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GAUSSIAN-HYBRID NUMBERS OBTAINED FROM PELL and PELL-LUCAS SEQUENCES

Abstract. In this study, we define a new type of Pell and Pell-Lucas numbers which are called Gaussian-hybrid Pell and Pell-Lucas numbers. We also give negaGaussian-hybrid Pell and Pell-Lucas numbers, the characteristic number and the type number of Gaussian-hybrid Pell and Pell-Lucas numbers. Also, some sum ve product properties of Pell and Pell-Lucas numbers are given. Moreover, we obtain the Binet's formula, generating function formula, d'Ocagne's identity, Catalan's identity, Cassini's identity and some sum formulas for the Gaussian-hybrid Pell and Pell-Lucas numbers. Some algebraic proporties of Gaussian-hybrid Pell and Pell-Lucas numbers are investigated. Futhermore, we give the matrix representation of Gaussian-hybrid Pell and Pell-Lucas numbers.
Keywords: Gaussian-hybrid number, Gaussian-hybrid Pell number, Gaussian-hybrid Pell-Lucas number.

1. Introduction

Complex numbers, Hyperbolic numbers and Dual numbers arise in many areas such as coordinate transformation, matrix modeling, displacement analysis, rigid body dynamics, velocity analysis, static analysis, dynamic analysis, transformation, mechanics, kinematics, physics, mathematics and geometry. Horadam[1] introduced the concept, the complex Fibonacci numbers, called the Gaussian Fibonacci numbers $GF_n = F_n + iF_{n-1}$ where $F_n \in \mathbb{R}, i^2 = -1$ and F_n, nth Fibonacci numbers. Fjelstad and Gal[2] defined the hyperbolic numbers $H = h + jh^*$ where $h, h^* \in \mathbb{R}, j^2 = 1$ and $j \neq \pm 1$. Clifford[3] described the dual numbers $D = d + \varepsilon d^*$ where $d, d^* \in \mathbb{R}, \varepsilon^2 = 0$ and $\varepsilon \neq 0$. Messelmi[4] expressed the dual-complex numbers $Z = z + \varepsilon z^*$ where $z, z^* \in \mathbb{C}, \varepsilon^2 = 0$ and $\varepsilon \neq 0$. Hybrid numbers are a new generalization of these numbers. Özdemir[5] presented the hybrid numbers $Z = a + ib + \varepsilon c + hd$, where $a, b, c, d \in \mathbb{R}$, $i^2 = -1$, $\varepsilon^2 = 0$ and $h^2 = 1$.

There are a number of studies in the literature that are concerned with these numbers[6-9]. Fjelstad and Gal[2] inspected the extensions of the hyperbolic complex numbers to n-dimensions and they gave n-dimensional dual complex numbers in algebra and analysis. Matsuda et al.[10] inspected the two-dimensional rigid transformation which is more concise and efficient than the standart matrix presentation, by modifying the ordinary dual number construction for the complex numbers. Majernik[11] gave three types of the four-component number systems which are formed by using the complex, binary and dual two-component numbers. Akar et al.[12] introduced arithmetical operations on dual-hyperbolic numbers. They investigated dual hyperbolic number and hyperbolic complex number valued functions. Soykan and Taşdemir[13] gave the generalized Tetranacci hybrid numbers. They presented Binet's formulas, generating functions and the summation formulas for those hybrid numbers. Catarino[14] presented a new sequence of numbers called *k*-Pell hybrid numbers and the presentation of some algebraic properties involving this sequence.

$\times$	1	i	ε	h
1	1	i	ε	h
i	i	-1	$1-h$	$\varepsilon+i$
ε	ε	$h+1$	0	$-\varepsilon$
h	h	$-\varepsilon-i$	ε	1

Table 1. Multiplication scheme of hybrid numbers

The conjugate of a hybrid number is defined by

$$\overline{Z}=\overline{x_1+ix_2+\varepsilon x_3+hx_4}=x_1-ix_2-\varepsilon x_3-hx_4$$

The real number

$$C(Z)=Z\overline{Z}=\overline{Z}Z=x_1^2+(x_2-x_3)^2-x_3^2-x_4^2$$

is called the characteristic number of *Z*.

The real number

$$\Delta(Z)=-(x_2-x_3)^2+x_3^2+x_4^2$$

is called the type number of *Z*. If $\Delta(Z)<0$, Z is elliptic; If $\Delta(Z)>0$, Z is hyperbolic and If $\Delta(Z)=0$, Z is parabolic.

The real number

$$\|Z\|=\sqrt{C(Z)}=\sqrt{|x_1^2+(x_2-x_3)^2-x_3^2-x_4^2|}$$

is called the norm of *Z*.

The inverse of *Z* is defined by

$$Z^{-1}=\frac{\overline{Z}}{C(Z)}$$

where $\|Z\| \neq 0$ [5]. For $n \in \mathbb{N}_0$, Pell and Pell-Lucas numbers are defined by the recurrence relations, respectively. $P_{n+2} = 2P_{n+1} + P_n$, $P_0 = 0$, $P_1 = 1$ and $Q_{n+2} = 2Q_{n+1} + Q_n$, $Q_0 = 2$, $Q_1 = 2$. Besides the nth Pell and Pell-Lucas number are formulized as $P_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$ and $Q_n = \alpha^n + \beta^n$, where $\alpha = 1 + \sqrt{2}$, $\beta = 1 - \sqrt{2}$. These formulas are called as Binet's formula[15, 16].

Many researchers studied several areas of this number sequence[17-19]. Halıcı and Çürük[20] examined the dual numbers and investigated the characteristic properties of them. They also gave equations about conjugates and some important features of these newly defined numbers. Azak and Güngör[21] defined the dual complex Fibonacci and Lucas numbers and gave the well-known properties for these numbers. Aydın[22] defined circular-hyperbolic Fibonacci numbers, circular-hyperbolic Fibonacci quaternions and also gave some algebraic properties of them. Szynal-Liana and Wloch[23] introduced the Jacobsthal and the Jacobsthal-Lucas numbers and presented some their properties.

The well-known number sequences have great importance as they are used in quantum physics, applied mathematics, kinematic, differential equations, cryptology and in many more areas. The new series of numbers can lead to the studies that will find application in the above-mentioned fields.

In the following sections, the Gaussian-hyperbolic Pell and Pell-Lucas numbers will be defined. In this work, a variety of algebraic properties of Gaussian-hyperbolic Pell and Pell-Lucas numbers are presented in a unified manner. Some identities will be given for Gaussian-hyperbolic Pell and Pell-Lucas numbers such as Binet's formula, generating function, d'Ocagne's identity, Catalan's identity, Cassini's identity and some sum formulas. The Gaussian-hyperbolic Pell and the Pell-Lucas numbers' properties will also be obtained using matrix representation.

2. Gaussian-Hybrid Pell and Pell-Lucas Numbers

Horadam [1] introduced the concept, the complex Fibonacci numbers, called the Gaussian Fibonacci numbers $GF_n = F_n + iF_{n-1}$ where $F_n \in \mathbb{R}, i^2 = -1$ and F_n, the nth Fibonacci numbers. In view of this definition, we will call Gaussian numbers as the numbers whose components are formed by ordering the consecutive terms of a number sequence from largest to smallest. After these explanations, we can give the following definition.

Definition 2.1. For $n \in \mathbb{N}_0$, the Gaussian-Hybrid Pell and Pell-Lucas numbers are defined by

$$GHP_{n+3} = P_{n+3} + iP_{n+2} + \varepsilon P_{n+1} + hP_n$$

$$GHQ_{n+3} = Q_{n+3} + iQ_{n+2} + \varepsilon Q_{n+1} + hQ_n$$

where P_n and Q_n, are the *nth* Pell and Pell-Lucas numbers. h, ε and i denotes ($h^2 = 1$), ($\varepsilon^2 = 0$) and ($i^2 = -1$).

$GHP_0 = i - 2\varepsilon + 5h$, $GHP_1 = 1 + \varepsilon - 2h$, $GHP_2 = 2 + i + h$, …

$GHQ_0 = 2 - 2i + 6\varepsilon - 14h$, $GHQ_1 = 2 + 2i - 2\varepsilon + 6h$, $GHQ_2 = 6 + 2i + 2\varepsilon - 2h$, …

Let GHQ_{n+3} and GHQ_{m+3} be two Gaussian-Hybrid Pell-Lucas numbers. The addition, substraction and multiplication of the Gaussian-Hybrid Pell-Lucas numbers are given by

$$\begin{aligned} GHQ_{n+3} \pm GHQ_{m+3} &= (Q_{n+3} \pm Q_{m+3}) + i(Q_{n+2} \pm Q_{m+2}) \\ &+ \varepsilon(Q_{n+1} \pm Q_{m+1}) + h(Q_n \pm Q_m) \end{aligned}$$

$$\begin{aligned} GHQ_{n+3} \times GHQ_{m+3} &= (Q_{n+3}Q_{m+3} - Q_{n+2}Q_{m+2} + Q_{n+2}Q_{m+1} + Q_{n+1}Q_{m+2} + Q_nQ_m) \\ &+ i(Q_{n+3}Q_{m+2} + Q_{n+2}Q_{m+3} + Q_{n+2}Q_m - Q_nQ_{m+2}) \\ &+ \varepsilon(Q_{n+3}Q_{m+1} + Q_{n+2}Q_m + Q_{n+1}Q_{m+3} - Q_nQ_{m+2} + Q_nQ_{m+1} - Q_{n+1}Q_m) \\ &+ h(Q_{n+3}Q_m - Q_{n+2}Q_{m+1} + Q_{n+1}Q_{m+2} + Q_nQ_{m+3}). \end{aligned}$$

Similarly, the properties for Gaussian-Hybrid Pell numbers is obtained.

Definition 2.2. For $n \in \mathbb{N}_0$, the negaGaussian-Hybrid Pell and the negaGaussian-Hybrid Pell-Lucas numbers are defined by

$$GHQ_{-n} = (-1)^n[Q_n - iQ_{n+1} + \varepsilon Q_{n+2} - hQ_{n+3}]$$

$$GHP_{-n} = (-1)^{n+1}[P_n - iP_{n+1} + \varepsilon P_{n+2} - hP_{n+3}]$$

where P_n and Q_n, are the *nth* Pell and Pell-Lucas numbers.

$$GHQ_{-n} = Q_{-n} + iQ_{-n-1} + \varepsilon Q_{-n-2} + hQ_{-n-3}$$

When the equality is established,

$$GHQ_{-n} = (-1)^nQ_n + i(-1)^{n+1}Q_{n+1} + \varepsilon(-1)^{n+2}Q_{n+2} + h(-1)^{n+3}Q_{n+3}$$

$$DGQ_{-n} = (-1)^n[Q_n - iQ_{n+1} + \varepsilon Q_{n+2} - hQ_{n+3}]$$

Similarly, GHP_{-n} is found.

Definition 2.3. Let GHP_n and GHQ_n be the Gaussian-Hybrid Pell numbers and the Gaussian-Hybrid Pell-Lucas numbers. The characteristic number of these numbers are as follows

$$C(GHP_n) = P_n^2 + (P_{n-1} - P_{n-2})^2 - P_{n-2}^2 - P_{n-3}^2$$

$$C(GHQ_n) = Q_n^2 + (Q_{n-1} - Q_{n-2})^2 - Q_{n-2}^2 - Q_{n-3}^2$$

where P_n and Q_n, are the *nth* Pell and Pell-Lucas numbers.

Definition 2.4. Let GHP_n and GHQ_n be the Gaussian-Hybrid Pell numbers and the Gaussian-Hybrid Pell-Lucas numbers. The type number of these numbers are as follows

$$\Delta(GHP_n) = -(P_{n-1} - P_{n-2})^2 + P_{n-2}^2 + P_{n-3}^2$$

$$\Delta(GHQ_n) = -(Q_{n-1} - Q_{n-2})^2 + Q_{n-2}^2 + Q_{n-3}^2$$

where P_n and Q_n, are the *nth* Pell and Pell-Lucas numbers.

Corollary 2.5. $\Delta(GHQ_n) = -(Q_{n-1} - Q_{n-2})^2 + Q_{n-2}^2 + Q_{n-3}^2$, when the equality is established, $\Delta(GHQ_n) = -(4P_{n-2})^2 + 8P_{2n-5}$.

a) If n<3, $\Delta(GHQ_n) > 0$ and GHQ_n is hyperbolic,

b) If n>2, $\Delta(GHQ_n) < 0$ and GHQ_n is elliptic.

A similar situation exists for the Gaussian-Hybrid Pell numbers.

Lemma 2.6. Let P_n and Q_n be the Pell and the Pell-Lucas numbers, respectively. The following relations are satisfied

$$Q_{n+1}^2 + Q_n^2 = 8P_{2n+1}$$

$$Q_{n+1}^2 - Q_n^2 = 8P_{2n+1} - 4(-1)^n$$

$$Q_{2n+2} + Q_{2n} = 8P_{2n+1}$$

$$Q_{2n+2} - Q_{2n} = 2Q_{2n+1}$$

$$Q_{n+1} - Q_n = 4P_n$$

$$Q_{n+r}Q_n = Q_{2n+r} + Q_r(-1)^n$$

$$Q_mQ_{n+r} + Q_{m+r}Q_n = 2Q_{m+n+r} + (-1)^n Q_{m-n}Q_r$$

$$Q_mQ_{n+r} - Q_{m+r}Q_n = (-8)(-1)^n P_{m-n}P_r$$

Proof: The proofs are carried out with the help of the Binet's formula.

Theorem 2.7. Let GHP_n and GHQ_n be the Gaussian-Hybrid Pell and the Gaussian-Hybrid Pell-Lucas numbers, respectively. The following relations are satisfied

- ✓ $2(GHP_{n+1}+GHP_n)=GHQ_{n+1}$
- ✓ $2(GHP_{n+1}-GHP_n)=GHQ_n$
- ✓ $GHP_{n+1}+GHP_{n-1}=GHQ_n$
- ✓ $GHP_{n+1}-GHP_{n-1}=2GHP_n$
- ✓ $GHP_{n+2}+GHP_{n-2}=6GHP_n$
- ✓ $GHP_{n+2}-GHP_{n-2}=2GHQ_n$
- ✓ $GHQ_{n+1}+GHQ_n=4GHP_{n+1}$
- ✓ $GHQ_{n+1}-GHQ_n=4GHP_n$
- ✓ $GHQ_{n+1}+GHQ_{n-1}=4GHP_n$
- ✓ $GHQ_{n+1}-GHQ_{n-1}=2GHQ_n$
- ✓ $GHQ_{n+2}+GHQ_{n-2}=6GHP_n$
- ✓ $GHQ_{n+2}-GHQ_{n-2}=16GHP_n$

Proof: By considering the definition 2.1., the theorem can be proved easily.

Theorem 2.8. (Generating Function Formula) Let GHQ_n be the Gaussian-Hybrid Pell-Lucas numbers. Generating function formula for this numbers is as follows

$$h(t)=\frac{(2-2i+6\varepsilon-14h)+t(-2+6i-14\varepsilon+34h)}{1-2t-t^2}.$$

Proof: Let $h(t)$ be the generating function for Gaussian-Hybrid Pell-Lucas numbers as

$h(t)=\sum_{n=0}^{\infty}GHQ_nt^n$. Using $h(t)$, $2th(t)$ and $t^2h(t)$, we get the following equations,

$th(t)=\sum_{n=0}^{\infty}GHQ_nt^{n+1}$, $t^2h(t)=\sum_{n=0}^{\infty}GHQ_nt^{n+2}$. After the needed calculations, the generating function for Gaussian-Hybrid Pell-Lucas numbers is obtained as

$$h(t)=\frac{GHQ_0+GHQ_1t-2GHQ_0t}{1-2t-t^2}$$

$$h(t)=\frac{(2-2i+6\varepsilon-14h)+t(-2+6i-14\varepsilon+34h)}{1-2t-t^2}.$$

Similarly, Generating function formula for Gaussian-Hybrid Pell numbers is obtained.

Theorem 2.9. (Binet's Formula) Let GHQ_n be the Gaussian-Hybrid Pell-Lucas numbers. Binet's formula for this numbers is as follows

$$GHQ_n=\hat{\alpha}\alpha^{n-3}+\hat{\beta}\beta^{n-3}$$

where $\hat{\alpha}=\alpha^3+i\alpha^2+\varepsilon\alpha^1+h$, $\alpha=1+\sqrt{2}$ and $\hat{\beta}=\beta^3+i\beta^2+\varepsilon\beta^1+h$, $\beta=1-\sqrt{2}$.

Proof:

$$GHQ_n = Q_n + iQ_{n-1} + \varepsilon Q_{n-2} + hQ_{n-3}$$

$$= (\alpha^n + \beta^n) + i(\alpha^{n-1} + \beta^{n-1}) + \varepsilon(\alpha^{n-2} + \beta^{n-2}) + h(\alpha^{n-3} + \beta^{n-3})$$

$$= \alpha^{n-3}(\alpha^3 + i\alpha^2 + \varepsilon\alpha^1 + h) + \beta^{n-3}(\beta^3 + i\beta^2 + \varepsilon\beta^1 + h)$$

$$GHQ_n = \hat{\alpha}\alpha^{n-3} + \hat{\beta}\beta^{n-3}.$$

Similarly, Binet's formula for Gaussian-Hybrid Pell numbers is obtained.

Theorem 2.10. (d'Ocagne's Identity) Let GHQ_n be the Gaussian-Hybrid Pell-Lucas numbers. d'Ocagne's identity for this numbers is as follows

$$\begin{aligned} GHQ_m GHQ_{n+1} - GHQ_{m+1} GHQ_n & \\ &= (-24)(-1)^n P_{m-n} + i[16(-1)^m(P_{n-m-1} + P_{n-m+1}) \\ &- 2(-1)^m(Q_{n-m-1} + Q_{n-m+1})] + \varepsilon[24(-1)^m(P_{n-m-1} + P_{n-m+1}) \\ &+ 2(-1)^m(Q_{n-m-1} + Q_{n-m+1})] + h[8(-1)^m(P_{n-m-1} + P_{n-m-3}) \\ &- 2(-1)^m(Q_{n-m+1} + Q_{n-m+3})]. \end{aligned}$$

Proof:

$$\begin{aligned} GHQ_m GHQ_{n+1} - GHQ_{m+1} GHQ_n & \\ &= (Q_m + iQ_{m-1} + \varepsilon Q_{m-2} + hQ_{m-3})(Q_{n+1} + iQ_n + \varepsilon Q_{n-1} + hQ_{n-2}) \\ &- (Q_{m+1} + iQ_m + \varepsilon Q_{m-1} + hQ_{m-2})(Q_n + iQ_{n-1} + \varepsilon Q_{n-2} + hQ_{n-3}) \\ &= (-24)(-1)^n P_{m-n} + i[16(-1)^m(P_{n-m-1} + P_{n-m+1}) \\ &- 2(-1)^m(Q_{n-m-1} + Q_{n-m+1})] + \varepsilon[24(-1)^m(P_{n-m-1} + P_{n-m+1}) \\ &+ 2(-1)^m(Q_{n-m-1} + Q_{n-m+1})] + h[8(-1)^m(P_{n-m-1} + P_{n-m-3}) \\ &- 2(-1)^m(Q_{n-m+1} + Q_{n-m+3})]. \end{aligned}$$

Similarly, d'Ocagne's identity for Gaussian-Hybrid Pell numbers is obtained.

Theorem 2.11. (Catalan's Identity) Let GHQ_n be the Gaussian-Hybrid Pell-Lucas numbers. Catalan's identity for this numbers is as follows

$$\begin{aligned} GHQ_n^2 - GHQ_{n+r} GHQ_{n-r} & \\ &= 8(-1)^n P_r^2 + 8(-1)^{n-r}[P_r P_{r-1} - P_r P_{r+1}] \\ &+ i[-4(-1)^n + 16(-1)^{n+r} P_{2r} + 2(-1)^{n+r} Q_{2r}] \\ &+ \varepsilon[4(-1)^n + 24(-1)^{n+r} P_{2r} - 2(-1)^{n+r} Q_{2r}] \\ &+ h[-28(-1)^n + 8(-1)^{n+r} P_{2r+2} - 2(-1)^{n+r} Q_{2r-2}]. \end{aligned}$$

Proof:

$$GHQ_n^2 - GHQ_{n+r}GHQ_{n-r}$$
$$= (Q_n + iQ_{n-1} + \varepsilon Q_{n-2} + hQ_{n-3})(Q_n + iQ_{n-1} + \varepsilon Q_{n-2} + hQ_{n-3})$$
$$- (Q_{n+r} + iQ_{n+r-1} + \varepsilon Q_{n+r-2} + hQ_{n+r-3})(Q_{n-r} + iQ_{n-r-1} + \varepsilon Q_{n-r-2}$$
$$+ hQ_{n-r-3})$$
$$= 8(-1)^n P_r^2 + 8(-1)^{n-r}[P_r P_{r-1} - P_r P_{r+1}]$$
$$+ i[-4(-1)^n + 16(-1)^{n+r}P_{2r} + 2(-1)^{n+r}Q_{2r}]$$
$$+ \varepsilon[4(-1)^n + 24(-1)^{n+r}P_{2r} - 2(-1)^{n+r}Q_{2r}]$$
$$+ h[-28(-1)^n + 8(-1)^{n+r}P_{2r+2} - 2(-1)^{n+r}Q_{2r-2}].$$

Similarly, Catalan's identity for Gaussian-Hybrid Pell numbers is obtained.

Theorem 2. 12. (Cassini's Identity) Let GHQ_n be the Gaussian-Hybrid Pell-Lucas numbers. Cassini's identity for this numbers is as follows

$$GHQ_n^2 - GHQ_{n+1}GHQ_{n-1} = 24(-1)^n + i[-48(-1)^n] + \varepsilon[-32(-1)^n] + h[-120(-1)^n].$$

Proof: If $r = 1$ is taken in the Catalan's identity, Cassini's identity is obtained. Similarly, Cassini's identity for Gaussian-Hybrid Pell numbers is obtained.

Theorem 2.13. Let GHQ_n be the Gaussian-Hybrid Pell-Lucas numbers. In this case

- ✓ $\sum_{k=1}^{n} GHQ_k = (2P_{n+1} - 2) + i(2P_n) + \varepsilon(2P_{n-1} - 2) + h(2P_{n-2} + 4)$
- ✓ $\sum_{k=1}^{n} GHQ_{2k-1} = \left(\frac{Q_{2n}-1}{2}\right) + i\left(\frac{Q_{2n-1}+3}{2}\right) + \varepsilon\left(\frac{Q_{2n-2}-5}{2}\right) + h\left(\frac{Q_{2n-3}+15}{2}\right)$
- ✓ $\sum_{k=1}^{n} GHQ_{2k} = \left(\frac{Q_{2n+1}-1}{2}\right) + i\left(\frac{Q_{2n}-1}{2}\right) + \varepsilon\left(\frac{Q_{2n-1}+3}{2}\right) + h\left(\frac{Q_{2n-2}-5}{2}\right)$

Proof:

$$\sum_{k=1}^{n} DGQ_k = \sum_{k=1}^{n}(Q_k + iQ_{k-1} + \varepsilon Q_{k-2} + hQ_{k-3})$$

$$= \sum_{k=1}^{n} Q_k + i\sum_{k=0}^{n-1} Q_k + \varepsilon\sum_{k=-1}^{n-2} Q_k + h\sum_{k=-2}^{n-3} Q_k$$

$$= (2P_{n+1} - 2) + i(2P_n) + \varepsilon(2P_{n-1} - 2) + h(2P_{n-2} + 4)$$

Other sums are proven through the same method. Similarly, Sums are proven for Gaussian-Hybrid Pell numbers is obtained.

Theorem 2.14. Let GHQ_n be the Gaussian-Hybrid Pell-Lucas numbers. For $n \geq 1$be integer. Then

$$\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} GHQ_2 & GHQ_1 \\ GHQ_1 & GHQ_0 \end{bmatrix} = \begin{bmatrix} GHQ_{n+2} & GHQ_{n+1} \\ GHQ_{n+1} & GHQ_n \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}^n \begin{bmatrix} GHQ_0 \\ GHQ_1 \end{bmatrix} = \begin{bmatrix} GHQ_n \\ GHQ_{n+1} \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}^n \begin{bmatrix} GHQ_2 & GHQ_1 \\ GHQ_1 & GHQ_0 \end{bmatrix} = \begin{bmatrix} GHQ_{-n+2} & GHQ_{-n+1} \\ GHQ_{-n+1} & GHQ_{-n} \end{bmatrix}$$

$$\begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} GHQ_0 \\ GHQ_1 \end{bmatrix} = \begin{bmatrix} GHQ_{-n} \\ GHQ_{-n+1} \end{bmatrix}$$

$$\begin{bmatrix} GHQ_1 & GHQ_0 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}^n = \begin{bmatrix} GHQ_{n+1} & GHQ_n \end{bmatrix}$$

$$\begin{bmatrix} GHQ_1 & GHQ_0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}^n = \begin{bmatrix} GHQ_{-n+1} & GHQ_{-n} \end{bmatrix}$$

Proof: The proof is seen by induction on n. Similarly, Matrix representations proven for Gaussian-Hybrid Pell numbers are obtained.

3. Conclusion

This study presents the Gaussian-hybrid Pell and Pell-Lucas numbers. We obtained these new numbers not defined in the literature before. These number sequences have great importance as they are used in quantum physics, applied mathematics, kinematic, differential equations and cryptology. Since this study includes some new results, it contributes to literature by providing essential information concerning these new numbers. Therefore, we hope that this new number system and properties that we have found will offer a new perspective to the researchers.

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