{(1-x)^{2n+1}}.$$

Proof. (a) We can see that

$$\sum_{k=0}^n \sum \frac{\alpha_k^j x^k}{(1-x)^{n+k+1}} = \frac{\sum_{k=0}^n \sum \alpha_k^j x^k (1-x)^{n-k}}{(1-x)^{2n+1}}$$

Since $0 \leq x < 1$ and $\alpha_k^j > 0$, we conclude that $\sum_{k=0}^n \sum \alpha_k^j x^k (1-x)^{n-k} \leq \sum_{k=0}^n \sum \alpha_k^j$. Hence the conclusion follows.

(b) We have

$$(1-x)^{2n+1} \sum_{k=0}^n \sum \frac{\alpha_k^j x^k}{(1-x)^{n+k+1}} = \sum_{k=0}^n \sum \alpha_k^j x^k (1-x)^{n-k} > 0.$$

Since $\sum_{k=0}^n \sum \alpha_k^j x^k (1-x)^{n-k}$ is a continuous function on $[0,1]$, there exists a positive number β^j such that $\sum_{k=0}^n \sum \alpha_k^j x^k (1-x)^{n-k} \geq \beta^j$. Hence the result follows. We have the following (see [5])

Lemma 3.2. Let n be a positive integer. Then

$$\sum_{m=n}^{\infty} [m(m-1) \dots (m-n+1)]^2 x^{m-n} = (n!)^2 \sum_{k=0}^n \frac{(n+k)!}{(k!)^2 (n-k)!} \frac{x^k}{(1-x)^{n+k+1}}$$

for $0 \leq x < 1$.

Proof. See [10, Lemma 1] and the general Leibniz rule.

A Hilbert-Schmidt operator on a separable Hilbert space H is a bounded operator A with finite Hilbert-Schmidt norm $\|A\|_{HS} = \left(\sum_{n=1}^{\infty} \|Ae_n\|^2\right)^{1/2}$, where $\{e_n\}$ is an orthonormal basis of H . These definitions are independent of the choice of the basis (see [1, Theorem 3.23]). Now we have the following (see [5])

Theorem 3.3. Let $D_{\Sigma \varphi_j, n}$ be a bounded operator on H^2 . Then $D_{\Sigma \varphi_j, n}$ is a HilbertSchmidt operator on H^2 if and only if

$$\lim_{r \rightarrow 1} \frac{1}{2\pi} \int_0^{2\pi} \sum \frac{1}{(1 - |\varphi_j(re^{i\theta})|^2)^{2n+1}} < \infty. \quad (3.1)$$

Proof. Suppose that (3.1) holds. Lemmas 3.1- 3.2- and [11, Theorem 1.27] imply that

$$\begin{aligned} \sum_{m=0}^{\infty} \sum \|D_{\varphi_j, n} z^m\| &= \sum_{m=n}^{\infty} \sum \|m(m-1) \dots (m-n+1) \varphi_j^{m-n}\| \\ &= \sum_{m=n}^{\infty} \lim_{r \rightarrow 1} \frac{1}{2\pi} \int_0^{2\pi} \sum |m(m-1) \dots (m-n+1) \varphi_j^{m-n}(r_j e^{i\theta})|^2 d\theta \end{aligned}$$

$$\begin{aligned}
&= \lim_{r \rightarrow 1} \sum_{m=n}^{\infty} \frac{1}{2\pi} \int_0^{2\pi} \sum |m(m-1) \dots (m-n+1) \varphi_j^{m-n}(r_j e^{i\theta})|^2 d\theta \\
&= \lim_{r \rightarrow 1} \frac{1}{2\pi} \int_0^{2\pi} \sum_{m=n}^{\infty} |m(m-1) \dots (m-n+1) \varphi_j^{m-n}(r_j e^{i\theta})|^2 d\theta \\
&= \lim_{r \rightarrow 1} \frac{1}{2\pi} \int_0^{2\pi} \sum_{k=0}^n \sum \frac{(n!)^2 (n+k)!}{(k!)^2 (n-k)!} \frac{|\varphi_j(r e^{i\theta})|^{2k}}{(1 - |\varphi_j(r e^{i\theta})|^2)^{n+k+1}} \\
&\leq \lim_{r \rightarrow 1} \frac{1}{2\pi} \int_0^{2\pi} \sum \frac{\alpha^j}{(1 - |\varphi_j(r e^{i\theta})|^2)^{2n+1}}, \tag{3.2}
\end{aligned}$$

where $\alpha^j = \sum_{k=0}^n \frac{(n!)^2 (n+k)!}{(k!)^2 (n-k)!}$ (note that the interchange of limit and summation is justified by [1 Corollary 2.23] and using Lebesgue's Monotone Convergence Theorem with counting measure). It follows that $\sum_{m=0}^{\infty} \sum \|D_{\varphi_j, n} z^m\| < \infty$ and so $D_{\Sigma \varphi_j, n}$ is a Hilbert-Schmidt operator on H^2 by [1 Theorem 3.23].

Conversely, suppose that $D_{\Sigma \varphi_j, n}$ is a Hilbert-Schmidt operator on H^2 . We infer from [1, Theorem 3.23] that

$$\sum_{m=0}^{\infty} \sum \|D_{\varphi_j, n} z^m\|^2 < \infty. \tag{3.3}$$

On the other hand, by the proof of (3.2) and Lemma 3.1 there exists a positive number β_j such that

$$\begin{aligned}
&\sum_{m=0}^{\infty} \sum \|D_{\varphi_j, n} z^m\|^2 \\
&= \lim_{r \rightarrow 1} \frac{1}{2\pi} \int_0^{2\pi} \sum_{k=0}^n \sum \frac{(n!)^2 (n+k)!}{(k!)^2 (n-k)!} \frac{|\varphi_j(r e^{i\theta})|^{2k}}{(1 - |\varphi_j(r e^{i\theta})|^2)^{n+k+1}} \\
&\geq \lim_{r \rightarrow 1} \frac{1}{2\pi} \int_0^{2\pi} \sum \frac{\beta_j}{(1 - |\varphi_j(r e^{i\theta})|^2)^{2n+1}}. \tag{3.4}
\end{aligned}$$

Hence the result follows from (3.3) and (3.4).

4. Conclusion

A theoretical approaches of the composition operators with series of chasuble symbols has boundedness, compactness and has a differential operator with an explicit characterization on the Hardy spaces. A full estimates of the Hilbert-Schmidt operator are classified and verified.

References

- [1] C. C. Cowen and B. D. MacCluer, Composition Operators on Spaces of Analytic Functions. Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1995.
- [2] J. B. Conway, Functions of One Complex Variable, Second Edition, Springer-Verlag, New York, 1978.
- [3] M. Fatehi and C. N. B. Hammond, Composition-differentiation operators on the Hardy space, Proc. Amer. Math. Soc. 148 (2020), 2893-2900.
- [4] M. Fatehi and C. N. B. Hammond, Normality and self-adjointness of weighted composition differentiation operators, Complex Anal. Oper. Theory, 15 (2021), 1-13.
- [5] Mahbube Moradi and Mahsa Fatehi, Products of Composition and Differentiation Operators, arXiv:2108.06774v1 [math.FA] 15 Aug 2021.
- [6] Massimo Sorella, Riccardo Tione, The four-state problem and convex integration for linear differential operators, Journal of Functional Analysis 284 (2023) 109785.
- [7] Naoki Hamada, Naoto Shida, and Naohito Tomita, On The Ranges of Bilinear Pseudo-Differential Operators of $S_{0,0}$ -Type on $L^2 \times L^2$, Journal of Functional Analysis 280 (2021) 108826.
- [8] R. A. Hirschweiler and N. Portnoy, Composition followed by differentiation between Bergman and Hardy spaces, Rocky Mountain J. Math, **35** (2005), 843-855.
- [9] S. Ohno, Products of composition and differentiation between Hardy spaces, Bull. Austral. Math. Soc. 73 (2006), 235-243.
- [10] S. Stević, Products of composition and differentiation operators on the weighted Bergman space, Bull. Belg. Math. Soc. Simon Stevin. 16 (2009), 623-635.
- [11] W. Rudin, Real and Complex Analysis, Third Edition, McGraw-Hill, New York, 1987.