The Orlicz Inequality for Series of Multilinear Forms

Abstract

The Orlicz (ℓ_2, ℓ_1)-mixed inequality who states that

$$\left(\sum_{j_1=1}^{n} \left(\sum_{j_2=1}^{n} \sum_{L} \left| A_L\left(e_{j_1}, e_{j_2}\right) \right| \right)^2 \right)^{\frac{1}{2}} \leq \sqrt{2} \sum_{L} \|A_L\|$$

for all sequences of bilinear forms $A_L: \mathbb{K}^n \times \mathbb{K}^n \to \mathbb{K}$ and all positive integers n, where $\mathbb{K}^n$ denotes $\mathbb{R}^n$ or $\mathbb{C}^n$ endowed with the supremum norm. We follow D. Núñez-Alarcón, D. Pellegrino, and D. Serrano-Rodríguez [19] to extend this inequality to series of multilinear forms, with $\mathbb{K}^n$ endowed with $\ell_{1+\epsilon}$ norms for all $0 \leq \epsilon \leq \infty$.

Keywords. Orlicz inequality, multilinear forms, Hölder inequality, Hardy-Littlewood inequalities, Maurey-Pisier factorization.

1. Introduction

The origins of the theory of summability of multilinear forms and absolutely summing multilinear operators are probably associated to Orlicz (ℓ_2, ℓ_1)-mixed inequality published in the 1930's (see [8, page 24]). It states that

$$\left(\sum_{j_1=1}^{n} \left(\sum_{j_2=1}^{n} \sum_{L} \left| A_L\left(e_{j_1}, e_{j_2}\right) \right| \right)^2 \right)^{\frac{1}{2}} \leq \sqrt{2} \sum_{L} \|A_L\|$$

for all bilinear forms $A_L: \mathbb{K}^n \times \mathbb{K}^n \to \mathbb{K}$, and all positive integers n. Here $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and $\mathbb{K}^n$ is endowed with the supremum norm. We also represent by e_k the canonical vectors in a sequence space and

$$\|A_L\| := \sup\{|A_L(x, y)|: \|x\| \leq 1 \text{ and } \|y\| \leq 1\}.$$

An equivalent formulation is the following:

$$\left(\sum_{j_1=1}^{\infty} \left(\sum_{j_2=1}^{\infty} \sum_{L} \left| A_L\left(e_{j_1}, e_{j_2}\right) \right| \right)^2 \right)^{\frac{1}{2}} \leq \sqrt{2} \sum_{L} \|A_L\| \qquad (1)$$

for all continuous sequences of bilinear forms $A_L: c_0 \times c_0 \to \mathbb{K}$. The exponents in (1) are optimal in the sense that, fixing the exponent 1, the exponent 2 cannot be replaced by smaller exponents (nor the exponent 1 can be replaced by smaller exponents) keeping the constant independent of n. The Orlicz inequality is closely related to Littlewood's (ℓ_1, ℓ_2)-mixed inequality (see [8, page 23]), which asserts that

$$\sum_{j_1=1}^{\infty}\left(\sum_{j_2=1}^{\infty}\sum_{L}\left|A_L\left(e_{j_1},e_{j_2}\right)\right|^2\right)^{\frac{1}{2}}\leq\sqrt{2}\sum_{L}\|A_L\|$$

for all continuous sequences of bilinear forms $A_L\colon c_0\times c_0\to\mathbb{K}$. Again, the exponents are optimal in the same sense above described. Combining these two inequalities, and using the Hölder inequality for mixed sums we recover Littlewood's $4/3$ inequality:

$$\left(\sum_{j_1,j_2=1}^{\infty}\sum_{L}\left|A_L\left(e_{j_1},e_{j_2}\right)\right|^{\frac{4}{3}}\right)^{\frac{3}{4}}\leq\sqrt{2}\sum_{L}\|A_L\|$$

for all continuous bilinear forms $A_L\colon c_0\times c_0\to\mathbb{K}$. For recent results on absolutely summing linear and multilinear operators see [6,11,17].

The exponent $4/3$ from the previous inequality cannot be replaced by smaller exponents keeping the constant independent of n. The constant $\sqrt{2}$ is optimal (in all the three inequalities) when $\mathbb{K}=\mathbb{R}$, but the optimal constants when $\mathbb{K}=\mathbb{C}$ are unknown.

In 1934 Hardy and Littlewood [10] (see also [13]) pushed the subject further, extending the above results to bilinear forms defined on $\ell_{1+\epsilon}$ spaces (when $\epsilon=\infty$ we consider c_0 instead of ℓ_∞). The investigation of extensions of the Hardy-Littlewood inequalities to multilinear forms were initiated by Praciano-Pereira [16] in 1981 and intensively investigated since then (see, for instance, [1,2,5,7,12,13,14,15]), but there are still several open problems regarding the optimal exponents and optimal constants involved.

We shall use the same notation from [1]:

$$X_{1+\epsilon}:=\begin{cases}\ell_{1+\epsilon}, \text{ if } 0\leq\epsilon<\infty\\ c_0, \text{ if } \epsilon=\infty\end{cases}$$

and, when $\epsilon=\infty$, the sum $\left(\sum_j\|x_j\|^{1+\epsilon/\epsilon}\right)^{1/\frac{1+\epsilon}{\epsilon}}$ shall represent the supremum of $\|x_j\|$. We also denote the conjugate index of $(1+\epsilon)$ by $(1+\epsilon)^*$, i.e., $1/(1+\epsilon)+1/(1+\epsilon)^*=1$. We find the optimal values of the exponents $(1+\epsilon)_1,\dots,(1+\epsilon)_m$ and of the constants $(1+\epsilon)_{(1+\epsilon)_1,\dots,(1+\epsilon)_m}^{(\mathbb{K})(1+\epsilon)_1,\dots,(1+\epsilon)_m}$ satisfying

$$\left(\sum_{j_1=1}^{\infty}\cdots\left(\sum_{j_{m-1}=1}^{\infty}\left(\sum_{j_m=1}^{\infty}\sum_{L}\left|A_L\left(e_{j_1},\dots e_{j_m}\right)\right|^{(1+\epsilon)_m}\right)^{\frac{(1+\epsilon)_{m-1}}{(1+\epsilon)_m}}\right)^{\frac{(1+\epsilon)_{m-2}}{(1+\epsilon)_{m-1}}}\right)^{\frac{1}{(1+\epsilon)_1}}$$
$$\leq(1+\epsilon)_{(1+\epsilon)_1,\dots,(1+\epsilon)_m}^{(\mathbb{K})(1+\epsilon)_1,\dots,(1+\epsilon)_m}\sum_{L}\|A_L\|$$

for all continuous m-linear forms $A_L\colon X_{(1+\epsilon)_1}\times\cdots\times X_{(1+\epsilon)_m}\to\mathbb{K}$. The answer is known in several cases (see [1,5,18] and the references therein), but a complete solution is still unknown. By [19] we shall be interested in investigating the optimal exponents $(1+\epsilon)_1,\dots,(1+\epsilon)_m$. It is simple to prove that the optimal exponent $(1+\epsilon)_m$ associated to the sum $\sum_{j_m=1}^{\infty}$ is $(1+$

$\epsilon)_m^*$. The main result provides the optimal exponents $(1+\epsilon)_1, \dots, (1+\epsilon)_{m-1}$ in the case that $(1+\epsilon)_m = (1+\epsilon)_m^*$.

From now on, let $\epsilon \geq 0$, and let $(1+\epsilon)_1, \dots, (1+\epsilon)_m \in [1, \infty]$. We define $\delta^{(1+\epsilon)_k, \dots, (1+\epsilon)_m}$ and $\lambda_{2+\epsilon}^{(1+\epsilon)_k, \dots, (1+\epsilon)_m}$ by

$$\delta^{(1+\epsilon)_k, \dots, (1+\epsilon)_m} := \frac{1}{\max\left\{1 - \left(\frac{1}{(1+\epsilon)_k} + \cdots + \frac{1}{(1+\epsilon)_m}\right), 0\right\}},$$

and

$$\lambda_{2+\epsilon}^{(1+\epsilon)_k, \dots, (1+\epsilon)_m} := \frac{1}{\max\left\{\frac{1}{2+\epsilon} - \left(\frac{1}{(1+\epsilon)_k} + \cdots + \frac{1}{(1+\epsilon)_m}\right), 0\right\}},$$

for all positive integers m and $k = 1, \dots, m$. Note that when $1/(1+\epsilon)_k + \cdots + 1/(1+\epsilon)_m \geq 1$ we have

$$\delta^{(1+\epsilon)_k, \dots, (1+\epsilon)_m} = \infty$$

and, also, when $1/(1+\epsilon)_k + \cdots + 1/(1+\epsilon)_m \geq \frac{1}{2+\epsilon}$ we have

$$\lambda_{2+\epsilon}^{(1+\epsilon)_k, \dots, (1+\epsilon)_m} = \infty.$$

The main result is, a generalization of the the Orlicz inequality. We consider the very particular case $(m, (1+\epsilon)_1, (1+\epsilon)_2) = (2, \infty, \infty)$ and σ as the identity map in its statement, we recover the Orlicz inequality (see [19]):

Theorem 1.1. Let $\epsilon \geq 0$ be an integer and $\sigma: \{1, \dots, m\} \to \{1, \dots, m\}$ be a bijection. If

$$\left(\left(\frac{1+\epsilon}{\epsilon}\right)_1, \dots, \left(\frac{1+\epsilon}{\epsilon}\right)_{m-1}\right) \quad \in (0, \infty]^{m-1}$$
$$((1+\epsilon)_1, \dots, (1+\epsilon)_m) \quad \in [1, \infty]^m$$

the following assertions are equivalent: (1) There is a constant $(1+\epsilon)_{(1+\epsilon)_1, \dots, (1+\epsilon)_m} \geq 1$ such that

$$\left(\sum_{j_{\sigma(1)}=1}^{\infty}\left(\sum_{j_{\sigma(2)}=1}^{\infty} \cdots \left(\sum_{j_{\sigma(m)}=1}^{\infty} \sum_L \sum \left|A_L\left(e_{j_{\sigma(1)}}, \dots, e_{j_{\sigma(m)}}\right)\right|^{(1+\epsilon)_{\sigma(m)}^*}\right)^{\frac{(1+\epsilon/\epsilon)_{m-1}}{(1+\epsilon)_{\sigma(m)}^*}} \cdots\right)^{\frac{(1+\epsilon/\epsilon)_1}{(1+\epsilon/\epsilon)_2}}\right)^{\frac{1}{(1+\epsilon/\epsilon)_1}}$$
$$\leq (1+\epsilon)_{(1+\epsilon)_1, \dots, (1+\epsilon)_m} \sum_L \|A_L\|$$

for all continuous m-linear forms $A_L: X_{(1+\epsilon)_1} \times \cdots \times X_{(1+\epsilon)_m} \to \mathbb{K}$.

(2) The exponents $(1+\epsilon/\epsilon)_1, \dots, (1+\epsilon/\epsilon)_{m-1}$ satisfy

$$(1+\epsilon/\epsilon)_1 \geq \delta^{(1+\epsilon)_{\sigma(1)}, \dots, (1+\epsilon)_{\sigma(m-1)}, \mu}, (1+\epsilon/\epsilon)_2 \geq \delta^{(1+\epsilon)_{\sigma(2)}, \dots, (1+\epsilon)_{\sigma(m-1)}, \mu}, \dots, (1+\epsilon/\epsilon)_{m-1} \geq \delta^{(1+\epsilon)_{\sigma(m-1)}, \mu},$$

where $\mu = \min\{(1+\epsilon)_{\sigma(m)}, 2\}$.

2. Preliminary Results

Let $0 < \epsilon < \infty$. Recall that a Banach space X has cotype $(2+\epsilon)$ if there is a constant $\epsilon > 0$ such that, we select finitely many vectors $x_1, \dots, x_n \in X$,

$$\left(\sum_{j=1}^{n} \|x_j\|^{(2+\epsilon)}\right)^{\frac{1}{(2+\epsilon)}} \leq (1+\epsilon)\left(\int_{[0,1]} \left\|\sum_{j=1}^{n} (2+\epsilon)_j(t)x_j\right\|^2 dt\right)^{\frac{1}{2}} \tag{2}$$

where $(2+\epsilon)_j$ denotes the j-th Rademacher function. The infimum of the cotypes of X is denoted by $\cot X$.

The following result was proved in [5] (see [19]):

Theorem 2.1. (see [5]) Let $((2+\epsilon)_1, \dots, (2+\epsilon)_m) \in (0,\infty)^m$, and Y be an infinite-dimensional Banach space with cotype $\cot Y$. If

$$\frac{1}{(1+\epsilon)_1} + \cdots + \frac{1}{(1+\epsilon)_m} < \frac{1}{\cot Y}, \tag{3}$$

then the following assertions are equivalent:

(a) There is a constant $(1+\epsilon)^Y_{(1+\epsilon)_1,\dots,(1+\epsilon)_m} \geq 1$ such that

$$\left(\sum_{j_1=1}^{\infty}\left(\sum_{j_2=1}^{\infty}\cdots\left(\sum_{j_m=1}^{\infty}\sum_{L}\|A_L(e_{j_1},\dots,e_{j_m})\|^{(2+\epsilon)_m}\right)^{\frac{(2+\epsilon)_{m-1}}{(2+\epsilon)_m}}\cdots\right)^{\frac{(2+\epsilon)_1}{(2+\epsilon)_2}}\right)^{\frac{1}{(2+\epsilon)_1}}$$

$$\leq (1+\epsilon)^Y_{(1+\epsilon)_1,\dots,(1+\epsilon)_m}\sum_{l}\|A_L\|$$

for all continuous m-linear operators $A_L: X_{(1+\epsilon)_1} \times \cdots \times X_{(1+\epsilon)_m} \to Y$.

(b) The exponents $(2+\epsilon)_1, \dots, (2+\epsilon)_m$ satisfy

$$(2+\epsilon)_1 \geq \lambda^{(1+\epsilon)_1,\dots,(1+\epsilon)_m}_{\cot Y}, (2+\epsilon)_2 \geq \lambda^{(1+\epsilon)_2,\dots,(1+\epsilon)_m}_{\cot Y}, \dots, (2+\epsilon)_{m-1} \geq \lambda^{(1+\epsilon)_{m-1},(1+\epsilon)_m}_{\cot Y}, (2+\epsilon)_m \geq \lambda^{(1+\epsilon)_m}_{\cot Y}.$$

We need the following extension of the previous theorem, relaxing the hypothesis (3). Besides, below we have $((2+\epsilon)_1, \dots, (2+\epsilon)_m) \in (0,\infty]^m$ while in Theorem 2.1 we have $((2+\epsilon)_1, \dots, (2+\epsilon)_m) \in (0,\infty)^m$ (see [19]).

Theorem 2.2. Let $((2+\epsilon)_1, \dots, (2+\epsilon)_m) \in (0,\infty]^m$, $((1+\epsilon)_1, \dots, (1+\epsilon)_m) \in [1,\infty]^m$ and Y be an infinite-dimensional Banach space with cotype $\cot Y$. The following assertions are equivalent:

(a) There is a constant $(1+\epsilon)^Y_{(1+\epsilon)_1,\dots,(1+\epsilon)_m} \geq 1$ such that

$$\left(\sum_{j_1=1}^{\infty}\left(\sum_{j_2=1}^{\infty}\cdots\left(\sum_{j_m=1}^{\infty}\sum_{L}\|A_L(e_{j_1},\dots,e_{j_m})\|^{(2+\epsilon)_m}\right)^{\frac{(2+\epsilon)_{m-1}}{(2+\epsilon)_m}}\cdots\right)^{\frac{(2+\epsilon)_1}{(1+\epsilon/\epsilon)_2}}\right)^{\frac{1}{(2+\epsilon)_1}}$$

$$\leq (1+\epsilon)^Y_{(1+\epsilon)_1,\dots,(1+\epsilon)_m}\sum_{L}\|A_L\| \tag{4}$$

for all continuous m-linear operators $A_L\colon X_{(1+\epsilon)_1} \times \cdots \times X_{(1+\epsilon)_m} \to Y$.

(b) The exponents $(2+\epsilon)_1, \dots, (2+\epsilon)_m$ satisfy

$$(2+\epsilon)_1 \geq \lambda_{\cot Y}^{(1+\epsilon)_1,\dots,(1+\epsilon)_m}, (2+\epsilon)_2 \geq \lambda_{\cot Y}^{(1+\epsilon)_2,\dots,(1+\epsilon)_m}, \dots, (2+\epsilon)_{m-1} \geq \lambda_{\cot Y}^{(1+\epsilon)_{m-1},(1+\epsilon)_m}, (2+\epsilon)_m \geq \lambda_{\cot Y}^{(1+\epsilon)_m}.$$

Proof. We begin by proving the direct implication. We just need to consider the case

$$\frac{1}{(1+\epsilon)_1} + \cdots + \frac{1}{(1+\epsilon)_m} \geq \frac{1}{\cot Y}, \tag{5}$$

since the other case is covered by Theorem 2.1. By the Maurey-Pisier factorization result (see [9, pages 286,287]), the Banach space Y finitely factors the formal inclusion $\ell_{\cot Y} \hookrightarrow \ell_\infty$, i.e., there are universal constants $\epsilon > 0$ such that, for all n, there are vectors $z_1^L, \dots, z_n^L \in Y$ satisfying

$$(1+\epsilon)\left\|(a_j)_{j=1}^n\right\|_\infty \leq \left\|\sum_{j=1}^n \sum_L a_j z_j^L\right\| \leq (1+2\epsilon)\left(\sum_{j=1}^n |a_j|^{\cot Y}\right)^{\frac{1}{\cot Y}}, \tag{6}$$

for all sequences of scalars $(a_j)_{j=1}^n$. Consider the continuous m-linear operator $(A_L)_n\colon X_{(1+\epsilon)_1} \times \cdots \times X_{(1+\epsilon)_m} \to Y$ given by

$$(A_L)_n\left(x^{(1)}, \dots, x^{(m)}\right) = \sum_{j=1}^n \sum_L x_j^{(1)} x_j^{(2)} \cdots x_j^{(m)} z_j^L. \tag{7}$$

By (6) and the Hölder inequality we have

$$\begin{aligned} \|(A_L)_n\| &= \sup_{\|x^{(1)}\|_{(1+\epsilon)_1} \leq 1, \dots, \|x^{(m)}\|_{(1+\epsilon)_m} \leq 1} \left\|\sum_{j=1}^n \sum_L x_j^{(1)} \dots x_j^{(m)} z_j^L\right\| \\ &\leq \sup_{\|x^{(1)}\|_{(1+\epsilon)_1} \leq 1, \dots, \|x^{(m)}\|_{(1+\epsilon)_m} \leq 1} (1+2\epsilon)\left(\sum_{j=1}^n \left|x_j^{(1)} \dots x_j^{(m)}\right|^{\cot Y}\right)^{1/\cot Y} \\ &\leq \sup_{\|x^{(1)}\|_{(1+\epsilon)_1} \leq 1, \dots, \|x^{(m)}\|_{(1+\epsilon)_m} \leq 1} (1+2\epsilon)\left(\prod_{k=1}^m \left(\sum_{j=1}^n \left|x_j^{(k)}\right|^{(1+\epsilon)_k}\right)^{1/(1+\epsilon)_k}\right) \\ &= (1+2\epsilon). \end{aligned} \tag{8}$$

Note that, by (7), we have

$$\left(\sum_{j_1=1}^n \left(\sum_{j_2=1}^n \cdots \left(\sum_{j_m=1}^n \sum_L \left\|(A_L)_n\left(e_{j_1}, \dots, e_{j_m}\right)\right\|^{(2+\epsilon)_m}\right)^{\frac{(2+\epsilon)_{m-1}}{(2+\epsilon)_m}} \cdots\right)^{\frac{(2+\epsilon)_1}{(2+\epsilon)_2}}\right)^{\frac{1}{(2+\epsilon)_1}}$$

$$= \left(\sum_{j=1}^{n} \sum_{L} \| z_j^L \|^{(2+\epsilon)_1} \right)^{\frac{1}{(2+\epsilon)_1}}.$$

Thus, by (6) we conclude that

$$\left(\sum_{j_1=1}^{n} \left(\sum_{j_2=1}^{n} \cdots \left(\sum_{j_m=1}^{n} \sum_{L} \| (A_L)_n (e_{j_1}, \dots, e_{j_m}) \|^{(2+\epsilon)_m} \right)^{\frac{(2+\epsilon)_{m-1}}{(2+\epsilon)_m}} \cdots \right)^{\frac{(2+\epsilon)_1}{(2+\epsilon)_2}} \right)^{\frac{1}{(2+\epsilon)_1}}$$

$$\geq (1+\epsilon) n^{\frac{1}{(2+\epsilon)_1}}$$

Combining the previous inequality with (4) and (8) we conclude that

$$(1+\epsilon) n^{1/(2+\epsilon)_1} \leq (1+\epsilon)^{Y}_{(1+\epsilon)_1, \dots, (1+\epsilon)_m} (1+2\epsilon).$$

Thus, since n is arbitrary, we have

$$(2+\epsilon)_1 = \infty = \lambda_{\cot Y}^{(1+\epsilon)_1, \dots, (1+\epsilon)_m}. \qquad (9)$$

If

$$\frac{1}{(1+\epsilon)_i} + \cdots + \frac{1}{(1+\epsilon)_m} \geq \frac{1}{\cot\, Y}$$

for all i, the proof is immediate. Otherwise, let $i_0 \in \{2,3, \dots, m\}$ be the smallest index such that

$$\begin{cases} \dfrac{1}{(1+\epsilon)_{i_0}} + \cdots + \dfrac{1}{(1+\epsilon)_m} < \dfrac{1}{\cot\, Y}, \\ \dfrac{1}{(1+\epsilon)_{i_0-1}} + \cdots + \dfrac{1}{(1+\epsilon)_m} \geq \dfrac{1}{\cot\, Y}. \end{cases}$$

If $i_0 = 2$, note that by (9) we have

$$\sup_{j_1} \left(\sum_{j_2=1}^{\infty} \left(\cdots \left(\sum_{j_m=1}^{\infty} \sum_{L} \| A_L (e_{j_1}, \dots, e_{j_m}) \|^{(2+\epsilon)_m} \right)^{\frac{(2+\epsilon)_{m-1}}{(2+\epsilon)_m}} \cdots \right)^{\frac{(2+\epsilon)_2}{(2+\epsilon)_3}} \right)^{\frac{1}{(2+\epsilon)_2}}$$

$$\leq (1+\epsilon)^{Y}_{(1+\epsilon)_1, \dots, (1+\epsilon)_m} \sum_{L} \| A_L \| \qquad (10)$$

for all continuous m-linear operators $A_L : X_{(1+\epsilon)_1} \times \cdots \times X_{(1+\epsilon)_m} \to Y$. From (10) it is simple to show that

$$\left(\sum_{j_2=1}^{\infty}\left(\cdots\left(\sum_{j_m=1}^{\infty}\sum_{L}\left\|A_L\left(e_{j_2},\ldots,e_{j_m}\right)\right\|^{(2+\epsilon)_m}\right)^{\frac{(2+\epsilon)_{m-1}}{(2+\epsilon)_m}}\cdots\right)^{\frac{(2+\epsilon)_2}{(2+\epsilon)_3}}\right)^{\frac{1}{(2+\epsilon)_2}}$$

$$\leq (1+\epsilon)^{Y}_{(1+\epsilon)_1,\ldots,(1+\epsilon)_m}\sum_{L}\|A_L\|,$$

for all continuous $(m-1)$-linear operators $A_L: X_{(1+\epsilon)_2}\times\cdots\times X_{(1+\epsilon)_m}\to Y$. Since

$$\frac{1}{(1+\epsilon)_2}+\cdots+\frac{1}{(1+\epsilon)_m}<\frac{1}{\cot Y},$$

by Theorem 2.1 we conclude that

$$(2+\epsilon)_2\geq\lambda^{(1+\epsilon)_2,\ldots,(1+\epsilon)_m}_{\cot Y},(2+\epsilon)_3\geq\lambda^{(1+\epsilon)_3,\ldots,(1+\epsilon)_m}_{\cot Y},\ldots,(2+\epsilon)_{m-1}\geq\lambda^{(1+\epsilon)_{m-1},(1+\epsilon)_m}_{\cot Y},(2+\epsilon)_m\geq\lambda^{(1+\epsilon)_m}_{\cot Y}.$$

If $i_0 = 3$, we consider

$$A_L\left(x^{(1)},\ldots,x^{(m)}\right)=x_1^{(1)}\sum_{j=1}^{n}\sum_{L}x_j^{(2)}\cdots x_j^{(m)}z_j^L$$

and we can imitate the previous arguments to conclude that

$$(2+\epsilon)_2=\infty=\lambda^{(1+\epsilon)_2,\ldots,(1+\epsilon)_m}_{\cot Y}.$$

and hence

$$\sup_{j_1,j_2}\left(\sum_{j_3=1}^{\infty}\left(\cdots\left(\sum_{j_m=1}^{\infty}\sum_{L}\left\|A_L\left(e_{j_1},\ldots,e_{j_m}\right)\right\|^{(2+\epsilon)_m}\right)^{\frac{(2+\epsilon)_{m-1}}{(2+\epsilon)_m}}\cdots\right)^{\frac{(2+\epsilon)_3}{(2+\epsilon)_4}}\right)^{\frac{1}{(2+\epsilon)_3}}$$

$$\leq (1+\epsilon)^{Y}_{(1+\epsilon)_1,\ldots,(1+\epsilon)_m}\sum_{L}\|A_L\|, \tag{11}$$

for all continuous m-linear operators $A_L: X_{(1+\epsilon)_1}\times\cdots\times X_{(1+\epsilon)_m}\to Y$. Again, it is plain that

$$\left(\sum_{j_3=1}^{\infty}\left(\cdots\left(\sum_{j_m=1}^{\infty}\sum_{L}\left\|A_L\left(e_{j_3},\ldots,e_{j_m}\right)\right\|^{(2+\epsilon)_m}\right)^{\frac{(2+\epsilon)_{m-1}}{(2+\epsilon)_m}}\cdots\right)^{\frac{(2+\epsilon)_{i_0+1}}{(2+\epsilon)_{i_0}}}\right)^{\frac{1}{(2+\epsilon)_{i_0}}}$$

$$\leq (1+\epsilon)^{Y}_{(1+\epsilon)_1,\ldots,(1+\epsilon)_m}\sum_{L}\|A_L\|$$

for all continuous $(m-2)$-linear operators $A_L: X_{(1+\epsilon)_3}\times\cdots\times X_{(1+\epsilon)_m}\to Y$. Since

$$\frac{1}{(1+\epsilon)_3}+\cdots+\frac{1}{(1+\epsilon)_m}<\frac{1}{\cot\ Y},$$

by Theorem 2.1 we have

$$(2+\epsilon)_3 \geq \lambda_{\cot Y}^{(1+\epsilon)_3,\ldots,(1+\epsilon)_m}, (2+\epsilon)_4 \geq \lambda_{\cot Y}^{(1+\epsilon)_4,\ldots,(1+\epsilon)_m}, \ldots, (2+\epsilon)_{m-1} \geq \lambda_{\cot Y}^{(1+\epsilon)_{m-1},(1+\epsilon)_m}, (2+\epsilon)_m \geq \lambda_{\cot Y}^{(1+\epsilon)_m}.$$

We conclude the proof in a similar fashion for $i_0 = 4, \ldots, m$.

Now we prove the reverse implication. The case

$$\frac{1}{(1+\epsilon)_1}+\cdots+\frac{1}{(1+\epsilon)_m}<\frac{1}{\cot Y},$$

is encompassed by Theorem 2.1. So, we shall consider

$$\frac{1}{(1+\epsilon)_1}+\cdots+\frac{1}{(1+\epsilon)_m}\geq\frac{1}{\cot Y}.$$

If

$$\frac{1}{(1+\epsilon)_i}+\cdots+\frac{1}{(1+\epsilon)_m}\geq\frac{1}{\cot\ Y}$$

for all i, the proof is immediate. Otherwise, let $i_0 \in \{2, \ldots, m\}$ be the smallest index such that

$$\begin{cases} \dfrac{1}{(1+\epsilon)_{i_0}}+\cdots+\dfrac{1}{(1+\epsilon)_m}<\dfrac{1}{\cot\ Y}, \\ \dfrac{1}{(1+\epsilon)_{i_0-1}}+\cdots+\dfrac{1}{(1+\epsilon)_m}\geq\dfrac{1}{\cot\ Y}. \end{cases}$$

We need to prove that there is a constant $(1+\epsilon)^Y_{(1+\epsilon)_1,\ldots,(1+\epsilon)_m} \geq 1$, such that

$$\sup_{j_1,\ldots,j_{i_0-1}}\left(\sum_{j_{i_0}=1}^{\infty}\left(\cdots\left(\sum_{j_m=1}^{\infty}\sum_{L}\left\|A_L(e_{j_1},\ldots,e_{j_m})\right\|^{(2+\epsilon)_m}\right)^{\frac{(2+\epsilon)_{m-1}}{(2+\epsilon)_m}}\cdots\right)^{\frac{(2+\epsilon)_{i_0}}{(2+\epsilon)_{i_0+1}}}\right)^{\frac{1}{(2+\epsilon)_{i_0}}} \leq (1+\epsilon)^Y_{(1+\epsilon)_1,\ldots,(1+\epsilon)_m}\sum_L\|A_L\|$$

for

$$(2+\epsilon)_{i_0} \geq \lambda_{\cot\ Y}^{(1+\epsilon)_{i_0},\ldots,(1+\epsilon)_m}, \ldots, (2+\epsilon)_m \geq \lambda_{\cot\ Y}^{(1+\epsilon)_m}$$

By Theorem 2.1, we know that for any fixed vectors $e_{j_1}, \ldots, e_{j_{i_0-1}}$, there is a constant $(1+\epsilon)^Y_{(1+\epsilon)_{i_0},\ldots,(1+\epsilon)_m} \geq 1$, such that

$$\left(\sum_{j_{i_0}=1}^{\infty}\left(\cdots\left(\sum_{j_m=1}^{\infty}\sum_{L}\left\|A_L\left(e_{j_1},\ldots,e_{j_m}\right)\right\|^{\lambda_{\cot Y}^{(1+\epsilon)_m}}\right)^{\frac{\lambda_{\cot Y}^{(1+\epsilon)_{m-1},(1+\epsilon)_m}}{\lambda_{\cot Y}^{(1+\epsilon)_m}}}\cdots\right)^{\frac{\lambda_{\cot Y}^{(1+\epsilon)_{i_0},\ldots,(1+\epsilon)_m}}{\lambda_{\cot Y}^{(1+\epsilon)_{i_0+1},\ldots,(1+\epsilon)_m}}}\right)^{\frac{1}{\lambda_{\cot Y}^{(1+\epsilon)_{i_0},\ldots,(1+\epsilon)_m}}}$$

$$\leq (1+\epsilon)^{Y}_{(1+\epsilon)_1,\ldots,(1+\epsilon)_m}\sum_{L}\|A_L\|$$

for all continuous m-linear operators $A\colon X_{(1+\epsilon)_1}\times\cdots\times X_{(1+\epsilon)_m}\to Y$. Then,

$$\sup_{j_1,\ldots,j_{i_0-1}}\left(\sum_{j_{i_0}=1}^{\infty}\left(\cdots\left(\sum_{j_m=1}^{\infty}\sum_{L}\left\|A_L\left(e_{j_1},\ldots,e_{j_m}\right)\right\|^{\lambda_{\cot Y}^{(1+\epsilon)_m}}\right)^{\frac{\lambda_{\cot Y}^{(1+\epsilon)_{m-1},(1+\epsilon)_m}}{\lambda_{\cot Y}^{(1+\epsilon)_m}}}\cdots\right)^{\frac{\lambda_{\cot Y}^{(1+\epsilon)_{i_0},\ldots,(1+\epsilon)_m}}{\lambda_{\cot Y}^{(1+\epsilon)_{i_0+1},\ldots,(1+\epsilon)_m}}}\right)^{\frac{1}{\lambda_{\cot Y}^{(1+\epsilon)_{i_0},\ldots,(1+\epsilon)_m}}}$$

$$\leq (1+\epsilon)^{Y}_{(1+\epsilon)_1,\ldots,(1+\epsilon)_m}\sum_{L}\|A_L\|$$

for all continuous m-linear operators $A_L\colon X_{(1+\epsilon)_1}\times\cdots\times X_{(1+\epsilon)_m}\to Y$.

To conclude the proof we just need to remark that

$$\sup_{j_1,\ldots,j_{i_0-1}}\left(\sum_{j_{i_0}=1}^{\infty}\left(\cdots\left(\sum_{j_m=1}^{\infty}\sum_{L}\left\|A_L\left(e_{j_1},\ldots,e_{j_m}\right)\right\|^{(2+\epsilon)_m}\right)^{\frac{(2+\epsilon)_{m-1}}{(2+\epsilon)_m}}\cdots\right)^{\frac{(2+\epsilon)_{i_0}}{(2+\epsilon)_{i_0+1}}}\right)^{\frac{1}{(2+\epsilon)_i}}$$

$$\leq \sup_{j_1,\ldots,j_{i_0-1}}\left(\sum_{j_{i_0}=1}^{\infty}\left(\cdots\left(\sum_{j_m=1}^{\infty}\sum_{L}\left\|A_L\left(e_{j_1},\ldots,e_{j_m}\right)\right\|^{\lambda_{\cot Y}^{(1+\epsilon)_m}}\right)^{\frac{\lambda_{\cot Y}^{(1+\epsilon)_{m-1},(1+\epsilon)_m}}{\lambda_{\cot Y}^{(1+\epsilon)_m}}}\cdots\right)^{\frac{\lambda_{\cot Y}^{(1+\epsilon)_{i_0},\ldots,(1+\epsilon)_m}}{\lambda_{\cot Y}^{(1+\epsilon)_{i_0+1},\ldots,(1+\epsilon)_m}}}\right)^{\frac{1}{\lambda_{\cot Y}^{(1+\epsilon)_{i_0},\ldots,(1+\epsilon)_m}}}$$

provided

$$(2+\epsilon)_{i_0}\geq\lambda_{\cot Y}^{(1+\epsilon)_{i_0},\ldots,(1+\epsilon)_m},\ldots,(2+\epsilon)_m\geq\lambda_{\cot Y}^{(1+\epsilon)_m}.$$

3. Proof of Theorem 1.1 (see [19])

Let the adjoint of a Banach space X be denoted by X^*. To simplify the notation we will consider $\sigma(j)=j$ for all j; the other cases are similar. Let $\mathcal{L}^m\left(X_{(1+\epsilon)_1},\ldots,X_{(1+\epsilon)_m};Y\right)$ denote the space of all continuous m-linear operators from $X_{p_1} \times \cdots \times X_{p_m}$ to Y. By the canonical isometric isomorphism

$$\Psi_L: \mathcal{L}^m\left(X_{(1+\epsilon)_1}, , X_{(1+\epsilon)_m}; \mathbb{K}\right) \to \mathcal{L}^{m-1}\left(X_{(1+\epsilon)_1}, \dots, X_{(1+\epsilon)_{m-1}}; \left(X_{(1+\epsilon)_m}\right)^*\right)$$

and duality in $X_{(1+\epsilon)_m}$, note that, if $R \in \mathcal{L}^m\left(X_{(1+\epsilon)_1}, \dots, X_{(1+\epsilon)_m}; \mathbb{K}\right)$, we have

$$R(x_1, \dots, x_{m-1}, e_n) = \Psi_L(R)(x_1, \dots, x_{m-1})(e_n) = \left(\Psi_L(R)(x_1, \dots, x_{m-1})\right)_n. \quad (12)$$

We start off by proving (1)⇒(2). Let us suppose that there is a constant $(1+\epsilon)_{(1+\epsilon)_1,\dots,(1+\epsilon)_m} \geq 1$ such that

$$\left(\sum_{j_1=1}^{\infty}\left(\sum_{j_2=1}^{\infty} \cdots \left(\sum_{j_m=1}^{\infty} \sum_L \left|T_L\left(e_{j_1}, \dots, e_{j_m}\right)\right|^{(1+\epsilon)_m^*}\right)^{\frac{\left(\frac{1+\epsilon}{\epsilon}\right)_{m-1}}{(1+\epsilon)_m^*}} \cdots\right)^{\frac{\left(\frac{1+\epsilon}{\epsilon}\right)_1}{\left(\frac{1+\epsilon}{\epsilon}\right)_2}}\right)^{\frac{1}{\left(\frac{1+\epsilon}{\epsilon}\right)_1}} \leq (1+\epsilon)_{(1+\epsilon)_1,\dots,(1+\epsilon)_m} \sum_L \|T_L\| \quad (13)$$

for all continuous m-linear forms $T_L: X_{(1+\epsilon)_1} \times \cdots \times X_{(1+\epsilon)_m} \to \mathbb{K}$

Consider a continuous $(m-1)$-linear operator $A_L: X_{(1+\epsilon)_1} \times \cdots \times X_{(1+\epsilon)_{m-1}} \to \left(X_{(1+\epsilon)_m}\right)^*$. Then, using our hypothesis, we have

$$\left(\sum_{j_1=1}^{\infty}\left(\sum_{j_2=1}^{\infty} \cdots \left(\sum_{j_{m-1}=1}^{\infty} \sum_L \left\|A_L\left(e_{j_1}, \dots, e_{j_{m-1}}\right)\right\|_{\left(X_{(1+\epsilon)_m}\right)^*}^{\left(\frac{1+\epsilon}{\epsilon}\right)_{m-1}}\right)^{\frac{\left(\frac{1+\epsilon}{\epsilon}\right)_{m-2}}{\left(\frac{1+\epsilon}{\epsilon}\right)_{m-1}}} \cdots\right)^{\frac{\left(\frac{1+\epsilon}{\epsilon}\right)_1}{\left(\frac{1+\epsilon}{\epsilon}\right)_2}}\right)^{\frac{1}{\left(\frac{1+\epsilon}{\epsilon}\right)_1}} \quad (14)$$

$$= \left(\sum_{j_1=1}^{\infty}\left(\sum_{j_2=1}^{\infty} \cdots \left(\sum_{j_{m-1}=1}^{\infty}\left(\sum_{j_m=1}^{\infty} \sum_L \left|\left(A_L\left(e_{j_1}, \dots, e_{j_{m-1}}\right)\right)_{j_m}\right|^{(1+\epsilon)_m^*}\right)^{\frac{\left(\frac{1+\epsilon}{\epsilon}\right)_{m-1}}{(1+\epsilon)_m^*}}\right)^{\frac{\left(\frac{1+\epsilon}{\epsilon}\right)_{m-2}}{\left(\frac{1+\epsilon}{\epsilon}\right)_{m-1}}}\right)^{\frac{\left(\frac{1+\epsilon}{\epsilon}\right)_1}{\left(\frac{1+\epsilon}{\epsilon}\right)_2}}\right)^{\frac{1}{\left(\frac{1+\epsilon}{\epsilon}\right)_1}}$$

$$\overset{(12)}{=} \sum_{j_1=1}^{\infty}\left(\sum_{j_2=1}^{\infty}\cdots\left(\sum_{j_{m-1}=1}^{\infty}\left(\sum_{j_m=1}^{\infty}\sum_{L}\left|\Psi_L^{-1}(A_L)\left(e_{j_1},\dots,e_{j_m}\right)\right|^{(1+\epsilon)_m^*}\right)^{\frac{\left(\frac{1+\epsilon}{\epsilon}\right)_{m-1}}{(1+\epsilon)_m^*}}\right)^{\frac{\left(\frac{1+\epsilon}{\epsilon}\right)_{m-2}}{\left(\frac{1+\epsilon}{\epsilon}\right)_{m-1}}}\cdots\right)^{\frac{\left(\frac{1+\epsilon}{\epsilon}\right)_{1}}{\left(\frac{1+\epsilon}{\epsilon}\right)_{2}}}\right)^{\frac{1}{\left(\frac{1+\epsilon}{\epsilon}\right)_{1}}}$$

$$\le (1+\epsilon)_{(1+\epsilon)_1,\dots,(1+\epsilon)_m}\sum_L \|\Psi_L^{-1}(A_L)\|$$

$$\le (1+\epsilon)_{(1+\epsilon)_1,\dots,(1+\epsilon)_m}\sum_L \|A_L\|$$

for all continuous $(m-1)$-linear operators $A_L: X_{(1+\epsilon)_1}\times\cdots\times X_{(1+\epsilon)_{m-1}} \to \left(X_{(1+\epsilon)_m}\right)^*$. Since $\left(X_{(1+\epsilon)_m}\right)^*$ has cotype $\max\{(1+\epsilon)_m^*, 2\}$, by Theorem 2.2, the exponents $\left(\frac{1+\epsilon}{\epsilon}\right)_1,\dots,\left(\frac{1+\epsilon}{\epsilon}\right)_{m-1}$ in (2.2) satisfy

$$\left(\frac{1+\epsilon}{\epsilon}\right)_1 \ge \lambda^{(1+\epsilon)_1,\dots,(1+\epsilon)_{m-1}}_{\max\{(1+\epsilon)_m^*,2\}},\left(\frac{1+\epsilon}{\epsilon}\right)_2 \ge \lambda^{(1+\epsilon)_2,\dots,(1+\epsilon)_{m-1}}_{\max\{(1+\epsilon)_m^*,2\}},\dots,\left(\frac{1+\epsilon}{\epsilon}\right)_{m-1} \ge \lambda^{(1+\epsilon)_{m-1}}_{\max\{(1+\epsilon)_m^*,2\}}. \quad (15)$$

Since

$$1-\frac{1}{\max\{(1+\epsilon)_m^*,2\}}=\frac{1}{\mu}$$

we have

$$\lambda^{(1+\epsilon)_i,\dots,(1+\epsilon)_{m-1}}_{\max\{(1+\epsilon)_m^*,2\}} = \frac{1}{\max\left\{\frac{1}{\max\{(1+\epsilon)_m^*,2\}}-\left(\frac{1}{(1+\epsilon)_i}+\cdots+\frac{1}{(1+\epsilon)_{m-1}}\right),0\right\}}$$

$$= \frac{1}{\max\left\{1-\left(\frac{1}{(1+\epsilon)_i}+\cdots+\frac{1}{(1+\epsilon)_{m-1}}+\frac{1}{\mu}\right),0\right\}}$$

$$= \delta^{(1+\epsilon)_i,\dots,(1+\epsilon)_{m-1},\mu}$$

for all $i\in\{1,\dots,m-1\}$. Then, (15) can be re-stated as

$$\left(\frac{1+\epsilon}{\epsilon}\right)_1 \ge \delta^{(1+\epsilon)_1,\dots,(1+\epsilon)_{m-1},\mu},\left(\frac{1+\epsilon}{\epsilon}\right)_2 \ge \delta^{(1+\epsilon)_2,\dots,(1+\epsilon)_{m-1},\mu},\dots,\left(\frac{1+\epsilon}{\epsilon}\right)_{m-1} \ge \delta^{(1+\epsilon)_{m-1},\mu}$$

and the proof is done.

(2)⇒(1). If the exponents $\left(\frac{1+\epsilon}{\epsilon}\right)_1,\dots,\left(\frac{1+\epsilon}{\epsilon}\right)_{m-1}$ satisfy

$$\left(\frac{1+\epsilon}{\epsilon}\right)_1 \ge \delta^{(1+\epsilon)_1,\dots,(1+\epsilon)_{m-1},\mu},\left(\frac{1+\epsilon}{\epsilon}\right)_2 \ge \delta^{(1+\epsilon)_2,\dots,(1+\epsilon)_{m-1},\mu},\dots,\left(\frac{1+\epsilon}{\epsilon}\right)_{m-1} \ge \delta^{(1+\epsilon)_{m-1},\mu},$$

we have, again, that the exponents $\left(\frac{1+\epsilon}{\epsilon}\right)_1,\dots,\left(\frac{1+\epsilon}{\epsilon}\right)_{m-1}$ satisfy

$$\left(\frac{1+\epsilon}{\epsilon}\right)_1 \geq \lambda_{2+\epsilon}^{(1+\epsilon)_1,\dots,(1+\epsilon)_{m-1}}, \left(\frac{1+\epsilon}{\epsilon}\right)_2 \geq \lambda_{2+\epsilon}^{(1+\epsilon)_2,\dots,(1+\epsilon)_{m-1}}, \dots, \left(\frac{1+\epsilon}{\epsilon}\right)_{m-1} \geq \lambda_{2+\epsilon}^{(1+\epsilon)_{m-1}},$$

with $(2+\epsilon) = \cot\left(X_{(1+\epsilon)_m}\right)^*$. Thus, by Theorem 2.2, there is a constant $(1+\epsilon)_{(1+\epsilon)_1,\dots,(1+\epsilon)_{m-1}}^{\left(X_{(1+\epsilon)_m}\right)^*} \geq 1$ such that

$$\left(\sum_{j_1=1}^{\infty}\left(\sum_{j_2=1}^{\infty}\cdots\left(\sum_{j_{m-1}=1}^{\infty}\sum_{L}\left\|T_L\left(e_{j_1},\dots,e_{j_{m-1}}\right)\right\|_{\left(X_{(1+\epsilon)_m}\right)^*}^{\left(\frac{1+\epsilon}{\epsilon}\right)_{m-1}}\right)^{\frac{\left(\frac{1+\epsilon}{\epsilon}\right)_{m-2}}{\left(\frac{1+\epsilon}{\epsilon}\right)_{m-1}}}\cdots\right)^{\frac{\left(\frac{1+\epsilon}{\epsilon}\right)_{1}}{\left(\frac{1+\epsilon}{\epsilon}\right)_{2}}}\right)^{\frac{1}{\left(\frac{1+\epsilon}{\epsilon}\right)_{1}}}$$

$$\leq (1+\epsilon)_{(1+\epsilon)_1,\dots,(1+\epsilon)_{m-1}}^{\left(X_{(1+\epsilon)_m}\right)^*}\sum_L\|T_L\|$$

for all continuous m-linear operators $T_L: X_{(1+\epsilon)_1} \times \cdots \times X_{(1+\epsilon)_{m-1}} \to \left(X_{(1+\epsilon)_m}\right)^*$.

We thus have

$$\left(\sum_{j_1=1}^{\infty}\left(\sum_{j_2=1}^{\infty}\cdots\left(\sum_{j_{m-1}=1}^{\infty}\left(\sum_{j_m=1}^{\infty}\sum_{L}\left|A_L\left(e_{j_1},\dots,e_{j_m}\right)\right|^{(1+\epsilon)_m^*}\right)^{\frac{\left(\frac{1+\epsilon}{\epsilon}\right)_{m-1}}{(1+\epsilon)_m^*}}\right)^{\frac{\left(\frac{1+\epsilon}{\epsilon}\right)_{m-2}}{\left(\frac{1+\epsilon}{\epsilon}\right)_{m-1}}}\cdots\right)^{\frac{\left(\frac{1+\epsilon}{\epsilon}\right)_{1}}{\left(\frac{1+\epsilon}{\epsilon}\right)_{2}}}\right)^{\frac{1}{\left(\frac{1+\epsilon}{\epsilon}\right)_{1}}}$$

$$= \left(\sum_{j_1=1}^{\infty}\left(\sum_{j_2=1}^{\infty}\cdots\left(\sum_{j_{m-1}=1}^{\infty}\sum_{L}\left\|\Psi_L(A_L)\left(e_{j_1},\dots,e_{j_m}\right)\right\|_{\left(X_{(1+\epsilon)_m}\right)^*}^{\left(\frac{1+\epsilon}{\epsilon}\right)_{m-1}}\right)^{\frac{\left(\frac{1+\epsilon}{\epsilon}\right)_{m-2}}{\left(\frac{1+\epsilon}{\epsilon}\right)_{m-1}}}\cdots\right)^{\frac{\left(\frac{1+\epsilon}{\epsilon}\right)_{1}}{\left(\frac{1+\epsilon}{\epsilon}\right)_{2}}}\right)^{\frac{1}{\left(\frac{1+\epsilon}{\epsilon}\right)_{1}}}$$

$$\leq (1+\epsilon)_{(1+\epsilon)_1,\dots,(1+\epsilon)_{m-1}}^{\left(X_{(1+\epsilon)_m}\right)^*}\sum_L\left\|\Psi_L(A_L)\right\|$$

$$= (1+\epsilon)_{(1+\epsilon)_1,\dots,(1+\epsilon)_{m-1}}\sum_L\|A_L\|$$

for all continuous m-linear forms $A_L: X_{(1+\epsilon)_1} \times \cdots \times X_{(1+\epsilon)_m} \to \mathbb{K}$.

Remark 3.1. [19] proved thathe determination of the exact values of the constants involved in the main theorem is probably a difficult task, as it happens with the Hardy-Littlewood inequalities (see [3,4] and the references therein). However when we are restricted to the bilinear case, with $(1+\epsilon)_1 = (1+\epsilon)_2 = \infty$ and σ as the identity map, it is not difficult to check that we recover the constant $\sqrt{2}$ from the Orlicz inequality.

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