Original Research Article

Products of Composition and Differentiation Operators

Abstract

We consider products of composition and differentiation operators on the Hardy space. We provide a complete characterization of boundedness and compactness of these operators. M. Moradi and M. Fatehi [9] obtain the explicit condition for these operators to be Hilbert-Schmidt operators. We have an application on [9].

Keywords. Composition operator, differentiation operators, boundedness, compactness.

1. Preliminaries

For $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$. The Hardy space H^2 is the Hilbert space of all analytic functions f_j on $\mathbb{D}$ such that

$$\| f_j \|^2 = \lim_{r\to 1} \frac{1}{2\pi}\int_0^{2\pi} \sum_j \left|f_j\left(re^{i\theta}\right)\right|^2 d\theta < \infty.$$

It is well known that the Hardy space H^2 is a reproducing kernel Hilbert space, with the inner product

$$\langle f_j, g_j \rangle = \frac{1}{2\pi}\int_0^{2\pi} \sum_j f_j\left(e^{i\theta}\right)\overline{g_j\left(e^{i\theta}\right)}d\theta,$$

and with kernel functions $K_{w_j}^{(n)}(z) = \sum_j \frac{n! z^n}{(1-\bar{w}_j z)^{n+1}}$, where n is a non-negative integer and $z, w_j \in \mathbb{D}$. These kernel functions satisfy $\left\langle f_j, K_{w_j}^{(n)} \right\rangle = f_j^{(n)}(w_j)$ for each $f_j \in H^2$. To simplify notation we write K_{w_j} in case $n = 0$. In particular note that $\sum_j \left\|K_{w_j}\right\|^2 = \sum_j K_{w_j}(w_j) = \sum_j \frac{1}{1-|w_j|^2}$. Let $\hat{f}_j(n)$ be the nth coefficient of f_j in its Maclaurin series. Moreover, we have another representation for the norm of f_j on H^2 as follows

$$\| f_j \|^2 = \sum_{n=0}^{\infty} \sum_j |\hat{f}_j(n)|^2 < \infty.$$

The space H^∞ is the Banach space of bounded analytic functions f_j on $\mathbb{D}$ with $\left\|f_j\right\|_\infty = \sup \sum\{|f_j(z)| : z \in \mathbb{D}\}$.

For φ_j an analytic self-map of $\mathbb{D}$, the composition operator $C_{\sum\varphi_j}$ is defined for analytic functions f_j on $\mathbb{D}$ by $\sum C_{\varphi_j}(f_j) = \sum f_j \circ \varphi_j$. It is well known that every composition operator $C_{\sum\varphi_j}$ is bounded on H^2 (see [2. Corollary 3.7]). For each positive integer k, the operator $D^{(k)}$ for any $f_j \in H^2$ is defined by the rule

$D^{(k)} \sum(f_j) = \sum f_j^{(k)}$. This operator is called the differentiation operator of order k. For convenience, we use the notation D when $k = 1$. The differentiation operators $D^{(k)}$ are unbounded on H^2, whereas Ohno [6] found a characterization for $C_{\sum \varphi_j} D$ and $DC_{\sum \varphi_j}$ to be bounded and compact on H^2. The study of operators $C_{\sum \varphi_j} D$ and $DC_{\sum \varphi_j}$ was initially addressed by Hibschweiler, Portnoy, and Ohno (see [5] and [6]) and has been noticed by many researchers ([3],[4], and [8]). [9] be considering a slightly broader class of these operators. For each positive integer n, we write $D_{\sum \varphi_j, n}$ to denote the operator on H^2 given by the rule $\sum D_{\varphi_j,n}(f_j) = \sum C_{\varphi_j} D^{(n)}(f_j) = \sum f_j^{(n)} \circ \varphi_j$. The main results provide complete characterizations of the boundedness and compactness of operators $D_{\sum \varphi_j, n}$ on H^2 (Theorems 2.1] and 2.2). In addition, we characterize the Hilbert-Schmidt operators $D_{\sum \varphi_j, n}$ on H^2 (Theorem 3.3). [9] use some ideas which are found in [6].

For φ_j be an analytic self-map of $\mathbb{D}$. The Nevanlinna counting function $N_{\sum \varphi_j}$ of φ_j is defined by

$$\sum N_{\varphi_j}(w_j) = \sum_{\varphi_j(z) = w_j} \sum \log\left(\frac{1}{|z|}\right) \qquad w_j \in \mathbb{D} \setminus \{\varphi_j(0)\}$$

and $\sum N_{\varphi_j}(\varphi_j(0)) = \infty$. Note that $\sum N_{\varphi_j}(w_j) = 0$ when w_j is not in $\varphi_j(\mathbb{D})$. For each $f_j \in H^2$, by using change of variables formula and Littlewood-Paley Identity, the norm of $\sum C_{\varphi_j} f_j$ is determined as follows:

$$\sum \| f_j \circ \varphi_j \|^2 = \sum |f_j(\varphi_j(0))|^2 + 2 \int_{\mathbb{D}} \sum \left|f_j'(w_j)\right|^2 N_{\varphi_j}(w_j) dA(w_j), \quad (1.1)$$

where dA is the normalized area measure on $\mathbb{D}$ (see [2. Theorem 2.31]). Moreover, to obtain the lower bound estimate on $\|D_{\sum \varphi_j, n}\|$ we need the following well known lemma as follows (see [2. p. 137]):

Suppose that φ_j is an analytic self-map of $\mathbb{D}$ and f is analytic in $\mathbb{D}$. Assume that Δ is any disk not containing $\{f_j^{-1}(\varphi_j(0))\}$ and centered at a. Then

$$\sum N_{\varphi_j}(f_j(a)) \leq \frac{1}{|\Delta|} \int_{\Delta} \sum N_{\varphi_j}\left(f_j(w_j)\right) dA(w_j), \quad (1.2)$$

where $|\Delta|$ is the normalized area measure of Δ.

2. Boundedness and Compactness of $D_{\sum \varphi_j, n}$

We determine which of these operators $D_{\sum \varphi_j, n}$ are bounded and compact (see [9]).

Theorem 2.1. Let φ_j be an analytic self-map of $\mathbb{D}$ and n be a positive integer. The operator $D_{\sum \varphi_j, n}$ is bounded on H^2 if and only if

$$\sum N_{\varphi_j}(w_j) = O\left([\sum \log(1/|w_j|)]^{2n+1}\right) \quad (|w_j| \to 1).$$

Proof. Suppose that $D_{\sum \varphi_j, n}$ is bounded on H^2. Let $f_j(z) = \frac{K_\lambda(z)}{\|K_\lambda\|} = \frac{\sqrt{1-|\lambda|^2}}{1-\bar{\lambda}z}$ for $\lambda \in \mathbb{D}$. By (1.1), we see that

$$\sum \left\|D_{\varphi_j,n}\right\|^2$$
$$\geq \sum \left\|D_{\varphi_j,n} f_j\right\|^2$$
$$= \sum \left\|C_{\varphi_j}\left(\frac{n!\,\bar{\lambda}^n\sqrt{1-|\lambda|^2}}{(1-\bar{\lambda}z)^{n+1}}\right)\right\|^2$$
$$= \sum \left|\frac{n!\,\bar{\lambda}^n\sqrt{1-|\lambda|^2}}{(1-\bar{\lambda}\varphi_j(0))^{n+1}}\right|^2 + 2\int_{\mathbb{D}} \sum \left|\frac{(n+1)!\,\bar{\lambda}^{n+1}\sqrt{1-|\lambda|^2}}{(1-\bar{\lambda}w_j)^{n+2}}\right|^2 N_{\varphi_j}(w_j)\,dA(w_j)$$
$$\geq \int_{\mathbb{D}} \sum \frac{2((n+1)!)^2|\lambda|^{2n+2}(1-|\lambda|^2)}{|1-\bar{\lambda}w_j|^{2n+4}} N_{\varphi_j}(w_j)\,dA(w_j). \qquad (2.1)$$

Substitute $w_j = \sum \alpha_\lambda(u_j) = \sum \frac{\lambda - u_j}{1-\bar{\lambda}u_j}$ back into (2.1) and using [7. Theorem 7.26] to obtain

$$\sum \left\|D_{\varphi_j,n}\right\|^2$$
$$\geq \int_{\mathbb{D}} \sum \frac{2((n+1)!)^2|\lambda|^{2n+2}(1-|\lambda|^2)}{\left|1-\bar{\lambda}\alpha_\lambda(u_j)\right|^{2n+4}} N_{\varphi_j}\left(\alpha_\lambda(u_j)\right)\left|\alpha_\lambda'(u_j)\right|^2 dA(u_j). \qquad (2.2)$$

Since $1 - \sum \bar{\lambda}\alpha_\lambda(u_j) = \sum \frac{1-|\lambda|^2}{1-\bar{\lambda}u_j}$ and $\sum \alpha_\lambda'(u_j) = \sum \frac{|\lambda|^2-1}{(1-\bar{\lambda}u_j)^2}$, by substituting α_λ' and $1-\bar{\lambda}\alpha_\lambda$ back into (2.2), we see that

$$\sum \left\|D_{\varphi_j,n}\right\|^2$$
$$\geq \int_{\mathbb{D}} \sum \frac{2((n+1)!)^2|\lambda|^{2n+2}|1-\bar{\lambda}u_j|^{2n}}{(1-|\lambda|^2)^{2n+1}} N_{\varphi_j}\left(\alpha_\lambda(u_j)\right) dA(u_j). \qquad (2.3)$$

Because $\sum |1-\bar{\lambda}u_j| \geq \frac{1}{2}$ for any $u_j \in \mathbb{D}/2$, we get from (2.3) that

$$\sum \left\|D_{\varphi_j,n}\right\|^2 \geq \int_{\frac{\mathbb{D}}{2}} \sum \frac{2((n+1)!)^2|\lambda|^{2n+2}}{2^{2n}(1-|\lambda|^2)^{2n+1}} N_{\varphi_j}\left(\alpha_\lambda(u_j)\right) dA(u_j). \qquad (2.4)$$

There exists $r < 1$ such that for each λ with $r < |\lambda| < 1, \sum \alpha_\lambda^{-1}(\varphi_j(0)) \notin \mathbb{D}/2$ because $\sum\left|\alpha_\lambda^{-1}(\varphi_j(0))\right| = \sum\left|\alpha_{\varphi_j(0)}(\lambda)\right|$ and $\sum \alpha_{\varphi_j(0)}$ is an automorphism of $\mathbb{D}$. By (1.2) and (2.4), we have

$$\begin{aligned}\sum \left\|D_{\varphi_j,n}\right\|^2 &\geq \frac{2((n+1)!)^2|\lambda|^{2n+2}}{2^{2n}(1-|\lambda|^2)^{2n+1}} \int_{\frac{\mathbb{D}}{2}} \sum N_{\varphi_j}\left(\alpha_\lambda(u_j)\right) dA(u_j) \\ &\geq \frac{2((n+1)!)^2|\lambda|^{2n+2}}{2^{2n}(1-|\lambda|^2)^{2n+1}} \cdot \sum \frac{N_{\varphi_j}(\alpha_\lambda(0))}{4} \\ &= \frac{((n+1)!)^2|\lambda|^{2n+2}}{2^{2n+1}(1-|\lambda|^2)^{2n+1}} \sum N_{\varphi_j}(\lambda) \qquad (2.5)\end{aligned}$$

for each λ with $r < |\lambda| < 1$. Since $D_{\sum \varphi_j,n}$ is bounded, there exists a constant number M so that

$$\lim_{|\lambda|\to 1}\sum \frac{((n+1)!)^2|\lambda|^{2n+2}}{2^{2n+1}(1-|\lambda|^2)^{2n+1}}N_{\varphi_j}(\lambda)\le M. \quad (2.6)$$

We know that $\log(1/|\lambda|)$ is comparable to $1-|\lambda|$ as $|\lambda|\to 1^-$. Note that

$$\lim_{|\lambda|\to 1}\sum \frac{((n+1)!)^2|\lambda|^{2n+2}}{2^{2n+1}(1-|\lambda|^2)^{2n+1}}N_{\varphi_j}(\lambda)$$
$$=\lim_{|\lambda|\to 1}\sum \frac{((n+1)!)^2|\lambda|^{2n+2}}{2^{2n+1}(1+|\lambda|)^{2n+1}}\left(\frac{\log(1/|\lambda|)}{1-|\lambda|}\right)^{2n+1}\frac{N_{\varphi_j}(\lambda)}{(\log(1/|\lambda|))^{2n+1}}$$
$$\ge \frac{((n+1)!)^2}{2^{6n+4}}\lim_{|\lambda|\to 1}\sum \frac{N_{\varphi_j}(\lambda)}{(\log(1/|\lambda|))^{2n+1}}. \quad (2.7)$$

By (2.6) and (2.7), we can see that

$$N_{\sum\varphi_j}(\lambda)=O([\log(1/|\lambda|)]^{2n+1})\ (|\lambda|\to 1).$$

Conversely, suppose that for some R with $0<R<1$, there exists a constant M satisfying

$$\sup_{R<|w_j|<1}\sum N_{\varphi_j}(w_j)/[\log(1/|w_j|)]^{2n+1}\le M.$$

Let f_j be an arbitrary function in H^2. It follows from (1.1) that

$$\sum \left\|D_{\varphi_j,n}f_j\right\|^2$$
$$=\sum\left|f_j^{(n)}(\varphi_j(0))\right|^2+2\int_{\mathbb{D}}\sum\left|f_j^{(n+1)}(w)_j\right|^2N_{\varphi_j}(w_j)dA(w_j)$$
$$=\left|f_j^{(n)}(\varphi_j(0))\right|^2$$
$$+2\sum\left(\int_{R\mathbb{D}}\left|f_j^{(n+1)}(w_j)\right|^2N_{\varphi_j}(w_j)dA(w_j)\,.\right.$$
$$\left.+\int_{\mathbb{D}\setminus R\mathbb{D}}\left|f_j^{(n+1)}(w_j)\right|^2N_{\varphi_j}(w_j)dA(w_j)\right). \quad (2.8)$$

First we estimate the first and the second terms in the right-hand of (2.8). Observe that

$$\sum f_j^{(n)}(z)=\sum\left\langle f_j,K_z^{(n)}\right\rangle=\int_0^{2\pi}\sum\frac{n!\,e^{-in\theta}f_j(e^{i\theta})}{(1-e^{-i\theta}z)^{n+1}}\frac{d\theta}{2\pi}$$

and hence

$$\sum\left|f_j^{(n)}(z)\right|\le\frac{n!}{(1-|z|)^{n+1}}\int_0^{2\pi}\sum\left|f_j(e^{i\theta})\right|\frac{d\theta}{2\pi}\le\frac{n!}{(1-|z|)^{n+1}}\sum\parallel f_j\parallel \quad (2.9)$$

for any $z\in\mathbb{D}$. It follows from (2.9) that

$$\sum\left|f_j^{(n)}(\varphi_j(0))\right|\le\sum\frac{n!\parallel f_j\parallel}{(1-|\varphi_j(0)|)^{n+1}}. \quad (2.10)$$

Moreover, we can see that

$$\sum\left|f_j^{(n+1)}(z)\right|=\sum\left|\left\langle f_j,K_z^{(n+1)}\right\rangle\right|=\frac{(n+1)!}{(1-|z|)^{n+2}}\sum\parallel f_j\parallel \quad (2.11)$$

for any $z\in\mathbb{D}$. Therefore by (2.11), we see that

$$\int_{R\mathbb{D}} \sum \left|f_j^{(n+1)}(w_j)\right|^2 N_{\varphi_j}(w_j)dA(w_j)$$
$$\leq \left(\frac{(n+1)!}{(1-R)^{n+2}}\right)^2 \sum \| f_j \|^2 \int_{R\mathbb{D}} N_{\varphi_j}(w_j)dA(w_j).$$

Since $\sum \| \varphi_j \|^2 = |\sum \varphi_j(0)|^2 + 2\int_{\mathbb{D}} N_{\varphi_j}(w_j)dA(w_j)$ by (1.1), we obtain

$$\int_{\mathbb{D}} \sum N_{\varphi_j}(w_j)dA(w_j) = \frac{1}{2}\sum\left(\| \varphi_j \|^2 - |\varphi_j(0)|^2\right) < 1. \qquad (2.12)$$

From (2.11) and (2.12), we see that

$$\int_{R\mathbb{D}} \sum \left|f_j^{(n+1)}(w_j)\right|^2 N_{\varphi_j}(w_j)dA(w_j) \leq \left(\frac{(n+1)!}{(1-R)^{n+2}}\right)^2 \sum \| f_j \|^2. \qquad (2.13)$$

Now we estimate the third term in the right-hand of (2.8). We have

$$\int_{\mathbb{D}\setminus R\mathbb{D}} \sum \left|f_j^{(n+1)}(w_j)\right|^2 N_{\varphi_j}(w_j)dA(w_j)$$
$$= \int_{\mathbb{D}\setminus R\mathbb{D}} \sum \left|f_j^{(n+1)}(w_j)\right|^2 (\log(1/|w_j|))^{2n+1} \frac{N_{\varphi_j}(w_j)}{(\log(1/|w_j|))^{2n+1}} dA(w_j)$$
$$\leq \sup_{R<|w_j|<1} \sum \frac{N_{\varphi_j}(w_j)}{(\log(1/|w_j|))^{2n+1}} \int_{\mathbb{D}\setminus R\mathbb{D}} \left|f_j^{(n+1)}(w_j)\right|^2 (\log(1/|w_j|))^{2n+1} dA(w_j)$$
$$\leq M \int_{\mathbb{D}\setminus R\mathbb{D}} \sum \left|f_j^{(n+1)}(w_j)\right|^2 (\log(1/|w_j|))^{2n+1} dA(w_j). \qquad (2.14)$$

Let $f_j(z) = \sum_{m=0}^{\infty} \sum a_m^j z^m$. We get

$$\int_{\mathbb{D}\setminus R\mathbb{D}} \sum \left|f_j^{(n+1)}(w_j)\right|^2 (\log(1/|w_j|))^{2n+1} dA(w_j)$$
$$\leq \int_{\mathbb{D}\setminus R\mathbb{D}} \sum \left|\sum_{m=n+1}^{\infty} m(m-1)\ldots(m-n)a_m^j (w_j)^{m-(n+1)}\right|^2 (\log(1/|w_j|))^{2n+1} dA(w_j)$$
$$\leq \sum_{m=n+1}^{\infty} \sum m^2(m-1)^2 \ldots (m-n)^2 \left|a_m^j\right|^2 \int_{\mathbb{D}\setminus R\mathbb{D}} \left|(w_j)^{m-(n+1)}\right|^2 (\log(1/|w_j|))^{2n+1} dA(w_j)$$
$$\leq \sum_{m=n+1}^{\infty} \sum m^2(m-1)^2 \ldots (m-n)^2 \left|a_m^j\right|^2 \int_{\mathbb{D}} \left|(w_j)^{m-(n+1)}\right|^2 (\log(1/|w_j|))^{2n+1} dA(w_j)$$
$$= \sum_{m=n+1}^{\infty} \sum m^2(m-1)^2 \ldots (m-n)^2 \left|a_m^j\right|^2 \int_0^1 \int_0^{2\pi} \left|r_j e^{i\theta}\right|^{2(m-(n+1))} (\log(1/r_j))^{2n+1} r_j dr_j \frac{d\theta}{\pi}$$
$$\leq \sum_{m=n+1}^{\infty} \sum m^2(m-1)^2 \ldots (m-n)^2 \left|a_m^j\right|^2$$
$$\int_0^1 (r_j)^{2(m-(n+1))} (\log(1/r_j))^{2n+1} 2r_j dr_j. \qquad (2.15)$$

Now substitute $t_j = r_j^2$ and $u_j = \log(1/t_j)$ to obtain

$$\int_0^1 \sum (r_j)^{2(m-(n+1))}(\log(1/r_j))^{2n+1}\, 2rdr_j$$

$$= \int_0^1 \sum t_j^{(m-(n+1))}\left(\frac{1}{2}\log\,(1/t_j)\right)^{2n+1} dt_j$$

$$= (1/2)^{2n+1}\int_0^\infty \sum e^{-u_j(m-n)}u_j^{2n+1}du_j. \qquad (2.16)$$

By substituting $x_j = (m-n)u_j$ back into (2.16), we have

$$(1/2)^{2n+1}\int_0^\infty \sum e^{-u_j(m-n)}u_j^{2n+1}du_j$$

$$= \frac{1}{2^{2n+1}(m-n)^{2n+2}}\int_0^\infty \sum e^{-x}x_j^{2n+1}dx_j$$

$$= \frac{\Gamma(2n+2)}{2^{2n+1}(m-n)^{2n+2}}. \qquad (2.17)$$

By (2.14), (2.15), (2.16) and (2.17), we can see that

$$\int_{\mathbb{D}\setminus R\mathbb{D}} \sum \left|f_j^{(n+1)}(w_j)\right|^2 N_{\varphi_j}(w_j)dA(w_j)$$

$$\leq M\sum_{m=n+1}^{\infty}\sum m^2(m-1)^2 \ldots (m-n)^2\left|a_m^j\right|^2 \frac{\Gamma(2n+2)}{2^{2n+1}(m-n)^{2n+2}}$$

$$= M\frac{(2n+1)!}{2^{2n+1}}\sum_{m=n+1}^{\infty}\sum \frac{m^2(m-1)^2 \ldots (m-n+1)^2}{(m-n)^{2n}}\left|a_m^j\right|^2$$

$$\leq M\lambda\frac{(2n+1)!}{2^{2n+1}}\sum_{m=n+1}^{\infty}\sum\left|a_m^j\right|^2$$

$$(2.18)\ \leq M\lambda\frac{(2n+1)!}{2^{2n+1}}\sum \| f_j \|^2, \qquad (2.18)$$

where λ is a constant so that $\frac{m^2(m-1)^2\ldots(m-n+1)^2}{(m-n)^{2n}} \leq \lambda$ for each $m \geq n+1$ (note that the function $f_j(x_j) = \frac{x_j^2(x-1)^2\ldots(x_j-n+1)^2}{(x-n)^{2n}}$ is bounded on $[n+1, +\infty)$). Then (2.8), (2.10), (2.13) and (2.18) show that $D_{\sum\varphi_j,n}$ is bounded. We have the following (see [9])

Theorem 2.2. Let φ_j be an analytic self-map of $\mathbb{D}$ and n be a positive integer. The operator $D_{\sum\varphi_j,n}$ is compact on H^2 if and only if

$$\sum N_{\varphi_j}(w_j) = o\sum\left([\log\,(1/|w_j|)]^{2n+1}\right) \qquad (|w_j| \to 1). \qquad (2.19)$$

Proof. Let $h_m(z) = \frac{\sqrt{1-|\lambda_m|^2}}{1-\bar{\lambda}_m z}$ for a sequence $\{\lambda_m\}$ in $\mathbb{D}$ so that $|\lambda_m| \to 1$ as $m \to \infty$. Then $h_m \to 0$ weakly as $m \to \infty$ by [2 Theorem 2.17]. First suppose that $D_{\sum\varphi_j,n}$ is compact. Hence $\sum\left\|D_{\varphi_j,n}h_m\right\| \to 0$ as $m \to \infty$. Therefore (2.5) shows that

$$\lim_{m\to\infty}\sum\frac{((n+1)!)^2|\lambda_m|^{2n+2}}{2^{2n+1}(1-|\lambda_m|^2)^{2n+1}}N_{\varphi_j}(\lambda_m)=0.$$

Since $\log(1/|\lambda_m|)$ is comparable to $1-|\lambda_m|$ as $m\to\infty$, the result follows.

Conversely, suppose that (2.19) holds. Let $\epsilon>0$. Then there exists $R, 0<R<1$, such that

$$\sup_{R<|w_j|<1}\sum N_{\varphi_j}(w_j)/[\log(1/|w_j|)]^{2n+1}<\epsilon. \qquad (2.20)$$

Let $\left\{(f_j)_m\right\}$ be any bounded sequence in H^2. By using the idea which was stated in the proof of [2. Proposition 3.11], we can see that $\left\{(f_j)_m\right\}$ is a normal family and there exists a subsequence $\left\{(f_j)_{m_k}\right\}$ which converges to some function $f_j\in H^2$ uniformly on all compact subsets of $\mathbb{D}$. Let $(g_j)_{m_k}=(f_j)_{m_k}-f_j$ for each positive integer k. Note that $\left\{(g_j)_{m_k}\right\}$ is a bounded sequence in H^2 which converges to 0 uniformly on all compact subsets of $\mathbb{D}$. By (2.8), we obtain

$$\sum\left\|D_{\varphi_j,n}(g_j)_{m_k}\right\|^2$$
$$=\sum\left|(g_j)_{m_k}^{(n)}(\varphi_j(0))\right|^2+2\int_{R\mathbb{D}}\sum\left|(g_j)_{m_k}^{(n+1)}(w_j)\right|^2N_{\varphi_j}(w_j)dA(w_j)$$
$$+2\int_{\mathbb{D}\setminus R\mathbb{D}}\sum\left|(g_j)_{m_k}^{(n+1)}(w_j)\right|^2N_{\varphi_j}(w_j)dA(w_j). \qquad (2.21)$$

By [1. Theorem 2.1, p. 151], we can choose k_ϵ so that

$$\sum\left|(g_j)_{m_k}^{(n)}(\varphi_j(0))\right|<\sqrt{\epsilon} \qquad (2.22)$$

and $\sum\left|(g_j)_{m_k}^{(n+1)}\right|<\sqrt{\epsilon}$ on $R\mathbb{D}$ whenever $k>k_\epsilon$. Substituting $f_j(z)=z$ into (1.1), we see that

$$\int_{R\mathbb{D}}\sum\left|(g_j)_{m_k}^{(n+1)}(w_j)\right|^2N_{\varphi_j}(w_j)dA(w_j)\quad\le\epsilon\int_{R\mathbb{D}}\sum N_{\varphi_j}(w_j)dA(w_j)$$
$$\le\frac{\epsilon}{2}\sum\left(\|\varphi_j\|^2-|\varphi_j(0)|^2\right) \qquad (2.23)$$

for $k>k_\epsilon$. On the other hand by (2.20) and the same idea as stated in the proof of (2.14) and (2.18), we see that

$$\int_{\mathbb{D}\setminus R\mathbb{D}}\sum\left|(g_j)(w)\right|^2N_{\varphi_j}(w_j)dA(w_j)$$
$$\le\sup_{R<|w_j|<1}\sum\frac{N_{\varphi_j}(w_j)}{[\log(1/|w_j|)]^{2n+1}}$$
$$\int_{\mathbb{D}\setminus R\mathbb{D}}\left|(g_j)_{m_k}^{(n+1)}(w_j)\right|^2[\log(1/|w_j|)]^{2n+1}dA(w_j)$$
$$\le C\epsilon\sum\left\|(g_j)_{m_k}\right\|, \qquad (2.24)$$

where C is a constant. Hence we conclude that $\sum \left\| D_{\varphi_j,n}(g_j)_{m_k} \right\|$ converges to zero as $k \to \infty$ by (2.21), (2.22) , (2.23) and (2.24) and so $D_{\varphi,n}$ is compact.

The preceding theorems lead to characterizations of all bounded and compact operators $D_{\varphi_j,n}$ when φ_j is a univalent self-map (see [9]).

Corollary 2.3. Let φ_j be a univalent self-map of $\mathbb{D}$ and n be a positive integer. Then the following hold.

(i) $D_{\sum \varphi_j,n}$ is bounded on H^2 if and only if

$$\sup_{w_j \in \mathbb{D}} \sum \frac{1 - |w_j|}{(1 - |\varphi_j(w_j)|)^{2n+1}} < \infty$$

(ii) $D_{\sum \varphi_j,n}$ is compact on H^2 if and only if

$$\lim_{|w_j| \to 1} \sum \sum \frac{1 - |w_j|}{(1 - |\varphi_j(w_j)|)^{2n+1}} = 0$$

Proof. Since $\sum \varphi_j$ is univalent, we can see that $N_{\varphi_j}(w_j) = \log(1/|z|)$, where $\varphi_j(z) = w_j$. We observe that

$$\sum \frac{N_{\varphi_j}(w_j)}{[\log(1/|w_j|)]^{2n+1}} = \sum \frac{-\log(|z|)}{(-\log(|\varphi_j(z)|))^{2n+1}}.$$

Moreover, we know that $\log(1/|z|)$ is comparable to $1 - |z|$ as $|z| \to 1^-$. Furthermore $|z| \to 1$ as $|\varphi(z)| \to 1$. Therefore the results follow immediately from Theorems 2.1 and 2.2

3. Hilbert-Schmidt Operator $D_{\sum \varphi_j,n}$

We begin with a few easy observations that help us in the proof of Theorem 3.3 In the proof of the following lemma, we assume that $0^0 = 1$ (see [9]).

Lemma 3.1. Let n be a positive integer and $\alpha_k > 0$ for each $0 \le k \le n$. Then for $0 \le x < 1$, the following statements hold.

(a) $\sum_{k=0}^{n} \sum \frac{\alpha_k^j x^k}{(1-x)^{n+k+1}} \le \frac{\sum_{k=0}^{n} \sum \alpha_k^j}{(1-x)^{2n+1}}$.

(b) There exists a positive number β^j such that $\sum_{k=0}^{n} \sum \frac{\alpha_k^j x^k}{(1-x)^{n+k+1}} \ge \sum \frac{\beta^j}{(1-x)^{2n+1}}$.

Proof. (a) We can see that

$$\sum_{k=0}^{n} \sum \frac{\alpha_k^j x^k}{(1-x)^{n+k+1}} = \frac{\sum_{k=0}^{n} \sum \alpha_k^j x^k (1-x)^{n-k}}{(1-x)^{2n+1}}$$

Since $0 \le x < 1$ and $\alpha_k^j > 0$, we conclude that $\sum_{k=0}^{n} \alpha_k^j x^k (1-x)^{n-k} \le \sum_{k=0}^{n} \alpha_k^j$. Hence the conclusion follows.

(b) We have

$$(1-x)^{2n+1} \sum_{k=0}^{n} \sum \frac{\alpha_k^j x^k}{(1-x)^{n+k+1}} = \sum_{k=0}^{n} \sum \alpha_k^j x^k (1-x)^{n-k} > 0.$$

Since $\sum_{k=0}^{n} \sum \alpha_k^j x^k (1-x)^{n-k}$ is a continuous function on $[0,1]$, there exists a positive number β^j such that $\sum_{k=0}^{n} \sum \alpha_k^j x^k (1-x)^{n-k} \ge \beta^j$. Hence the result follows. We have the following (see [9])

Lemma 3.2. Let n be a positive integer. Then

$$\sum_{m=n}^{\infty} [m(m-1)\dots(m-n+1)]^2 x^{m-n} = (n!)^2 \sum_{k=0}^{n} \frac{(n+k)!}{(k!)^2(n-k)!}\frac{x^k}{(1-x)^{n+k+1}}$$

for $0 \leq x < 1$.

Proof. See [8, Lemma 1] and the general Leibniz rule.

A Hilbert-Schmidt operator on a separable Hilbert space H is a bounded operator A with finite Hilbert-Schmidt norm $\| A \|_{HS} = \left(\sum_{n=1}^{\infty} \|Ae_n\|^2\right)^{1/2}$, where $\{e_n\}$ is an orthonormal basis of H. These definitions are independent of the choice of the basis (see [2. Theorem 3.23]). Now we have the following (see [9])

Theorem 3.3. Let $D_{\sum \varphi_j, n}$ be a bounded operator on H^2. Then $D_{\sum \varphi_j, n}$ is a HilbertSchmidt operator on H^2 if and only if

$$\lim_{r\to 1} \frac{1}{2\pi}\int_0^{2\pi} \sum \frac{1}{\left(1-\left|\varphi_j(re^{i\theta})\right|^2\right)^{2n+1}} < \infty. \qquad (3.1)$$

Proof. Suppose that (3.1) holds. Lemmas 3.1- 3.2- and [7. Theorem 1.27] imply that

$$\sum_{m=0}^{\infty} \sum \left\| D_{\varphi_j, n} z^m \right\|$$

$$= \sum_{m=n}^{\infty} \sum \| m(m-1)\dots(m-n+1)\varphi_j^{m-n} \|$$

$$= \sum_{m=n}^{\infty} \lim_{r\to 1} \frac{1}{2\pi}\int_0^{2\pi} \sum \left| m(m-1)\dots(m-n+1)\varphi_j^{m-n}(r_j e^{i\theta})\right|^2 d\theta$$

$$= \lim_{r\to 1} \sum_{m=n}^{\infty} \frac{1}{2\pi}\int_0^{2\pi} \sum \left| m(m-1)\dots(m-n+1)\varphi_j^{m-n}(r_j e^{i\theta})\right|^2 d\theta$$

$$= \lim_{r\to 1} \frac{1}{2\pi}\int_0^{2\pi} \sum_{m=n}^{\infty} \left| m(m-1)\dots(m-n+1)\varphi_j^{m-n}(r_j e^{i\theta})\right|^2 d\theta$$

$$= \lim_{r\to 1} \frac{1}{2\pi}\int_0^{2\pi} \sum_{k=0}^{n} \sum \frac{(n!)^2(n+k)!}{(k!)^2(n-k)!} \frac{\left|\varphi_j(re^{i\theta})\right|^{2k}}{\left(1-\left|\varphi_j(re^{i\theta})\right|^2\right)^{n+k+1}}$$

$$\leq \lim_{r\to 1} \frac{1}{2\pi}\int_0^{2\pi} \sum \frac{\alpha^j}{\left(1-\left|\varphi_j(re^{i\theta})\right|^2\right)^{2n+1}}, \qquad (3.2)$$

where $\alpha^j = \sum_{k=0}^{n} \frac{(n!)^2(n+k)!}{(k!)^2(n-k)!}$ (note that the interchange of limit and summation is justified by [2 Corollary 2.23] and using Lebesgue's Monotone Convergence Theorem with counting measure). It follows that $\sum_{m=0}^{\infty} \sum \left\| D_{\varphi_j, n} z^m \right\| < \infty$ and so $D_{\sum \varphi_j, n}$ is a Hilbert-Schmidt operator on H^2 by [2 Theorem 3.23].

Conversely, suppose that $D_{\sum \varphi_j,n}$ is a Hilbert-Schmidt operator on H^2. We infer from [2, Theorem 3.23] that

$$\sum_{m=0}^{\infty} \sum \left\| D_{\varphi_j,n} z^m \right\|^2 < \infty. \tag{3.3}$$

On the other hand, by the proof of (3.2) and Lemma 3.1 there exists a positive number β_j such that

$$\sum_{m=0}^{\infty} \sum \left\| D_{\varphi_j,n} z^m \right\|^2$$

$$= \lim_{r \to 1} \frac{1}{2\pi} \int_0^{2\pi} \sum_{k=0}^{n} \sum \frac{(n!)^2 (n+k)!}{(k!)^2 (n-k)!} \frac{\left|\varphi_j\left(re^{i\theta}\right)\right|^{2k}}{\left(1 - \left|\varphi_j\left(re^{i\theta}\right)\right|^2\right)^{n+k+1}}$$

$$\geq \lim_{r \to 1} \frac{1}{2\pi} \int_0^{2\pi} \sum \frac{\beta_j}{\left(1 - \left|\varphi_j\left(re^{i\theta}\right)\right|^2\right)^{2n+1}}. \tag{3.4}$$

Hence the result follows from (3.3) and (3.4).

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