

## The Orlicz Inequality for Series of Multilinear Forms

### Abstract

The Orlicz  $(\ell_2, \ell_1)$ -mixed inequality who states that

$$\left( \sum_{j_1=1}^n \left( \sum_{j_2=1}^n \sum_L |A_L(e_{j_1}, e_{j_2})| \right)^2 \right)^{\frac{1}{2}} \leq \sqrt{2} \sum_L \|A_L\|$$

for all sequences of bilinear forms  $A_L: \mathbb{K}^n \times \mathbb{K}^n \rightarrow \mathbb{K}$  and all positive integers  $n$ , where  $\mathbb{K}^n$  denotes  $\mathbb{R}^n$  or  $\mathbb{C}^n$  endowed with the supremum norm. We follow D. Núñez-Alarcón, D. Pellegrino, and D. Serrano-Rodríguez [19] to extend this inequality to series of multilinear forms, with  $\mathbb{K}^n$  endowed with  $\ell_{1+\epsilon}$  norms for all  $0 \leq \epsilon \leq \infty$ .

**Keywords.** Orlicz inequality, multilinear forms, Hölder inequality, Hardy-Littlewood inequalities, Maurey-Pisier factorization.

### 1. Introduction

The origins of the theory of summability of multilinear forms and absolutely summing multilinear operators are probably associated to Orlicz  $(\ell_2, \ell_1)$ -mixed inequality published in the 1930's ( see [8, page 24]). It states that

$$\left( \sum_{j_1=1}^n \left( \sum_{j_2=1}^n \sum_L |A_L(e_{j_1}, e_{j_2})| \right)^2 \right)^{\frac{1}{2}} \leq \sqrt{2} \sum_L \|A_L\|$$

for all bilinear forms  $A_L: \mathbb{K}^n \times \mathbb{K}^n \rightarrow \mathbb{K}$ , and all positive integers  $n$ . Here  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$  and  $\mathbb{K}^n$  is endowed with the supremum norm. We also represent by  $e_k$  the canonical vectors in a sequence space and

$$\|A_L\| := \sup\{|A_L(x, y)|: \|x\| \leq 1 \text{ and } \|y\| \leq 1\}.$$

An equivalent formulation is the following:

$$\left( \sum_{j_1=1}^{\infty} \left( \sum_{j_2=1}^{\infty} \sum_L |A_L(e_{j_1}, e_{j_2})| \right)^2 \right)^{\frac{1}{2}} \leq \sqrt{2} \sum_L \|A_L\| \quad (1)$$

for all continuous sequences of bilinear forms  $A_L: c_0 \times c_0 \rightarrow \mathbb{K}$ . The exponents in (1) are optimal in the sense that, fixing the exponent 1, the exponent 2 cannot be replaced by smaller exponents (nor the exponent 1 can be replaced by smaller exponents) keeping the constant independent of  $n$ . The Orlicz inequality is closely related to Littlewood's  $(\ell_1, \ell_2)$ -mixed inequality (see [8, page 23]), which asserts that

$$\sum_{j_1=1}^{\infty} \left( \sum_{j_2=1}^{\infty} \sum_L |A_L(e_{j_1}, e_{j_2})|^2 \right)^{\frac{1}{2}} \leq \sqrt{2} \sum_L \|A_L\|$$

for all continuous sequences of bilinear forms  $A_L: c_0 \times c_0 \rightarrow \mathbb{K}$ . Again, the exponents are optimal in the same sense above described. Combining these two inequalities, and using the Hölder inequality for mixed sums we recover Littlewood's 4/3 inequality:

$$\left( \sum_{j_1, j_2=1}^{\infty} \sum_L |A_L(e_{j_1}, e_{j_2})|^{\frac{4}{3}} \right)^{\frac{3}{4}} \leq \sqrt{2} \sum_L \|A_L\|$$

for all continuous bilinear forms  $A_L: c_0 \times c_0 \rightarrow \mathbb{K}$ . For recent results on absolutely summing linear and multilinear operators see [6, 11, 17].

The exponent 4/3 from the previous inequality cannot be replaced by smaller exponents keeping the constant independent of  $n$ . The constant  $\sqrt{2}$  is optimal (in all the three inequalities) when  $\mathbb{K} = \mathbb{R}$ , but the optimal constants when  $\mathbb{K} = \mathbb{C}$  are unknown.

In 1934 Hardy and Littlewood [10] (see also [13]) pushed the subject further, extending the above results to bilinear forms defined on  $\ell_{1+\epsilon}$  spaces (when  $\epsilon = \infty$  we consider  $c_0$  instead of  $\ell_{\infty}$ ). The investigation of extensions of the Hardy-Littlewood inequalities to multilinear forms were initiated by Praciano-Pereira [16] in 1981 and intensively investigated since then (see, for instance, [1, 2, 5, 7, 12, 13, 14, 15]), but there are still several open problems regarding the optimal exponents and optimal constants involved.

We shall use the same notation from [1]:

$$X_{1+\epsilon} := \begin{cases} \ell_{1+\epsilon}, & \text{if } 0 \leq \epsilon < \infty \\ c_0, & \text{if } \epsilon = \infty \end{cases}$$

and, when  $\epsilon = \infty$ , the sum  $(\sum_j \|x_j\|^{1+\epsilon/\epsilon})^{1/(1+\epsilon)}$  shall represent the supremum of  $\|x_j\|$ . We also denote the conjugate index of  $(1+\epsilon)$  by  $(1+\epsilon)^*$ , i.e.,  $1/(1+\epsilon) + 1/(1+\epsilon)^* = 1$ . We find the optimal values of the exponents  $(1+\epsilon)_1, \dots, (1+\epsilon)_m$  and of the constants  $(1+\epsilon)_{(1+\epsilon)_1, \dots, (1+\epsilon)_m}^{(\mathbb{K})}$  satisfying

$$\left( \sum_{j_1=1}^{\infty} \dots \left( \sum_{j_{m-1}=1}^{\infty} \left( \sum_{j_m=1}^{\infty} \sum_L |A_L(e_{j_1}, \dots, e_{j_m})|^{(1+\epsilon)_m} \right)^{\frac{(1+\epsilon)_{m-1}}{(1+\epsilon)_m}} \right)^{\frac{(1+\epsilon)_{m-2}}{(1+\epsilon)_{m-1}}} \right)^{\frac{1}{(1+\epsilon)_1}} \leq (1+\epsilon)_{(1+\epsilon)_1, \dots, (1+\epsilon)_m}^{(\mathbb{K})} \sum_L \|A_L\|$$

for all continuous  $m$ -linear forms  $A_L: X_{(1+\epsilon)_1} \times \dots \times X_{(1+\epsilon)_m} \rightarrow \mathbb{K}$ . The answer is known in several cases (see [1, 5, 18] and the references therein), but a complete solution is still unknown. By [19] we shall be interested in investigating the optimal exponents  $(1+\epsilon)_1, \dots, (1+\epsilon)_m$ . It is simple to prove that the optimal exponent  $(1+\epsilon)_m$  associated to the sum  $\sum_{j_m=1}^{\infty}$  is  $(1+\epsilon)_m^*$ .

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The main result provides the optimal exponents  $(1 + \epsilon)_1, \dots, (1 + \epsilon)_{m-1}$  in the case that  $(1 + \epsilon)_m = (1 + \epsilon)_m^*$ .

From now on, let  $\epsilon \geq 0$ , and let  $(1 + \epsilon)_1, \dots, (1 + \epsilon)_m \in [1, \infty]$ . We define  $\delta^{(1+\epsilon)_{k, \dots, (1+\epsilon)_m}}$  and  $\lambda_{2+\epsilon}^{(1+\epsilon)_{k, \dots, (1+\epsilon)_m}}$  by

$$\delta^{(1+\epsilon)_{k, \dots, (1+\epsilon)_m}} := \frac{1}{\max\left\{1 - \left(\frac{1}{(1+\epsilon)_k} + \dots + \frac{1}{(1+\epsilon)_m}\right), 0\right\}},$$

and

$$\lambda_{2+\epsilon}^{(1+\epsilon)_{k, \dots, (1+\epsilon)_m}} := \frac{1}{\max\left\{\frac{1}{2+\epsilon} - \left(\frac{1}{(1+\epsilon)_k} + \dots + \frac{1}{(1+\epsilon)_m}\right), 0\right\}},$$

for all positive integers  $m$  and  $k = 1, \dots, m$ . Note that when  $1/(1 + \epsilon)_k + \dots + 1/(1 + \epsilon)_m \geq 1$  we have

$$\delta^{(1+\epsilon)_{k, \dots, (1+\epsilon)_m}} = \infty$$

and, also, when  $1/(1 + \epsilon)_k + \dots + 1/(1 + \epsilon)_m \geq \frac{1}{2+\epsilon}$  we have

$$\lambda_{2+\epsilon}^{(1+\epsilon)_{k, \dots, (1+\epsilon)_m}} = \infty.$$

The main result is, a generalization of the Orlicz inequality. We consider the very particular case  $(m, (1 + \epsilon)_1, (1 + \epsilon)_2) = (2, \infty, \infty)$  and  $\sigma$  as the identity map in its statement, we recover the Orlicz inequality (see [19]):

**Theorem 1.1.** Let  $\epsilon \geq 0$  be an integer and  $\sigma: \{1, \dots, m\} \rightarrow \{1, \dots, m\}$  be a bijection. If

$$\left(\left(\frac{1 + \epsilon}{\epsilon}\right)_1, \dots, \left(\frac{1 + \epsilon}{\epsilon}\right)_{m-1}\right) \in (0, \infty]^{m-1}$$

$$\left((1 + \epsilon)_1, \dots, (1 + \epsilon)_m\right) \in [1, \infty]^m$$

the following assertions are equivalent: (1) There is a constant  $(1 + \epsilon)_{(1+\epsilon)_1, \dots, (1+\epsilon)_m} \geq 1$  such that

$$\left(\sum_{j_{\sigma(1)}=1}^{\infty} \left(\sum_{j_{\sigma(2)}=1}^{\infty} \dots \left(\sum_{j_{\sigma(m)}=1}^{\infty} \sum_L |A_L(e_{j_{\sigma(1)}}, \dots, e_{j_{\sigma(m)}})| \right)^{\frac{(1+\epsilon)_{\sigma(m)}}{(1+\epsilon)_{\sigma(m)}^*}} \dots \right)^{\frac{(1+\epsilon/\epsilon)_1}{(1+\epsilon/\epsilon)_2}} \right)^{\frac{1}{(1+\epsilon/\epsilon)_1}}$$

$$\leq (1 + \epsilon)_{(1+\epsilon)_1, \dots, (1+\epsilon)_m} \sum_L \|A_L\|$$

for all continuous  $m$ -linear forms  $A_L: X_{(1+\epsilon)_1} \times \dots \times X_{(1+\epsilon)_m} \rightarrow \mathbb{K}$ .

(2) The exponents  $(1 + \epsilon/\epsilon)_1, \dots, (1 + \epsilon/\epsilon)_{m-1}$  satisfy  $(1 + \epsilon/\epsilon)_1 \geq \delta^{(1+\epsilon)_{\sigma(1), \dots, (1+\epsilon)_{\sigma(m-1)}, \mu}}$ ,  $(1 + \epsilon/\epsilon)_2 \geq \delta^{(1+\epsilon)_{\sigma(2), \dots, (1+\epsilon)_{\sigma(m-1)}, \mu}}, \dots, (1 + \epsilon/\epsilon)_{m-1} \geq \delta^{(1+\epsilon)_{\sigma(m-1), \mu}}$ ,

where  $\mu = \min\{(1 + \epsilon)_{\sigma(m)}, 2\}$ .

## 2. Preliminary Results

Let  $0 < \epsilon < \infty$ . Recall that a Banach space  $X$  has cotype  $(2 + \epsilon)$  if there is a constant  $\epsilon > 0$  such that, we select finitely many vectors  $x_1, \dots, x_n \in X$ ,

$$\left( \sum_{j=1}^n \|x_j\|^{(2+\epsilon)} \right)^{\frac{1}{(2+\epsilon)}} \leq (1+\epsilon) \left( \int_{[0,1]} \left\| \sum_{j=1}^n (2+\epsilon)_j(t)x_j \right\|^2 dt \right)^{\frac{1}{2}} \quad (2)$$

where  $(2+\epsilon)_j$  denotes the  $j$ -th Rademacher function. The infimum of the cotypes of  $X$  is denoted by  $\text{cot } X$ .

The following result was proved in [5] (see [19]):

**Theorem 2.1.** (see[5]) Let  $((2+\epsilon)_1, \dots, (2+\epsilon)_m) \in (0, \infty)^m$ , and  $Y$  be an infinite-dimensional Banach space with cotype  $\text{cot } Y$ . If

$$\frac{1}{(1+\epsilon)_1} + \dots + \frac{1}{(1+\epsilon)_m} < \frac{1}{\text{cot } Y}, \quad (3)$$

then the following assertions are equivalent:

(a) There is a constant  $(1+\epsilon)_{(1+\epsilon)_1, \dots, (1+\epsilon)_m}^Y \geq 1$  such that

$$\left( \sum_{j_1=1}^{\infty} \left( \sum_{j_2=1}^{\infty} \dots \left( \sum_{j_m=1}^{\infty} \sum_L \|A_L(e_{j_1}, \dots, e_{j_m})\|^{(2+\epsilon)_m} \right)^{\frac{(2+\epsilon)_{m-1}}{(2+\epsilon)_m}} \dots \right)^{\frac{(2+\epsilon)_1}{(2+\epsilon)_2}} \right)^{\frac{1}{(2+\epsilon)_1}} \leq (1+\epsilon)_{(1+\epsilon)_1, \dots, (1+\epsilon)_m}^Y \sum_L \|A_L\|$$

for all continuous  $m$ -linear operators  $A_L: X_{(1+\epsilon)_1} \times \dots \times X_{(1+\epsilon)_m} \rightarrow Y$ .

(b) The exponents  $(2+\epsilon)_1, \dots, (2+\epsilon)_m$  satisfy

$$\begin{aligned} (2+\epsilon)_1 &\geq \lambda_{\text{cot } Y}^{(1+\epsilon)_1, \dots, (1+\epsilon)_m}, (2+\epsilon)_2 \geq \lambda_{\text{cot } Y}^{(1+\epsilon)_2, \dots, (1+\epsilon)_m}, \dots, (2+\epsilon)_{m-1} \\ &\geq \lambda_{\text{cot } Y}^{(1+\epsilon)_{m-1}, (1+\epsilon)_m}, (2+\epsilon)_m \geq \lambda_{\text{cot } Y}^{(1+\epsilon)_m}. \end{aligned}$$

We need the following extension of the previous theorem, relaxing the hypothesis (3). Besides, below we have  $((2+\epsilon)_1, \dots, (2+\epsilon)_m) \in (0, \infty]^m$  while in Theorem 2.1 we have  $((2+\epsilon)_1, \dots, (2+\epsilon)_m) \in (0, \infty)^m$  (see [19]).

**Theorem 2.2.** Let  $((2+\epsilon)_1, \dots, (2+\epsilon)_m) \in (0, \infty]^m$ ,  $((1+\epsilon)_1, \dots, (1+\epsilon)_m) \in [1, \infty]^m$  and  $Y$  be an infinite-dimensional Banach space with cotype  $\text{cot } Y$ . The following assertions are equivalent:

(a) There is a constant  $(1+\epsilon)_{(1+\epsilon)_1, \dots, (1+\epsilon)_m}^Y \geq 1$  such that

$$\left( \sum_{j_1=1}^{\infty} \left( \sum_{j_2=1}^{\infty} \dots \left( \sum_{j_m=1}^{\infty} \sum_L \|A_L(e_{j_1}, \dots, e_{j_m})\|^{(2+\epsilon)_m} \right)^{\frac{(2+\epsilon)_{m-1}}{(2+\epsilon)_m}} \dots \right)^{\frac{(2+\epsilon)_1}{(1+\epsilon)_2}} \right)^{\frac{1}{(2+\epsilon)_1}} \leq (1+\epsilon)_{(1+\epsilon)_1, \dots, (1+\epsilon)_m}^Y \sum_L \|A_L\| \quad (4)$$

for all continuous  $m$ -linear operators  $A_L: X_{(1+\epsilon)_1} \times \dots \times X_{(1+\epsilon)_m} \rightarrow Y$ .

(b) The exponents  $(2 + \epsilon)_1, \dots, (2 + \epsilon)_m$  satisfy  
 $(2 + \epsilon)_1 \geq \lambda_{\cot Y}^{(1+\epsilon)_1, \dots, (1+\epsilon)_m}, (2 + \epsilon)_2 \geq \lambda_{\cot Y}^{(1+\epsilon)_2, \dots, (1+\epsilon)_m}, \dots, (2 + \epsilon)_{m-1}$   
 $\geq \lambda_{\cot Y}^{(1+\epsilon)_{m-1}, (1+\epsilon)_m}, (2 + \epsilon)_m \geq \lambda_{\cot Y}^{(1+\epsilon)_m}.$

**Proof.** We begin by proving the direct implication. We just need to consider the case

$$\frac{1}{(1 + \epsilon)_1} + \dots + \frac{1}{(1 + \epsilon)_m} \geq \frac{1}{\cot Y}, \quad (5)$$

since the other case is covered by Theorem 2.1. By the Maurey-Pisier factorization result (see [9, pages 286,287]), the Banach space  $Y$  finitely factors the formal inclusion  $\ell_{\cot Y} \hookrightarrow \ell_\infty$ , i.e., there are universal constants  $\epsilon > 0$  such that, for all  $n$ , there are vectors  $z_1^L, \dots, z_n^L \in Y$  satisfying

$$(1 + \epsilon) \|(a_j)_{j=1}^n\|_\infty \leq \left\| \sum_{j=1}^n \sum_L a_j z_j^L \right\| \leq (1 + 2\epsilon) \left( \sum_{j=1}^n |a_j|^{\cot Y} \right)^{\frac{1}{\cot Y}}, \quad (6)$$

for all sequences of scalars  $(a_j)_{j=1}^n$ . Consider the continuous  $m$ -linear operator  $(A_L)_n: X_{(1+\epsilon)_1} \times \dots \times X_{(1+\epsilon)_m} \rightarrow Y$  given by

$$(A_L)_n(x^{(1)}, \dots, x^{(m)}) = \sum_{j=1}^n \sum_L x_j^{(1)} x_j^{(2)} \dots x_j^{(m)} z_j^L. \quad (7)$$

By (6) and the Hölder inequality we have

$$\begin{aligned} \|(A_L)_n\| &= \sup_{\|x^{(1)}\|_{(1+\epsilon)_1} \leq 1, \dots, \|x^{(m)}\|_{(1+\epsilon)_m} \leq 1} \left\| \sum_{j=1}^n \sum_L x_j^{(1)} \dots x_j^{(m)} z_j^L \right\| \\ &\leq \sup_{\|x^{(1)}\|_{(1+\epsilon)_1} \leq 1, \dots, \|x^{(m)}\|_{(1+\epsilon)_m} \leq 1} (1 + 2\epsilon) \left( \sum_{j=1}^n |x_j^{(1)} \dots x_j^{(m)}|^{\cot Y} \right)^{1/\cot Y} \\ &\leq \sup_{\|x^{(1)}\|_{(1+\epsilon)_1} \leq 1, \dots, \|x^{(m)}\|_{(1+\epsilon)_m} \leq 1} (1 + 2\epsilon) \left( \prod_{k=1}^m \left( \sum_{j=1}^n |x_j^{(k)}|^{(1+\epsilon)_k} \right)^{1/(1+\epsilon)_k} \right) \\ &= (1 + 2\epsilon). \end{aligned} \quad (8)$$

Note that, by (7), we have

$$\begin{aligned} &\left( \sum_{j_1=1}^n \left( \sum_{j_2=1}^n \dots \left( \sum_{j_m=1}^n \sum_L \|(A_L)_n(e_{j_1}, \dots, e_{j_m})\|^{(2+\epsilon)_m} \right)^{\frac{(2+\epsilon)_{m-1}}{(2+\epsilon)_m}} \dots \right)^{\frac{(2+\epsilon)_1}{(2+\epsilon)_2}} \right)^{\frac{1}{(2+\epsilon)_1}} \\ &= \left( \sum_{j=1}^n \sum_L \|z_j^L\|^{(2+\epsilon)_1} \right)^{\frac{1}{(2+\epsilon)_1}}. \end{aligned}$$

Thus, by (6) we conclude that

$$\left( \sum_{j_1=1}^n \left( \sum_{j_2=1}^n \cdots \left( \sum_{j_m=1}^n \sum_L \| (A_L)_n(e_{j_1}, \dots, e_{j_m}) \|^{(2+\epsilon)_m} \right)^{\frac{(2+\epsilon)_{m-1}}{(2+\epsilon)_m}} \cdots \right)^{\frac{(2+\epsilon)_1}{(2+\epsilon)_2}} \right)^{\frac{1}{(2+\epsilon)_1}}$$

$$\geq (1 + \epsilon) n^{\frac{1}{(2+\epsilon)_1}}$$

Combining the previous inequality with (4) and (8) we conclude that

$$(1 + \epsilon) n^{1/(2+\epsilon)_1} \leq (1 + \epsilon)_{(1+\epsilon)_1, \dots, (1+\epsilon)_m}^Y (1 + 2\epsilon).$$

Thus, since  $n$  is arbitrary, we have

$$(2 + \epsilon)_1 = \infty = \lambda_{\cot Y}^{(1+\epsilon)_1, \dots, (1+\epsilon)_m}. \quad (9)$$

If

$$\frac{1}{(1 + \epsilon)_i} + \cdots + \frac{1}{(1 + \epsilon)_m} \geq \frac{1}{\cot Y}$$

for all  $i$ , the proof is immediate. Otherwise, let  $i_0 \in \{2, 3, \dots, m\}$  be the smallest index such that

$$\begin{cases} \frac{1}{(1 + \epsilon)_{i_0}} + \cdots + \frac{1}{(1 + \epsilon)_m} < \frac{1}{\cot Y}, \\ \frac{1}{(1 + \epsilon)_{i_0-1}} + \cdots + \frac{1}{(1 + \epsilon)_m} \geq \frac{1}{\cot Y}. \end{cases}$$

If  $i_0 = 2$ , note that by (9) we have

$$\begin{aligned} \sup_{j_1} \left( \sum_{j_2=1}^{\infty} \left( \cdots \left( \sum_{j_m=1}^{\infty} \sum_L \| A_L(e_{j_1}, \dots, e_{j_m}) \|^{(2+\epsilon)_m} \right)^{\frac{(2+\epsilon)_{m-1}}{(2+\epsilon)_m}} \cdots \right)^{\frac{(2+\epsilon)_2}{(2+\epsilon)_3}} \right)^{\frac{1}{(2+\epsilon)_2}} \\ \leq (1 + \epsilon)_{(1+\epsilon)_1, \dots, (1+\epsilon)_m}^Y \sum_L \| A_L \| \end{aligned} \quad (10)$$

for all continuous  $m$ -linear operators  $A_L: X_{(1+\epsilon)_1} \times \cdots \times X_{(1+\epsilon)_m} \rightarrow Y$ . From (10) it is simple to show that

$$\begin{aligned} \left( \sum_{j_2=1}^{\infty} \left( \cdots \left( \sum_{j_m=1}^{\infty} \sum_L \| A_L(e_{j_2}, \dots, e_{j_m}) \|^{(2+\epsilon)_m} \right)^{\frac{(2+\epsilon)_{m-1}}{(2+\epsilon)_m}} \cdots \right)^{\frac{(2+\epsilon)_2}{(2+\epsilon)_3}} \right)^{\frac{1}{(2+\epsilon)_2}} \\ \leq (1 + \epsilon)_{(1+\epsilon)_1, \dots, (1+\epsilon)_m}^Y \sum_L \| A_L \|, \end{aligned}$$

for all continuous  $(m - 1)$ -linear operators  $A_L: X_{(1+\epsilon)_2} \times \cdots \times X_{(1+\epsilon)_m} \rightarrow Y$ . Since

$$\frac{1}{(1+\epsilon)_2} + \dots + \frac{1}{(1+\epsilon)_m} < \frac{1}{\cot Y'}$$

by Theorem 2.1 we conclude that

$$(2+\epsilon)_2 \geq \lambda_{\cot Y}^{(1+\epsilon)_2, \dots, (1+\epsilon)_m}, (2+\epsilon)_3 \geq \lambda_{\cot Y}^{(1+\epsilon)_3, \dots, (1+\epsilon)_m}, \dots, (2+\epsilon)_{m-1} \geq \lambda_{\cot Y}^{(1+\epsilon)_{m-1}, (1+\epsilon)_m}, (2+\epsilon)_m \geq \lambda_{\cot Y}^{(1+\epsilon)_m}.$$

If  $i_0 = 3$ , we consider

$$A_L(x^{(1)}, \dots, x^{(m)}) = x_1^{(1)} \sum_{j=1}^n \sum_L x_j^{(2)} \dots x_j^{(m)} z_j^L$$

and we can imitate the previous arguments to conclude that

$$(2+\epsilon)_2 = \infty = \lambda_{\cot Y}^{(1+\epsilon)_2, \dots, (1+\epsilon)_m}.$$

and hence

$$\sup_{j_1, j_2} \left( \sum_{j_3=1}^{\infty} \left( \dots \left( \sum_{j_m=1}^{\infty} \sum_L \|A_L(e_{j_1}, \dots, e_{j_m})\|^{(2+\epsilon)_m} \right)^{\frac{(2+\epsilon)_{m-1}}{(2+\epsilon)_m}} \dots \right)^{\frac{(2+\epsilon)_3}{(2+\epsilon)_4}} \right)^{\frac{1}{(2+\epsilon)_3}} \leq (1+\epsilon)_{(1+\epsilon)_1, \dots, (1+\epsilon)_m} \sum_L \|A_L\|, \quad (11)$$

for all continuous  $m$ -linear operators  $A_L: X_{(1+\epsilon)_1} \times \dots \times X_{(1+\epsilon)_m} \rightarrow Y$ . Again, it is plain that

$$\left( \sum_{j_3=1}^{\infty} \left( \dots \left( \sum_{j_m=1}^{\infty} \sum_L \|A_L(e_{j_3}, \dots, e_{j_m})\|^{(2+\epsilon)_m} \right)^{\frac{(2+\epsilon)_{m-1}}{(2+\epsilon)_m}} \dots \right)^{\frac{(2+\epsilon)_{i_0+1}}{(2+\epsilon)_{i_0}}} \right)^{\frac{1}{(2+\epsilon)_{i_0}}} \leq (1+\epsilon)_{(1+\epsilon)_1, \dots, (1+\epsilon)_m} \sum_L \|A_L\|$$

for all continuous  $(m-2)$ -linear operators  $A_L: X_{(1+\epsilon)_3} \times \dots \times X_{(1+\epsilon)_m} \rightarrow Y$ . Since

$$\frac{1}{(1+\epsilon)_3} + \dots + \frac{1}{(1+\epsilon)_m} < \frac{1}{\cot Y},$$

by Theorem 2.1 we have

$$(2+\epsilon)_3 \geq \lambda_{\cot Y}^{(1+\epsilon)_3, \dots, (1+\epsilon)_m}, (2+\epsilon)_4 \geq \lambda_{\cot Y}^{(1+\epsilon)_4, \dots, (1+\epsilon)_m}, \dots, (2+\epsilon)_{m-1} \geq \lambda_{\cot Y}^{(1+\epsilon)_{m-1}, (1+\epsilon)_m}, (2+\epsilon)_m \geq \lambda_{\cot Y}^{(1+\epsilon)_m}.$$

We conclude the proof in a similar fashion for  $i_0 = 4, \dots, m$ .

Now we prove the reverse implication. The case

$$\frac{1}{(1+\epsilon)_1} + \dots + \frac{1}{(1+\epsilon)_m} < \frac{1}{\cot Y'}$$

is encompassed by Theorem 2.1. So, we shall consider

$$\frac{1}{(1+\epsilon)_1} + \dots + \frac{1}{(1+\epsilon)_m} \geq \frac{1}{\cot Y}.$$

If

$$\frac{1}{(1+\epsilon)_i} + \dots + \frac{1}{(1+\epsilon)_m} \geq \frac{1}{\cot Y}$$

for all  $i$ , the proof is immediate. Otherwise, let  $i_0 \in \{2, \dots, m\}$  be the smallest index such that

$$\begin{cases} \frac{1}{(1+\epsilon)_{i_0}} + \dots + \frac{1}{(1+\epsilon)_m} < \frac{1}{\cot Y}, \\ \frac{1}{(1+\epsilon)_{i_0-1}} + \dots + \frac{1}{(1+\epsilon)_m} \geq \frac{1}{\cot Y}. \end{cases}$$

We need to prove that there is a constant  $(1+\epsilon)_{(1+\epsilon)_1, \dots, (1+\epsilon)_m}^Y \geq 1$ , such that

$$\begin{aligned} & \sup_{j_1, \dots, j_{i_0-1}} \left( \sum_{j_{i_0}=1}^{\infty} \left( \dots \left( \sum_{j_m=1}^{\infty} \sum_L \|A_L(e_{j_1}, \dots, e_{j_m})\|^{(2+\epsilon)_m} \right)^{\frac{(2+\epsilon)_{m-1}}{(2+\epsilon)_m}} \dots \right)^{\frac{(2+\epsilon)_{i_0}}{(2+\epsilon)_{i_0+1}}} \right)^{\frac{1}{(2+\epsilon)_{i_0}}} \\ & \leq (1+\epsilon)_{(1+\epsilon)_1, \dots, (1+\epsilon)_m}^Y \sum_L \|A_L\| \end{aligned}$$

for

$$(2+\epsilon)_{i_0} \geq \lambda_{\cot Y}^{(1+\epsilon)_{i_0}, \dots, (1+\epsilon)_m}, \dots, (2+\epsilon)_m \geq \lambda_{\cot Y}^{(1+\epsilon)_m}$$

By Theorem 2.1, we know that for any fixed vectors  $e_{j_1}, \dots, e_{j_{i_0-1}}$ , there is a constant  $(1+\epsilon)_{(1+\epsilon)_{i_0}, \dots, (1+\epsilon)_m}^Y \geq 1$ , such that

$$\begin{aligned} & \left( \sum_{j_{i_0}=1}^{\infty} \left( \dots \left( \sum_{j_m=1}^{\infty} \sum_L \|A_L(e_{j_1}, \dots, e_{j_m})\| \lambda_{\cot Y}^{(1+\epsilon)_m} \right)^{\frac{\lambda_{\cot Y}^{(1+\epsilon)_{m-1}, (1+\epsilon)_m}}{\lambda_{\cot Y}^{(1+\epsilon)_m}}} \dots \right)^{\frac{\lambda_{\cot Y}^{(1+\epsilon)_{i_0}, \dots, (1+\epsilon)_m}}{\lambda_{\cot Y}^{(1+\epsilon)_{i_0+1}, \dots, (1+\epsilon)_m}}} \right)^{\frac{1}{\lambda_{\cot Y}^{(1+\epsilon)_{i_0}, \dots, (1+\epsilon)_m}}} \\ & \leq (1+\epsilon)_{(1+\epsilon)_1, \dots, (1+\epsilon)_m}^Y \sum_L \|A_L\| \end{aligned}$$

for all continuous  $m$ -linear operators  $A: X_{(1+\epsilon)_1} \times \dots \times X_{(1+\epsilon)_m} \rightarrow Y$ . Then,

$$\sup_{j_1, \dots, j_{i_0-1}} \left( \sum_{j_{i_0}=1}^{\infty} \left( \dots \left( \sum_{j_m=1}^{\infty} \sum_L \|A_L(e_{j_1}, \dots, e_{j_m})\|^{\lambda_{\cot Y}^{(1+\epsilon)m}} \right)^{\frac{\lambda_{\cot Y}^{(1+\epsilon)m-1}(1+\epsilon)m}{\lambda_{\cot Y}^{(1+\epsilon)m}}} \dots \right)^{\frac{\lambda_{\cot Y}^{(1+\epsilon)i_0, \dots, (1+\epsilon)m}}{\lambda_{\cot Y}^{(1+\epsilon)i_0+1, \dots, (1+\epsilon)m}}} \right)^{\frac{1}{\lambda_{\cot Y}^{(1+\epsilon)i_0, \dots, (1+\epsilon)m}}}$$

$$\leq (1 + \epsilon)_{(1+\epsilon)_1, \dots, (1+\epsilon)_m}^Y \sum_L \|A_L\|$$

for all continuous  $m$ -linear operators  $A_L: X_{(1+\epsilon)_1} \times \dots \times X_{(1+\epsilon)_m} \rightarrow Y$ .

To conclude the proof we just need to remark that

$$\sup_{j_1, \dots, j_{i_0-1}} \left( \sum_{j_{i_0}=1}^{\infty} \left( \dots \left( \sum_{j_m=1}^{\infty} \sum_L \|A_L(e_{j_1}, \dots, e_{j_m})\|^{(2+\epsilon)_m} \right)^{\frac{(2+\epsilon)_{m-1}}{(2+\epsilon)_m}} \dots \right)^{\frac{(2+\epsilon)_{i_0}}{(2+\epsilon)_{i_0+1}}} \right)^{\frac{1}{(2+\epsilon)_i}}$$

$$\leq \sup_{j_1, \dots, j_{i_0-1}} \left( \sum_{j_{i_0}=1}^{\infty} \left( \dots \left( \sum_{j_m=1}^{\infty} \sum_L \|A_L(e_{j_1}, \dots, e_{j_m})\|^{\lambda_{\cot Y}^{(1+\epsilon)m}} \right)^{\frac{\lambda_{\cot Y}^{(1+\epsilon)m-1}(1+\epsilon)m}{\lambda_{\cot Y}^{(1+\epsilon)m}}} \dots \right)^{\frac{\lambda_{\cot Y}^{(1+\epsilon)i_0, \dots, (1+\epsilon)m}}{\lambda_{\cot Y}^{(1+\epsilon)i_0+1, \dots, (1+\epsilon)m}}} \right)^{\frac{1}{\lambda_{\cot Y}^{(1+\epsilon)i_0, \dots, (1+\epsilon)m}}}$$

provided

$$(2 + \epsilon)_{i_0} \geq \lambda_{\cot Y}^{(1+\epsilon)_{i_0}, \dots, (1+\epsilon)_m}, \dots, (2 + \epsilon)_m \geq \lambda_{\cot Y}^{(1+\epsilon)_m}.$$

### 3. Proof of Theorem 1.1 (see [19])

Let the adjoint of a Banach space  $X$  be denoted by  $X^*$ . To simplify the notation we will consider  $\sigma(j) = j$  for all  $j$ ; the other cases are similar. Let  $\mathcal{L}^m(X_{(1+\epsilon)_1}, \dots, X_{(1+\epsilon)_m}; Y)$  denote the space of all continuous  $m$ -linear operators from  $X_{p_1} \times \dots \times X_{p_m}$  to  $Y$ . By the canonical isometric isomorphism

$$\Psi_L: \mathcal{L}^m(X_{(1+\epsilon)_1}, X_{(1+\epsilon)_m}; \mathbb{K}) \rightarrow \mathcal{L}^{m-1}(X_{(1+\epsilon)_1}, \dots, X_{(1+\epsilon)_{m-1}}; (X_{(1+\epsilon)_m})^*)$$

and duality in  $X_{(1+\epsilon)_m}$ , note that, if  $R \in \mathcal{L}^m(X_{(1+\epsilon)_1}, \dots, X_{(1+\epsilon)_m}; \mathbb{K})$ , we have

$$R(x_1, \dots, x_{m-1}, e_n) = \Psi_L(R)(x_1, \dots, x_{m-1})(e_n) = (\Psi_L(R)(x_1, \dots, x_{m-1}))_n. \quad (12)$$

We start off by proving (1) $\Rightarrow$ (2). Let us suppose that there is a constant  $(1 + \epsilon)_{(1+\epsilon)_1, \dots, (1+\epsilon)_m} \geq 1$  such that

$$\left( \sum_{j_1=1}^{\infty} \left( \sum_{j_2=1}^{\infty} \cdots \left( \sum_{j_m=1}^{\infty} \sum_L |T_L(e_{j_1}, \dots, e_{j_m})|^{(1+\epsilon)_m^*} \right)^{\frac{(1+\epsilon)_{m-1}}{(1+\epsilon)_m}} \right)^{\frac{(1+\epsilon)_1}{(1+\epsilon)_2}} \right)^{\frac{1}{(1+\epsilon)_1}} \leq (1 + \epsilon)_{(1+\epsilon)_1, \dots, (1+\epsilon)_m} \sum_L \|T_L\| \quad (13)$$

for all continuous  $m$ -linear forms  $T_L: X_{(1+\epsilon)_1} \times \cdots \times X_{(1+\epsilon)_m} \rightarrow \mathbb{K}$

Consider a continuous  $(m-1)$ -linear operator  $A_L: X_{(1+\epsilon)_1} \times \cdots \times X_{(1+\epsilon)_{m-1}} \rightarrow (X_{(1+\epsilon)_m})^*$ . Then, using our hypothesis, we have

$$\left( \sum_{j_1=1}^{\infty} \left( \sum_{j_2=1}^{\infty} \cdots \left( \sum_{j_{m-1}=1}^{\infty} \sum_L \|A_L(e_{j_1}, \dots, e_{j_{m-1}})\|_{(X_{(1+\epsilon)_m})^*}^{\frac{(1+\epsilon)_{m-1}}{(1+\epsilon)_m}} \right)^{\frac{(1+\epsilon)_{m-2}}{(1+\epsilon)_{m-1}}} \right)^{\frac{(1+\epsilon)_1}{(1+\epsilon)_2}} \right)^{\frac{1}{(1+\epsilon)_1}} \quad (14)$$

$$= \left( \sum_{j_1=1}^{\infty} \left( \sum_{j_2=1}^{\infty} \cdots \left( \sum_{j_{m-1}=1}^{\infty} \left( \sum_{j_m=1}^{\infty} \sum_L |(A_L(e_{j_1}, \dots, e_{j_{m-1}}))_{j_m}|^{(1+\epsilon)_m^*} \right)^{\frac{(1+\epsilon)_{m-1}}{(1+\epsilon)_m}} \right)^{\frac{(1+\epsilon)_{m-2}}{(1+\epsilon)_{m-1}}} \right)^{\frac{(1+\epsilon)_1}{(1+\epsilon)_2}} \right)^{\frac{1}{(1+\epsilon)_1}}$$

$$\stackrel{(12)}{=} \sum_{j_1=1}^{\infty} \left( \sum_{j_2=1}^{\infty} \cdots \left( \sum_{j_{m-1}=1}^{\infty} \left( \sum_{j_m=1}^{\infty} \sum_L |\Psi_L^{-1}(A_L)(e_{j_1}, \dots, e_{j_m})|^{(1+\epsilon)_m^*} \right)^{\frac{(1+\epsilon)_{m-1}}{(1+\epsilon)_m}} \right)^{\frac{(1+\epsilon)_{m-2}}{(1+\epsilon)_{m-1}}} \right)^{\frac{(1+\epsilon)_1}{(1+\epsilon)_2}} \right)^{\frac{1}{(1+\epsilon)_1}}$$

$$\leq (1 + \epsilon)_{(1+\epsilon)_1, \dots, (1+\epsilon)_m} \sum_L \|\Psi_L^{-1}(A_L)\|$$

$$\leq (1 + \epsilon)_{(1+\epsilon)_1, \dots, (1+\epsilon)_m} \sum_L \|A_L\|$$

Missing the :[4PK]Comment continuation dots on the right side of the system' brackets

for all continuous  $(m-1)$ -linear operators  $A_L: X_{(1+\epsilon)_1} \times \cdots \times X_{(1+\epsilon)_{m-1}} \rightarrow (X_{(1+\epsilon)_m})^*$ . Since  $(X_{(1+\epsilon)_m})^*$  has cotype  $\max\{(1+\epsilon)_m^*, 2\}$ , by Theorem 2.2, the exponents  $\left(\frac{1+\epsilon}{\epsilon}\right)_1, \dots, \left(\frac{1+\epsilon}{\epsilon}\right)_{m-1}$  in (2.2) satisfy

$$\begin{aligned} \left(\frac{1+\epsilon}{\epsilon}\right)_1 &\geq \lambda_{\max\{(1+\epsilon)_m^*, 2\}}^{(1+\epsilon)_1, \dots, (1+\epsilon)_{m-1}} \left(\frac{1+\epsilon}{\epsilon}\right)_2 \geq \lambda_{\max\{(1+\epsilon)_m^*, 2\}}^{(1+\epsilon)_2, \dots, (1+\epsilon)_{m-1}} \left(\frac{1+\epsilon}{\epsilon}\right)_{m-1} \\ &\geq \lambda_{\max\{(1+\epsilon)_m^*, 2\}}^{(1+\epsilon)_{m-1}}. \end{aligned} \quad (15)$$

Since

$$1 - \frac{1}{\max\{(1+\epsilon)_m^*, 2\}} = \frac{1}{\mu}$$

we have

$$\begin{aligned} \lambda_{\max\{(1+\epsilon)_m^*, 2\}}^{(1+\epsilon)_1, \dots, (1+\epsilon)_{m-1}} &= \frac{1}{\max\left\{\frac{1}{\max\{(1+\epsilon)_m^*, 2\}} - \left(\frac{1}{(1+\epsilon)_1} + \cdots + \frac{1}{(1+\epsilon)_{m-1}}\right), 0\right\}} \\ &= \frac{1}{\max\left\{1 - \left(\frac{1}{(1+\epsilon)_1} + \cdots + \frac{1}{(1+\epsilon)_{m-1}} + \frac{1}{\mu}\right), 0\right\}} \\ &= \delta^{(1+\epsilon)_1, \dots, (1+\epsilon)_{m-1}, \mu} \end{aligned}$$

for all  $i \in \{1, \dots, m-1\}$ . Then, (15) can be re-stated as

$$\left(\frac{1+\epsilon}{\epsilon}\right)_1 \geq \delta^{(1+\epsilon)_1, \dots, (1+\epsilon)_{m-1}, \mu}, \left(\frac{1+\epsilon}{\epsilon}\right)_2 \geq \delta^{(1+\epsilon)_2, \dots, (1+\epsilon)_{m-1}, \mu}, \dots, \left(\frac{1+\epsilon}{\epsilon}\right)_{m-1} \geq \delta^{(1+\epsilon)_{m-1}, \mu}$$

and the proof is done.

(2) $\Rightarrow$ (1). If the exponents  $\left(\frac{1+\epsilon}{\epsilon}\right)_1, \dots, \left(\frac{1+\epsilon}{\epsilon}\right)_{m-1}$  satisfy

$$\left(\frac{1+\epsilon}{\epsilon}\right)_1 \geq \delta^{(1+\epsilon)_1, \dots, (1+\epsilon)_{m-1}, \mu}, \left(\frac{1+\epsilon}{\epsilon}\right)_2 \geq \delta^{(1+\epsilon)_2, \dots, (1+\epsilon)_{m-1}, \mu}, \dots, \left(\frac{1+\epsilon}{\epsilon}\right)_{m-1} \geq \delta^{(1+\epsilon)_{m-1}, \mu},$$

we have, again, that the exponents  $\left(\frac{1+\epsilon}{\epsilon}\right)_1, \dots, \left(\frac{1+\epsilon}{\epsilon}\right)_{m-1}$  satisfy

$$\left(\frac{1+\epsilon}{\epsilon}\right)_1 \geq \lambda_{2+\epsilon}^{(1+\epsilon)_1, \dots, (1+\epsilon)_{m-1}}, \left(\frac{1+\epsilon}{\epsilon}\right)_2 \geq \lambda_{2+\epsilon}^{(1+\epsilon)_2, \dots, (1+\epsilon)_{m-1}}, \dots, \left(\frac{1+\epsilon}{\epsilon}\right)_{m-1} \geq \lambda_{2+\epsilon}^{(1+\epsilon)_{m-1}},$$

with  $(2+\epsilon) = \cot(X_{(1+\epsilon)_m})^*$ . Thus, by Theorem 2.2, there is a constant

$(1+\epsilon)_{(1+\epsilon)_1, \dots, (1+\epsilon)_{m-1}}^{(X_{(1+\epsilon)_m})^*} \geq 1$  such that

Fit math expressions :[5PK]Comment  
!!inone line

$$\begin{aligned} &\left( \sum_{j_1=1}^{\infty} \left( \sum_{j_2=1}^{\infty} \cdots \left( \sum_{j_{m-1}=1}^{\infty} \sum_L \|T_L(e_{j_1}, \dots, e_{j_{m-1}})\|_{(X_{(1+\epsilon)_m})^*}^{\left(\frac{1+\epsilon}{\epsilon}\right)_{m-1}} \right)^{\left(\frac{1+\epsilon}{\epsilon}\right)_{m-2}} \cdots \right)^{\left(\frac{1+\epsilon}{\epsilon}\right)_2} \right)^{\frac{1}{\left(\frac{1+\epsilon}{\epsilon}\right)_1}} \\ &\leq (1+\epsilon)_{(1+\epsilon)_1, \dots, (1+\epsilon)_{m-1}}^{(X_{(1+\epsilon)_m})^*} \sum_L \|T_L\| \end{aligned}$$

for all continuous  $m$ -linear operators  $T_L: X_{(1+\epsilon)_1} \times \cdots \times X_{(1+\epsilon)_{m-1}} \rightarrow (X_{(1+\epsilon)_m})^*$ .

We thus have

$$\begin{aligned}
 & \left( \sum_{j_1=1}^{\infty} \left( \sum_{j_2=1}^{\infty} \cdots \left( \sum_{j_{m-1}=1}^{\infty} \left( \sum_{j_m=1}^{\infty} \sum_L |A_L(e_{j_1}, \dots, e_{j_m})|^{(1+\epsilon)_m^*} \right)^{\frac{(1+\epsilon)_{m-1}}{(1+\epsilon)_m^*}} \right)^{\frac{(1+\epsilon)_{m-2}}{(1+\epsilon)_{m-1}}} \right)^{\frac{(1+\epsilon)_1}{(1+\epsilon)_2}} \right)^{\frac{1}{(1+\epsilon)_1}} \\
 &= \left( \sum_{j_1=1}^{\infty} \left( \sum_{j_2=1}^{\infty} \cdots \left( \sum_{j_{m-1}=1}^{\infty} \sum_L \|\Psi_L(A_L)(e_{j_1}, \dots, e_{j_m})\|_{(X_{(1+\epsilon)_m})^*}^{\frac{(1+\epsilon)_{m-1}}{(1+\epsilon)_m^*}} \right)^{\frac{(1+\epsilon)_{m-2}}{(1+\epsilon)_{m-1}}} \right)^{\frac{(1+\epsilon)_1}{(1+\epsilon)_2}} \right)^{\frac{1}{(1+\epsilon)_1}} \\
 &\leq (1 + \epsilon)_{(1+\epsilon)_1, \dots, (1+\epsilon)_{m-1}}^{(X_{(1+\epsilon)_m})^*} \sum_L \|\Psi_L(A_L)\| \\
 &= (1 + \epsilon)_{(1+\epsilon)_1, \dots, (1+\epsilon)_{m-1}} \sum_L \|A_L\|
 \end{aligned}$$

for all continuous  $m$ -linear forms  $A_L: X_{(1+\epsilon)_1} \times \cdots \times X_{(1+\epsilon)_m} \rightarrow \mathbb{K}$ .

**Remark 3.1.**[19] proved that the determination of the exact values of the constants involved in the main theorem is probably a difficult task, as it happens with the Hardy-Littlewood inequalities ( see [3,4] and the references therein). However when we are restricted to the bilinear case, with  $(1 + \epsilon)_1 = (1 + \epsilon)_2 = \infty$  and  $\sigma$  as the identity map, it is not difficult to check that we recover the constant  $\sqrt{2}$  from the Orlicz inequality.

.Typing error :[6PK]Comment

What !No conclusion :[7PK]Comment  
?are the benefits of the new statements

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