

Coupling the SBA method and the Elzaki transformation to solve nonlinear fractional differential equations

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Abstract

In this paper, we propose a new technique well adapted to the solution of nonlinear fractional differential equations. This technique combines the Elzaki transform and the Some Blaise-Abbo (SBA) method. It allows to find the exact solution or an acceptable approximate solution of the equation.

Keywords: SBA method, Elzaki transform, SBATEM, Fractional differential equation, Caputo fractional derivative.

AMS Subject Classification: 26A33, 35R11, 34A08

1 Introduction

Fractional derivatives are used in the modeling of many physical phenomena, such as heat diffusion through a semi-infinite solid, flow in oil reservoirs, rheological properties of solids etc. In general, it is difficult to find the exact solution of a nonlinear fractional differential equation. Many numerical methods are used to find an approximate solution. Commonly used numerical methods are the variational iteration method (VIM) [18, 19], the Adomian decomposition method (ADM) [4, 5, 7], and the generalized differential transformation method (GDTM) [3, 6]. Recently, the SBA method [9, 20, 21] which is a combination of the Adomian method, the method of successive approximations [14, 24] and the Picard principle, is also used. The nonlinear fractional differential equations are also solved with techniques combining numerical methods with integral transformations, such as the Homotopy perturbation method combined with the Elzaki transformation (EHTPM) [12, 13], the Homotopy perturbation method combined with the Sumudu transformation (HPSTM) [16], the Adomian decomposition method combined with the Elzaki transformation (EADM) [17]. The discretization methods are also used [22, 23]. In this paper, we propose a new technique to find the exact solution or an approximate solution of nonlinear fractional differential equations. This technique is a combination of the SBA method and the Elzaki transform (SBATEM). After having recalled some notions on fractional calculus and on the Elzaki transform, we will give the principle of this new technique, then we will apply it on some examples of nonlinear fractional differential equations.

2 Definitions and basic properties

In this section, we recall some definitions and properties of fractional calculus and the Elzaki transformation.

1

2.1 Gamma function and Mittag-Leffler function

Gamma function The Euler Gamma function [1, 11] is defined on the half-plane $P = \{z \in \mathbb{C} \mid \text{Re}(z) > 0\}$ by

$$\Gamma(z) =$$

$$\int_0^{\infty} t^{z-1} e^{-t} dt; \quad (1)$$

$$\int_0^{\infty} t^{z-1} e^{-t} dt; \quad (1)$$

For any natural number n : $\Gamma(n+1) = n!$:

Mittag-Leffler function For any complex number z ; we define the one-parameter Mittag-Leffler function [2, 11] by

$$E_{\alpha}(z) =$$

$$\sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}$$

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$$\sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}; \quad \alpha \in \mathbb{C}; \text{Re}(\alpha) > 0; \quad (2)$$

In particular, when $\alpha = 1$; this function coincides with the exponential function:

$$E_1(z) =$$

$$\sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k + 1)}$$

$$\sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k + 1)}$$

$$z^k$$

$$\int_0^x (k+1)$$

$$=$$

$$\int_{k=0}^1 x$$

$$z^k$$

$$k!$$

$$= e^z: (3)$$

2.2 Caputo fractional derivative

Definition 2.1 Let $[a; b]$ be a finite interval of \mathbb{R} and $f \in L_1([a; b])$. The fractional Riemann-Liouville left-handed integral of order $\alpha > 0$ of the function f is defined by [2]

$$I_{a+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt: (4)$$

Definition 2.2 The fractional Caputo left derivative of order $\alpha > 0$ of the function $f(x)$, $x \in [a; b]$ is defined by [15]

$${}^C D_{a+}^\alpha f(x) = I_{a+}^{m-\alpha} f^{(m)}(x)$$

$$=$$

$$\frac{1}{\Gamma(m-\alpha)} \int_a^x (x-t)^{m-\alpha-1} f^{(m)}(t) dt: (5)$$

where $m = [\alpha] + 1$ if $\alpha \neq 2N$ and $m = \alpha$ if $\alpha = 2N$.

2.3 Elzaki Transform

Consider the following set of functions of exponential order

$$A = \{f(t) : \exists M; k_1; k_2 > 0; |f(t)| < M e^{k_1 t}; \text{ if } t \in [0; 1) \}$$

$$: (6)$$

Definition 2.3 For $f \in A$; the Elzaki transform of f is given by the following formula [8]

$$E[f(t)] = T(s) = \int_0^1 e^{-st} f(t) dt; k_1 \leq s \leq k_2 (7)$$

From the formula (7), we obtain the following Elzaki transforms:

$$E[1] = s^{-2}; E[t] = s^{-3}; E[t^n] = \frac{n!}{s^{n+2}}; n > 0 (8)$$

The Elzaki transform verifies the linearity property: If $f; g \in A$ and $a; b \in \mathbb{R}$, we have

$$E[af(t) + bg(t)] = aE[f(t)] + bE[g(t)] (9)$$

Theorem 2.1 The Elzaki transform of the fractional Caputo derivative is [10]:

$$E[D_{a+}^\alpha f(t)] = s^{-\alpha} E[f(t)]$$

$$\int_{k=0}^m x^{\alpha-1}$$

$$s^2 \square + kf(k)(0); m \square 1 < _ _ m \quad (10)$$

Theorem 2.2 [8] Let $T(u)$ be the Elzaki transform of $f(t)$ such that

(i) sT

$$\frac{1}{s}$$

$$s$$

is a meromorphic function, with singularities having $\text{Re}(s) < _;$ and

(ii) There exists a circular region d with radius R and positive constants, M and K with

$$sT$$

$$\frac{1}{s}$$

$$s$$

$$< MR \square K \quad (11)$$

Then the function $f(t)$ is given by

$$E \square 1 [T(s)] =$$

$$\frac{1}{s}$$

$$\frac{2_i}{Z _ + i:1}$$

$$\frac{_ \square i:1}{e^{st} s T}$$

$$e^{st} s T$$

$$\frac{1}{s}$$

$$s$$

$$ds =$$

$$X$$

residues of

$$e^{st} s T$$

$$\frac{1}{s}$$

$$s$$

$$(12)$$

3 Description of the SBATEM technique

Consider the following nonlinear and inhomogeneous fractional differential equation

$$D _$$

$${}^c u(x; t) = Lu(x; t) + Nu(x; t) + g(x; t); _ > 0; \quad (13)$$

with the initial conditions:

$$u(x; 0) = h_0(x);$$

$${}^c_k u(x; 0)$$

$${}^c_{tk} = h_k(x); k \geq 1; \dots; m \square 1g; \quad (14)$$

where $D _$

$$t =$$

$${}^c _$$

${}^c _$ is the fractional derivative of Caputo with respect to t of order $_ > 0$; L

and N are linear and nonlinear differential operators, respectively.

Applying the Elzaki transform to (13), we obtain

$$E [D _$$

$${}^c u(x; t)] = E [Lu(x; t)] + E [Nu(x; t)] + E [g(x; t)] \quad (15)$$

Using Theorem 1 and the initial conditions (14), we obtain

$$E [u(x; t)] =$$

$$mX \square 1$$

$$\sum_{k=0}^{2+k} h_k(x) + s_E [g(x; t)] + s_E [Lu(x; t)] + s_E [Nu(x; t)] : (16)$$

Applying the inverse Elzaki transform to (16), we obtain

$$u(x; t) = H(x; t) + E_{\square 1} [s_E [Lu(x; t)]] + E_{\square 1} [s_E [Nu(x; t)]]; (17)$$

where

$$H(x; t) = E_{\square 1}$$

"

$$\sum_{k=0}^{m} X_{\square 1}$$

$$\sum_{k=0}^{2+k} h_k(x)$$

$$\#$$

$$+ E_{\square 1} [s_E [g(x; t)]]; (18)$$

Apply the method of successive approximations to (17), we get:

$$u_k(x; t) = H(x; t) + E_{\square 1}$$

$$s_E$$

$$\bar{L}u_k(x; t)$$

$$+ E_{\square 1}$$

$$s_E$$

$$\bar{N}u_{k\square 1}(x; t)$$

$$: (19)$$

The Adomian algorithm associated with (19) is the following:

(

$$u_{k0}$$

$$(x; t) = H(x; t) + E_{\square 1}$$

$$s_E$$

$$\bar{N}u_{k\square 1}(x; t)$$

$$; k _ 1$$

$$u_k$$

$$n(x; t) = E_{\square 1}$$

$$s_E$$

$$\bar{L}u_k$$

$$n_{\square 1}(x; t)$$

$$; n _ 1:$$

$$(20)$$

We will call the above algorithm the SBATEM algorithm.

Let us apply Picard's principle to (20): we choose $u_0 \in V$ any root of the equation $Nu = 0$:

Step 1. For $k = 1$, we compute u_1 using the following algorithm

(

$$u_{10}$$

$$(x; t) = H(x; t)$$

$$u_1$$

$$n(x; t) = E_{\square 1}$$

$$s_E$$

-

$$Lu_1 = n_{\square 1}(x; t)$$

$$; n_{\square 1}: \quad (21)$$

If the series

$$P_{n_{\square 0}} u_1$$

is convergent, then we get:

$$u_1 =$$

$$X_{n_{\square 0}} u_1$$

$$n; \quad (22)$$

approximate solution of the problem (13)-(14) in step 1.

Step 2. For $k = 2$; we compute u_2 using the following algorithm:

(

$$u_{20}$$

$$(x; t) = H(x; t) + E_{\square 1}$$

$$\bar{s}_E$$

$$\bar{N}u_1(x; t)$$

$$u_2$$

$$n(x; t) = E_{\square 1}$$

$$\bar{s}_E$$

$$\bar{L}u_2$$

$$n_{\square 1}(x; t)$$

$$; n_{\square 1}: \quad (23)$$

$$(23)$$

If the series

$$P_{n_{\square 0}} u_2$$

is convergent, then we get:

$$u_2 =$$

$$X_{n_{\square 0}} u_2$$

$$n; \quad (24)$$

approximate solution of the problem (13)-(14) in step 2.

Step k. Recursively, if the series

$$P_{n_{\square 0}} u_k$$

is convergent for $k_{\square 1}$; then we get:

$$u_k =$$

$$X_{n_{\square 0}} u_k$$

$$n; \quad (25)$$

approximate solution of the problem (13)-(14) in step k: The solution of the problem (13)-(14) is then:

$$u = \lim_{k \rightarrow \infty} u_k \quad (26)$$

Proposition 3.1 Consider the following nonlinear and inhomogeneous fractional differential equation:

$$D_{\alpha} u(x; t) = Lu(x; t) + Nu(x; t) + g(x; t); \quad \alpha > 0 \quad (27)$$

with the initial conditions:

$$u(x; 0) = h_0(x);$$

$${}_{@k}u(x; 0)$$

$${}_{@k}t_k = h_k(x); \quad k = 2, 1, \dots, m \quad (28)$$

where D_{α}

$${}_{@}t_{\alpha}$$

$${}_{@}t_{\alpha}$$

$${}_{@}t_{\alpha}$$

where D_{α}

$${}_{@}t_{\alpha}$$

$${}_{@}t_{\alpha}$$

${}_{@}t_{\alpha}$ is the fractional derivative of Caputo with respect to t of order $\alpha > 0$; L is

a linear operator and N a nonlinear operator defined in a suitably chosen space V ; $g \in V$ and u

the unknown function.

Let be the SBATEM algorithm associated to (27)-(28) :

(

$$u_{k0}$$

$$u_{k+1}(x; t) = H(x; t) + E_{\alpha}^{-1}$$

$$s_{\alpha} E$$

$$N_{k+1}(x; t)$$

$$; k = 1$$

$$u_k$$

$$u_{k+1}(x; t) = E_{\alpha}^{-1}$$

$$s_{\alpha} E$$

$$L u_k$$

$$u_{k+1}(x; t)$$

$$; n = 1:$$

$$(29)$$

By Picard's principle, we choose $u_0 \in V$ such that $Nu_0 = 0$:

(a) If $Nu_1 = 0$; then the problem (27)-(28) admits a unique solution $u = u_1$:

(b) If for a fixed integer p , $u_p = u_{p+1}$; $p \geq 2$; then the problem (27)-(28) admits a unique solution $u = u_{p+1}$:

Proof.

Existence

(a) Let u_1 be the approximate solution in step 1. Assume that $Nu_1 = 0$; so the scheme in step 2 is written:

(

$$u_{20}$$

$$u_{2+1}(x; t) = H(x; t)$$

$$u_2$$

$$u_{2+1}(x; t) = E_{\alpha}^{-1}$$

$$s_{\alpha} E$$

$$L u_2$$

$$u_{2+1}(x; t)$$

n_{k-1} :

(30)

This scheme is identical to the scheme in step 1. So the approximate solution in step 2 is $u_2 = u_1$:

We have $Nu_2 = Nu_1 = 0$; therefore the scheme at step 3 is also identical to the scheme at step 2. Therefore, the solution at

step 3 is $u_3 = u_2 = u_1$:

Recursively, the approximate solution at step k ($k \geq 2$) is $u_k = u_{k-1} = \dots = u_1$:

The solution of the problem (27)-(28) is

$u = \lim$

$k \rightarrow \infty$

$u_k = u_1$ (31)

(b) Suppose that for a fixed integer p ; $u_p = u_{p-1}$; $p \geq 2$; then we have $Nu_p = Nu_{p-1}$:

At step $p + 1$; the algorithm is written:

(

u_{p+1}

$u_{p+1}(x; t) = H(x; t) + E_{p+1} [s_E [Nu_p(x; t)]]$

u_{p+1}

$u_{p+1}(x; t) = E_{p+1}$

h

s_E

h

Lu_{p+1}

$u_{p+1}(x; t)$

ii

n_{k-1} :

(32)

From this algorithm, we obtain:

u_{p+1}

$u_{p+1}(x; t) = H(x; t) + E_{p+1} [s_E [Nu_p(x; t)]] = H(x; t) + E_{p+1}$

s_E

h

h

$Nu_{p+1}(x; t)$

h

u_p

$u_p(x; t)$;

u_{p+1}

$u_{p+1}(x; t) = E_{p+1}$

h

s_E

h

Lu_{p+1}

$u_{p+1}(x; t)$

ii

$= E_{p+1} [s_E [Lu_p$

$u_p(x; t)]] = u_p$

$u_p(x; t)$;

u_{p+1}

$u_{p+1}(x; t) = E_{p+1}$

h

s_E

h

Lu_{p+1}

$u_{p+1}(x; t)$

$$\begin{aligned}
 & \text{ii} \\
 & = E_{\square 1} [s_E [Lu_p \\
 & 1(x; t)]] = u_p \\
 & 2(x; t); \\
 & \vdots \\
 & 5 \\
 & u_{p+1} \\
 & n(x; t) = E_{\square 1} \\
 & h \\
 & s_E \\
 & h \\
 & Lu_{p+1} \\
 & n_{\square 1}(x; t) \\
 & \text{ii} \\
 & = E_{\square 1}
 \end{aligned}$$

$$s_E$$

$$Lu_p$$

$$n_{\square 1}(x; t)$$

$$= u_p$$

$$n(x; t):$$

$$\text{So}$$

$$u_{p+1} =$$

$$X$$

$$n_0$$

$$u_{p+1}$$

$$n =$$

$$X$$

$$n_0$$

$$u_p$$

$$n = u$$

$$p$$

Similarly, we show that at step $p + 2$; $u_{p+2} = u_{p+1}$:
 Recursively, the approximate solution at step k ($k \geq p + 1$) is $u_k = u_{k-1} = \dots = u_p = u_{p+1}$:
 The solution of the problem (27)-(28) is thus

$$u = \lim_{k \rightarrow \infty} u_k = u_{p+1} \quad (34)$$

Uniqueness suppose that the problem (27)-(28) admits by the SBA method two distinct solutions u and v : Let $w = u - v$. Then we have:

$$(u - v)_k(x; t) = H(x; t) + E_{\square 1}$$

$$s_E$$

$$Nu_{k-1}(x; t)$$

$$; k \geq 1$$

$$u_k$$

$$n(x; t) = E_{\square 1}$$

$$s_E$$

$$-$$

$L_{n-1}^k(x; t)$

(35)

and

$$v_k(x; t) = H(x; t) + E_{n-1}$$

s_{n-1}

$N_{k-1}^k(x; t)$

(36)

$$v_{kn}(x; t) = E_{n-1}$$

s_{n-1}

$L_{k-1}^k(x; t)$

(36)

Making the difference (35)-(36), we get:

$$v_k(x; t) - v_{kn}(x; t) = E_{n-1}$$

s_{n-1}

$N_{k-1}^k(x; t)$

E_{n-1}

s_{n-1}

$N_{k-1}^k(x; t)$

(37)

$$v_k(x; t) - v_{kn}(x; t) = E_{n-1}$$

s_{n-1}

$L_{k-1}^k(x; t)$

(37)

where v_k

$$v_k = U_k$$

$$v_k = V_{kn}$$

:

Step 1. For $k = 1$; we have:

$$\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} = 0$$

$$n = E_{n \times 1}$$

$$s_E$$

$$\bar{L}'_1(x; t)$$

$$; n = 1: \quad (38)$$

- For $n = 1$; we have:

$$\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} = E_{n \times 1}$$

$$s_E$$

$$\bar{L}'_{10}(x; t)$$

$$= 0: (39)$$

- For $n = 2$; we have:

$$\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} = E_{n \times 1}$$

$$s_E$$

$$\bar{L}'_{11}(x; t)$$

$$= 0: (40)$$

- We find that for all $n \geq 0$; $u_1 = v_1$

$n = 0$. Therefore, we have:

$$u_1 = v_1$$

$$n = 0: (41)$$

Therefore, we obtain $u_1 = v_1$:

Step 2. For $k = 2$; we have:

$$\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} = L_{n \times 1}$$

$$tNu_1 = L_{n \times 1}$$

$$tNv_1$$

$$= L_{n \times 1}$$

$$tR'(2)$$

$$n \times 1; n = 1: \quad (42)$$

6

Since $u_1 = v_1$; then $Nu_1 = Nv_1$: As a result, the scheme (42) is written

$$\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} = 0$$

$$= 0$$

$$n = L_{n \times 1}$$

$$t R('2$$

$$n_{\square 1}); n_{\square 1};$$

$$(43)$$

This scheme is identical to the scheme in step 1; thus for all $n_{\square 0}; '2$
 $n = 0$: Hence

$$'2 =$$

$$X$$

$$n_{\square 0}$$

$$'2$$

$$n = 0: (44)$$

Therefore, we get $u_2 = v_2$:

Recursively, for all $k_{\square 1}; u_k = v_k$. Therefore $u = v$; which is absurd. So the problem
 (27)-(28) admits a unique solution.

4 Applications

In this section, we apply the SBATEM technique to solve four examples of nonlinear fractional
 differential equations.

Example 1. Consider the following nonlinear fractional partial differential equation

$$D_{\square} u \square 3 (u_x)^2 + u_{xxx} = 0; t > 0; 0 < _ _ 1 (45)$$

with the initial condition

$$u(x; 0) = 6x: (46)$$

We have: $Lu(x; t) = \square u_{xxx}(x; t)$; $Nu(x; t) = 3 (u_x(x; t))^2$ and $g(x; t) = 0$:

The SBATEM algorithm associated to the problem (45)-(46) is

$$($$

$$u_{k0}$$

$$(x; t) = H(x; t) + E_{\square 1}$$

$$_ s_E$$

$$_ Nu_{k\square 1}(x; t)$$

$$; k_{\square 1}$$

$$u_k$$

$$n(x; t) = E_{\square 1}$$

$$_ s_E$$

$$_ Lu_k$$

$$n_{\square 1}(x; t)$$

$$; n_{\square 1}$$

$$(47)$$

with

$$H(x; t) = E_{\square 1}$$

"

$$mX_{\square 1}$$

$$k=0$$

$$s_{2+khk}(x)$$

#

$$+ E_{\square 1} [s_E [g(x; t)]] (48)$$

$$= E_{\square 1} _$$

$$s_{2ho}(x)$$

$$= h_0(x) = 6x:$$

The algorithm (47) is again written

(

$$u_{k0}$$

$$(x; t) = 6x + E_{\square 1}$$

\bar{s}_E

$$\bar{N}u_{k\square 1}(x; t)$$

$\bar{; k_1}$

$$u_k$$

$$n(x; t) = E_{\square 1}$$

\bar{s}_E

$$\bar{L}u_k$$

$$n\square 1(x; t)$$

$\bar{; n_1:$

(49)

Let us apply to (49) Picard's principle: we take $u_0 = 0$; then $Nu_0 = 0$:

Step 1. For $k = 1$, we compute u_1 using the following algorithm

(

$$u_{10}$$

$$(x; t) = 6x$$

$$u_1$$

$$n(x; t) = E_{\square 1}$$

\bar{s}_E

$$\bar{L}u_1$$

$$n\square 1(x; t)$$

$\bar{; n_1:$

(50)

We have:

7

8>>>><

>>>>:

$$u_{11}$$

$$(x; t) = E_{\square 1}$$

\bar{s}_E

$$\bar{L}u_{10}$$

$$(x; t)$$

$\bar{= 0}$

$$u_{12}$$

$$(x; t) = E_{\square 1}$$

\bar{s}_E

$$\bar{L}u_{11}$$

$$(x; t)$$

$\bar{= 0}$

...

$$u_1$$

$$n(x; t) = E_{01}$$

\bar{s}_E

$$\bar{L}u_1$$

$$n_{01}(x; t)$$

$$= 0; 8 n_{-1}: \quad (51)$$

So

$$u_1(x; t) =$$

$$X$$

$$n_{-0}$$

$$u_1$$

$$n(x; t) = u_{10}$$

$$(x; t) = 6x \quad (52)$$

is approximate solution of the problem (45)-(46) in step 1.

Step 2. For $k = 2$; we compute u_2 using the following algorithm:

$$($$

$$u_{20}$$

$$(x; t) = 6x + E_{01}$$

\bar{s}_E

$$\bar{N}u_1(x; t)$$

$$\bar{u}_2$$

$$n(x; t) = E_{01}$$

\bar{s}_E

$$\bar{L}u_2$$

$$n_{01}(x; t)$$

$$; n_{-1}: \quad (53)$$

We have:

$$\bar{N}u_1(x; t) = 3(6)^2 = 108 \quad (54)$$

and

$$8 >>>>>>>>><$$

$$>>>>>>>>:$$

$$u_{20}$$

$$(x; t) = 6x + E_{01}$$

\bar{s}_E

$$\bar{N}u_1(x; t)$$

$$= 6x +$$

$$108t_{-}$$

$$\square(_ + 1)$$

$$u_{21}$$

$$(x; t) = E_{01}$$

\bar{s}_E

$$\bar{L}u_{20}$$

$$(x; t)$$

$$= 0$$

$$u_{22}$$

$$(x; t) = E_{\square 1}$$

$$\bar{s}_E$$

$$\bar{L}u_{21}$$

$$(x; t)$$

$$= 0$$

...

$$u_2$$

$$n(x; t) = E_{\square 1}$$

$$\bar{s}_E$$

$$\bar{L}u_2$$

$$n_{\square 1}(x; t)$$

$$= 0; 8 n_{\square 1}$$

(55)

So

$$u_2(x; t) =$$

X

$n_{\square 0}$

u_2

$$n(x; t) = u_{20}$$

$$(x; t) = 6x +$$

$$108t_{\square}$$

$$\square(\square + 1)$$

(56)

is approximate solution of the problem (45)-(46) in step 2.

Step 3. For $k = 3$; we compute u_3 using the following algorithm:

(

u_{30}

$$(x; t) = 6x + E_{\square 1}$$

$$\bar{s}_E$$

$$\bar{N}u_2(x; t)$$

u_3

$$n(x; t) = E_{\square 1}$$

$$\bar{s}_E$$

$\bar{L}u_3$

$$n_{\square 1}(x; t)$$

; $n_{\square 1}$

(57)

We have:

$$Nu_2(x; t) = 3(6)^2 = 108 \quad (58)$$

and

8

8>>>>>>>>><

>>>>>>>>:

u_{30}

$$(x; t) = 6x + E_{\square 1}$$

\bar{s}_E

$$\bar{Nu}_2(x; t)$$

$$= 6x +$$

$$108t$$

$$\square(_ + 1)$$

u_{31}

$$(x; t) = E_{\square 1}$$

\bar{s}_E

$$\bar{Lu}_{30}$$

$$(x; t)$$

$$= 0$$

u_{32}

$$(x; t) = E_{\square 1}$$

\bar{s}_E

$$\bar{Lu}_{31}$$

$$(x; t)$$

$$= 0$$

...

u_3

$$n(x; t) = E_{\square 1}$$

\bar{s}_E

$$\bar{Lu}_3$$

$$n_{\square 1}(x; t)$$

$$= 0; 8 n_{\square 1}:$$

$$(59)$$

So

$$u_3(x; t) =$$

X

$n_{\square 0}$

u_3

$$n(x; t) = u_{30}$$

$$(x; t) = 6x +$$

$$108t$$

$$\square(_ + 1)$$

$$(60)$$

We have $u_3 = u_2$; so by Proposition 1 (b), the exact solution of the problem (45)-(46) is:

$$u(x; t) = u_2(x; t) = 6x +$$

108t_

$\square(_ + 1)$

: (61)

Example 2. Consider the following nonlinear diffusion problem:

(

D_

$$t u = k u_{xx} + u^3 + (u_{xx})^3$$

$$u(x; 0) = \sin x$$

(62)

where $0 < _ < 1$; $x \in \mathbb{R}$ and $t > 0$:

We have: $Lu(x; t) = k u_{xx}(x; t)$; $Nu(x; t) = (u(x; t))^3 + (u_{xx}(x; t))^3$ and $g(x; t) = 0$:

The SBATEM algorithm associated to the problem (62) is

(

u_{k0}

$$(x; t) = H(x; t) + E_{\square 1}$$

$_ s_ E$

$_ Nu_{k\square 1}(x; t)$

$_ ; k_ 1$

u_k

$$n(x; t) = E_{\square 1}$$

$_ s_ E$

$_ Lu_k$

$$n_{\square 1}(x; t)$$

$_ ; n_ 1$

(63)

with

$$H(x; t) = E_{\square 1}$$

"

$m X_{\square 1}$

$k=0$

$$s_{2+k} h_k(x)$$

#

$$+ E_{\square 1} [s_ E [g(x; t)]] \quad (64)$$

$$= E_{\square 1} _ s_2 h_0(x)$$

$_ = h_0(x) = \sin x$:

The algorithm (64) is again written

(

u_{k0}

$$(x; t) = \sin x + E_{\square 1}$$

$_ s_ E$

$_ Nu_{k\square 1}(x; t)$

$_ ; k_ 1$

u_k

$$n(x; t) = E_{\square 1}$$

-

\bar{s}_E

$\bar{L}u_{12}$
 $(x; t)$

$= E_{\square 1}$

\bar{s}_E

$\square k_3 \sin x t_2$
 $\square (2_ + 1)$

$= \square k_3 \sin x t_3$
 $\square (3_ + 1)$

...

u_1
 $n(x; t) = E_{\square 1}$

\bar{s}_E

$\bar{L}u_1$
 $n_{\square 1}(x; t)$

$=$

$\sin x (\square kt)_n$
 $\square (n_ + 1)$
; $8 n_ - 1$
(67)

So

$u_1(x; t) =$

X
 n_0

u_1
 $n(x; t) = \sin x$

X
 n_0

$(\square kt)_{n_}$
 $\square (n_ + 1)$
 $= \sin x E_-(\square kt_)$ (68)

is approximate solution of the problem (62) in step 1.

Step 2. For $k = 2$; we compute u_2 using the following algorithm:

(

u_{20}

$(x; t) = \sin x + E_{\square 1}$

\bar{s}_E

$\bar{N}u_1(x; t)$

u_2

$n(x; t) = E_{\square 1}$

\bar{s}_E

$\bar{L}u_2$

$n_{\square 1}(x; t)$

;

(69)

We have

$Nu_1(x; t) =$

\square

$u_1(x; t)$

\square

+

\square

u_1

$xx(x; t)$

\square

$= (\sin x E_{\square}(\square kt))_{\square 3} + (\square \sin x E_{\square}(\square kt))_{\square 3} = 0; (70)$

so by Proposition 1 (a), the exact solution of the problem (45)-(46) is:

$u(x; t) = \sin x E_{\square}(\square kt); (71)$

Example 3. Consider the following fractional Riccati differential equation:

$d_{\square} y(t)$

$dt_{\square} = 2y(t) \square y_2(t) + 1; 0 < \square \leq 1 (72)$

with the following initial condition

$y(0) = 0; (73)$

We have: $Ly(t) = 2y(t); Ny(t) = \square y_2(t)$ and $g(t) = 1$:

The SBATEM algorithm associated to the problem (72)-(73) is

$8 > <$

$>:$

y_k

$0(t) = H(t) + E_{\square 1}$

\square
 $s_{\square} E$

\square
 $Ny_{k \square 1}(t)$

;

$k_{\square 1}$

y_{kn}

$(t) = E_{\square 1}$

\square
 $s_{\square} E$

\square
 Ly_{kn}

$\square 1(t)$

;

$n_{\square 1}$

(74)

with

$H(t) = E_{\square 1}$

"

$m X_{\square 1}$

$k=0$

$s_{2+k} h_k(t)$

#

$+ E_{\square 1} [s_{\square} E [g(t)]]$

$= E_{\square 1} \square$

$s_2 h_0(t)$

\square

$+ E_{\square 1} [s_{\square} E [1]] = h_0(t) +$

$\mathcal{L}y_1$
 $0(t)$

$$= \frac{1}{s} E_{01}$$

$$\frac{1}{s} E$$

$$\frac{1}{2} t_1$$
$$\frac{1}{s} (s + 1)$$

$$=$$

$$\frac{1}{2}$$

$$\frac{(2t_1)^2}{2}$$

$$\frac{1}{s} (2s + 1)$$

$$y_1$$

$$2(t) = E_{01}$$

$$\frac{1}{s} E$$

$\mathcal{L}y_1$
 $1(t)$

$$= \frac{1}{s} E_{01}$$

$$\frac{1}{s} E$$

$$\frac{1}{4}$$

$$t_2$$

$$\frac{1}{s} (2s + 1)$$

$$=$$

$$\frac{1}{2}$$

$$\frac{(2t_2)^3}{3}$$

$$\frac{1}{s} (3s + 1)$$

$$y_1$$

$$3(t) = E_{01}$$

$$\frac{1}{s} E$$

$\mathcal{L}y_1$
 $2(t)$

$$= \frac{1}{s} E_{01}$$

$$\frac{1}{s} E$$

$$\frac{1}{8}$$

$$t_3$$

$$\frac{1}{s} (3s + 1)$$

$$=$$

$$\frac{1}{2}$$

$$\frac{1}{2}$$

$$\begin{aligned}
 & (2t_{-})^4 \\
 & \square(4_{-} + 1) \\
 & \dots \\
 & y_{1n} \\
 & (t) = E_{\square 1} \\
 & \bar{s}_E \\
 & \bar{L}y_{1n} \\
 & \square_1(t) \\
 & \text{---} \\
 & = \\
 & 1 \\
 & 2 \\
 & (2t_{-})_{n+1} \\
 & \square((n+1)_{-} + 1) \\
 & ; n_{-} 1:
 \end{aligned}$$

(78)

So

$$\begin{aligned}
 & y_1(t) = \\
 & X \\
 & n_{-0} \\
 & y_{1n} \\
 & (t) = \\
 & 1 \\
 & 2 \\
 & X \\
 & n_{-0} \\
 & (2t_{-})_{n+1} \\
 & \square((n+1)_{-} + 1) \\
 & = \\
 & 1 \\
 & 2 \\
 & X \\
 & n_{-1} \\
 & (2t_{-})_n \\
 & \square(n_{-} + 1) \\
 & = \\
 & 1 \\
 & 2 \\
 & E_{-}(2t_{-}) \square \\
 & 1 \\
 & 2 \\
 & (79)
 \end{aligned}$$

is approximate solution of the problem (72)-(73) in step 1.
 Step 2. For $k = 2$; we compute y_2 using the following algorithm:

$$\begin{aligned}
 & 8 >> < \\
 & >> : \\
 & y_2 \\
 & o(t) = \\
 & t_{-} \\
 & \square(_ + 1) \\
 & + E_{\square 1} \\
 & \bar{s}_E \\
 & \bar{N}y_1(t)
 \end{aligned}$$

$$; k_{-1} \\ y_{2n} \\ (t) = E_{\square 1}$$

$$\bar{s}_E$$

$$\bar{L}y_{2n} \\ \square_1(t)$$

$$; n_{-1}: \\ (80) \\ \text{We have:} \\ Ny_1(t) = \square$$

$$\bar{1} \\ 2 \\ E_{-}(2t_{-}) \square \\ t_{-} \\ \square(\bar{-} + 1) \\ \bar{-}_2 \\ = \square$$

$$\bar{-} \\ t_{-} \\ a_1 \\ + 2 \\ t_2 \\ a_2 \\ + 4 \\ t_3 \\ a_3 \\ + 8 \\ t_4 \\ a_4 \\ + 16 \\ t_5 \\ a_5 \\ + 32 \\ t_3 \\ a_6 \\ + 64 \\ t_7 \\ a_7 \\ \vdots \\ \bar{-}_2 \\ : (81)$$

For $t_{-} 1$; we approximate $Ny_1(t)$ by:

$$Ny_1(t)' \square \\ t_2 \\ a_{21} \\ \square 4 \\ t_3 \\ a_1 a_2 \square 4$$

$$\bar{-} \\ 1 \\ a_{22}$$

+
2
a₁a₃

-
t₄ □

-
16
a₁a₄
+
16
a₂a₃

-
t₅ □

-
32
a₁a₅
+
32
a₂a₄

-
t₆
□

-
64
a₁a₆
+
64
a₂a₅

-
t₇ □

-
128
a₁a₇
+
128
a₂a₆

-
t₈ (82)
11

- Calculation of y₂

0:

y₂

o(t) =

t₁

a₁

+ E □₁ □

s₁E

-
Ny₁(t)

-
=

t₁

a₁ □

a₂t₃

a₂

a₃ □

4a3t4_
a1a2a4
□ 4

—
1
a22
+
2
a1a3

—
a4
a5
t5_ □

—
16
a1a4
+
16
a2a3

—
a5
a6
t6_ □

—
32
a1a5
+
32
a2a4

—
a6
a7
t7_
□

—
64
a1a6
+
64
a2a5

—
a7
a8
t8_ □

—
128
a1a7
+
128
a2a6

—
a8
a9
t9_ (83)

-Calculation of y2

1:

$$y_2$$
$$1(t) = E_{\square 1} _$$
$$s_E$$

$$_$$
$$Ly_2$$
$$0(t)$$

$$_$$
$$=$$
$$2t_2 _$$
$$a_2 \square$$
$$2a_2t_4 _$$
$$a_{21}$$
$$a_4 \square$$
$$8a_3t_5 _$$
$$a_1a_2a_5$$
$$\square 8$$

$$_$$
$$1$$
$$a_{22}$$
$$+$$
$$2$$
$$a_1a_3$$

$$_$$
$$a_4$$
$$a_6$$
$$t_6 \square 2$$

$$_$$
$$16$$
$$a_1a_4$$
$$+$$
$$16$$
$$a_2a_3$$

$$_$$
$$a_5$$
$$a_7$$
$$t_7 \square 2$$

$$_$$
$$32$$
$$a_1a_5$$
$$+$$
$$32$$
$$a_2a_4$$

$$_$$
$$a_6$$
$$a_8$$
$$t_8 _$$
$$\square 2$$

$$_$$
$$64$$
$$a_1a_6$$
$$+$$
$$64$$
$$a_2a_5$$

$$_$$
$$a_7$$
$$a_9$$

$t_9 \square 2$

—
128

$a_1 a_7$

+

128

$a_2 a_6$

—
 a_8

a_{10}

$t_{10} (84)$

- Calculation of y_2

2:

y_2

$z(t) = E_{\square 1} _$

s_E

—
 Ly_2

$1(t)$

—
=

$4t_3 _$

$a_3 \square$

$4a_2 t_5 _$

a_{21}

$a_5 \square$

$16a_3 t_6 _$

$a_1 a_2 a_6$

$\square 16$

—
1

a_{22}

+

2

$a_1 a_3$

—
 a_4

a_7

$t_7 \square 4$

—
16

$a_1 a_4$

+

16

$a_2 a_3$

—
 a_5

a_8

$t_8 \square 4$

—
32

$a_1 a_5$

+

32

$a_2 a_4$

—

a6
a9
t9_
□ 4

—
64
a1a6
+
64
a2a5

—
a7
a10
t10_ □ 4

—
128
a1a7
+
128
a2a6

—
a8
a11
t11_(85)
- Calculation of y2

3:
y2
3(t) = E□1_
s_E

—
Ly2
2(t)

—
=
8t4_
a4 □
8a2t6_
a21
a6 □
32a3t7_
a1a2a7
□ 32

—
1
a22
+
2
a1a3

—
a4
a8
t8_ □ 8

—
16
a1a4
+

16

$a_2 a_3$

—
 a_5

a_9

$t_9 \square 8$

—
32

$a_1 a_5$

+

32

$a_2 a_4$

—
 a_6

a_{10}

t_{10}

$\square 8$

—
64

$a_1 a_6$

+

64

$a_2 a_5$

—
 a_7

a_{11}

$t_{11} \square 8$

—
128

$a_1 a_7$

+

128

$a_2 a_6$

—
 a_8

a_{12}

$t_{12} (86)$

We find that for any $n \geq 1$:

y_{2n}

$(t) = E_{\square 1}$

s_E

—
 $L y_{2n}$

$\square 1(t)$

—
=

$2_{n+1} t_{(n+1)}$

$2 a_{n+1} \square$

$2_{n+3} a_2 t_{(n+3)}$

$8 a_2$

$a_{n+3} \square$

$2_{n+4} a_3 t_{(n+4)}$

$4 a_1 a_2 a_{n+4}$

\square

—
1

$$a_2^2 + 2a_1a_3 - 2_{n+5}a_4t_{(n+5)} - 8a_{n+5} \square$$

$$1 + a_1a_4 + a_2a_3$$

$$2_{n+6}a_5 + 4a_{(n+6)}t_{(n+6)} \square$$

$$1 + a_1a_5 + a_2a_4$$

$$2_{n+6}a_6 + 4a_{n+7}t_{(n+7)} \square$$

$$1 + a_1a_6 + a_2a_5$$

$$2_{n+8}a_7 + 4a_{n+8}t_{(n+8)} \square$$

$$1 + a_1a_7 + a_2a_6$$

$$2_{n+9}a_8 + 4a_{n+9}t_{(n+9)} \quad (87)$$

So
 $y_2(t) = X_{n_0} y_{2n}(t)$
 12

is an approximate solution of the problem (72)-(73) in step 2.

Numerical analysis

When $\alpha = 1$; the exact solution of the problem (72)-(73) is given by $y_{ex}(t) = 1 + p^2 \tanh(p^2 t + (1-2) \log((p^2 - 1)/(p^2 + 1)))$: We will compare this exact solution with the approximate solution

$$y_{ap}(t) = y_2(t) =$$

P

n_0

y_2^n

(t) for $\alpha = 1$:

If $\alpha = 1$; then we have:

$$y_{ap}(t) = y_2(t) =$$

X

n_0

$(2t)^{(n+1)}$

$2^{n+1} (n+2)^{-n}$

X

n_0

$2(2t)^{(n+3)}$

$8^{n+3} (n+4)^{-n}$

X

n_0

$3(2t)^{(n+4)}$

$4^{n+4} (n+5)^{-n}$

X

n_0

$7(2t)^{(n+5)}$

$4^{n+5} (n+6)^{-n}$

□

X

n_0

$15(2t)^{(n+6)}$

$4^{n+6} (n+7)^{-n}$

X

n_0

$21(2t)^{(n+7)}$

$4^{n+7} (n+8)^{-n}$

X

n_0

$28(2t)^{(n+8)}$

$4^{n+8} (n+9)^{-n}$

X

n_0

$36(2t)^{(n+9)}$

$4^{n+9} (n+10)^{-n}$

: (88)

By simplifying (88), we obtain

$$y_{ap}(t) = y_2(t) =$$

$109e^{2t}$

4

+

109

4

+

$111t$

2

+

$111t^2$

2

+

$110t^3$

$$\begin{aligned}
& 3 \\
& + \\
& 107t^4 \\
& 6 \\
& + \\
& 20t^5 \\
& 3 \\
& + \\
& 17t^6 \\
& 9 \\
& + \\
& 128t^7 \\
& 315 \\
& + \\
& 2t^8 \\
& 35 \\
& : (89)
\end{aligned}$$

The following comparison table (Table 1) gives the deviation between the exact solution and the approximated solution for values of t between 0 and 0.5 for $\alpha = 1$: We represent graphically the exact solution and the approximate solution for $\alpha = 1$ in the following figure (Figure 1).

t	$y_{ex}(t)$	$y_{ap}(t)$	$ y_{ex}(t) - y_{ap}(t) $
0	0	0	0
0:10	0:1103	0:1103	$1:7600 \times 10^{-6}$
0:15	0:1734	0:1734	$1:5295 \times 10^{-5}$
0:20	0:2420	0:2419	$7:3574 \times 10^{-5}$
0:25	0:3159	0:3157	$2:5567 \times 10^{-4}$
0:30	0:3951	0:3944	$7:2254 \times 10^{-4}$
0:35	0:4792	0:4774	0:0018
0:40	0:5678	0:5639	0:0039
0:45	0:6603	0:6524	0:0079
0:50	0:7560	0:7410	0:0150

Table 1 : Comparison of the exact solution with the approximate solution of the Riccati problem (72) (73)

for $\alpha = 1$:

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Figure 1: Exact solution and approximate solution of the Riccati problem (72)-(73) for $\alpha = 1$:

Example 4. Consider the following fractional Korteweg-de Vries (KDV) equation

$$D_t u + u^2 = 3$$

(90)

$$u(x, 0) = 6x$$

$$x + u_{xxx} = 0; t > 0; 0 < \alpha < 1 \quad (90)$$

with the initial condition

$$u(x; 0) = 6x \quad (91)$$

$$\text{We have: } Lu(x; t) = -u_{xxx}(x; t); Nu(x; t) = 3$$

(92)

$$u_2(x; t)$$

$$g(x; t) = 0:$$

The SBATEM algorithm associated to the problem (90)-(91) is

$$8 > <$$

>:

$$u_{k0}$$

$$(x; t) = H(x; t) + E_{\alpha, 1}$$

$$s_E$$

$\bar{N}_{u_{k+1}}(x; t)$

; $k = 1$

u_k

$n(x; t) = E_{\epsilon}$

\bar{s}_E

\bar{L}_{u_k}

$n_{k+1}(x; t)$

; $n = 1$

(92)

with

$H(x; t) = E_{\epsilon}$

"

mX_{k+1}

$k=0$

$s_{2+k}h_k(x)$

#

+ $E_{\epsilon} [s_E [g(x; t)]]$ (93)

= $E_{\epsilon} \bar{s}_E$

$s_{2h_0}(x)$

= $h_0(x) = 6x$:

14

The algorithm (92) is again written

\bar{s}_E

>

u_{k0}

$(x; t) = 6x + E_{\epsilon}$

\bar{s}_E

$\bar{N}_{u_{k+1}}(x; t)$

; $k = 1$

u_k

$n(x; t) = E_{\epsilon}$

\bar{s}_E

\bar{L}_{u_k}

$n_{k+1}(x; t)$

; $n = 1$:

(94)

Let us apply to (94) Picard's principle: we take $u_0 = 0$; then $N_{u_0} = 0$:

Step 1. For $k = 1$, we compute u_1 using the following algorithm

\bar{s}_E

>

u_{10}

$(x; t) = 6x$

u_1

$n(x; t) = E_{\epsilon}$

\bar{s}_E

$\bar{L}u_1$

$n_{\square 1}(x; t)$

; $n_{\square 1}$:

(95)

We have:

$\bar{8} > > > > <$

$> > > > <$:

u_{11}

$(x; t) = E_{\square 1}$

\bar{s}_E

$\bar{L}u_{10}$

$(x; t)$

$= 0$

u_{12}

$(x; t) = E_{\square 1}$

\bar{s}_E

$\bar{L}u_{11}$

$(x; t)$

$= 0$

...

u_1

$n(x; t) = E_{\square 1}$

\bar{s}_E

$\bar{L}u_1$

$n_{\square 1}(x; t)$

$= 0; \bar{8} n_{\square 1}$:

(96)

So

$u_1(x; t) =$

X

$n_{\square 0}$

u_1

$n(x; t) = u_{10}$

$(x; t) = 6x$ (97)

is approximate solution of the problem (90)-(91) in step 1.

Step 2. For $k = 2$; we compute u_2 using the following algorithm:

$\bar{8} > <$

$>$:

u_{20}

$(x; t) = 6x + E_{\square 1}$

\bar{s}_E

-

$$Nu_1(x; t)$$

$$\overline{u_2} \\ n(x; t) = E_{\square 1}$$

$$\overline{s_E}$$

$$\overline{Lu_2} \\ n_{\square 1}(x; t)$$

$$; n_{\square 1}: \\ (98)$$

We have

$$Nu_1(x; t) = 3(36x^2)_x = 216x \quad (99)$$

and

$$8 >>>>>>>>><$$

$$>>>>>>>>:$$

$$u_{20} \\ (x; t) = 6x + E_{\square 1}$$

$$\overline{s_E}$$

$$\overline{Nu_1(x; t)}$$

$$\overline{= 6x +} \\ \overline{216xt_{\square}} \\ \overline{\square(\square + 1)} \\ u_{21} \\ (x; t) = E_{\square 1}$$

$$\overline{s_E}$$

$$\overline{Lu_{20}} \\ (x; t)$$

$$\overline{= 0} \\ u_{22} \\ (x; t) = E_{\square 1}$$

$$\overline{s_E}$$

$$\overline{Lu_{21}} \\ (x; t)$$

$$\overline{= 0}$$

$$\dots \\ u_2 \\ n(x; t) = E_{\square 1}$$

$$\overline{s_E}$$

$$\overline{Lu_2} \\ n_{\square 1}(x; t)$$

$$\overline{= 0}; 8 n_{\square 1}: \\ (100)$$

So

$$u_2(x; t) =$$

X

n_0

u_2

$$n(x; t) = u_2$$

$$(x; t) = 6x +$$

$$216xt_$$

$$\square(_ + 1)$$

(101)

is approximate solution of the problem (90)-(91) in step 2.

15

Step 3. For $k = 3$; we compute u_3 using the following algorithm:

$8 > <$

$> :$

u_{30}

$$(x; t) = 6x + E_{\square 1}$$

\bar{s}_E

$$\bar{N}u_2(x; t)$$

\bar{u}_3

$$n(x; t) = E_{\square 1}$$

\bar{s}_E

$\bar{L}u_3$

$$n_{\square 1}(x; t)$$

$; n_{\square 1} :$

(102)

We have:

$$\bar{N}u_2(x; t) = 3$$

"

$$\bar{6}x +$$

$$216xt_$$

$$\square(_ + 1)$$

$$_2$$

#

x

$$= 6x$$

$$\bar{6} +$$

$$216t_$$

$$\square(_ + 1)$$

$$_2$$

$$= 216x$$

$$\bar{1} +$$

$$36t_$$

$$\square(_ + 1)$$

$$_2$$

$$= 216x$$

$$\bar{1} +$$

Lu_3
 $(x; t)$

$= 0$

...

u_3

$n(x; t) = E_{n-1}$

s_E

Lu_3

$n(x; t)$

$= 0; 8 n - 1:$

(104)

So

$u_3(x; t) =$

X

n_0

u_3

$n(x; t) = u_3$

$(x; t)$

$= 6x + 216x$

t

$(n + 1)$

+

$72t_2$

$(2n + 1)$

+

$362(2n + 1)t_3$

$(n + 1)^2(3n + 1)$

(105)

is approximate solution of the problem (90)-(91) in step 3.

Step 4. For $k = 4$; we compute u_4 using the following algorithm:

$8 > <$

$>:$

u_4

$(x; t) = 6x + E_{n-1}$

s_E

$Nu_3(x; t)$

u_4

$n(x; t) = E_{n-1}$

s_E

Lu_4

$n(x; t)$

$; n - 1:$

(106)

To simplify the expressions, let's put $a_n = a_n(n) = (n + 1)$. We have:

16

$$\text{Nu}_3(x; t) = 3$$

$$\frac{1}{6x + 216x}$$

—
t

a₁

+

$$72t_2$$

a₂

+

$$36_2a_2t_3$$

a₂₁

a₃

$$\frac{1}{\#}$$

x

$$= 216x$$

—
1 + 36

—
t

a₁

+

$$72t_2$$

a₂

+

$$36_2a_2t_3$$

a₂₁

a₃

$$\frac{1}{\#}$$

x

$$= 216x$$

—
1 +

72t

a₁

+

$$4_36_2t_2$$

a₂

+

$$2_36_3a_2t_3$$

a₂₁

a₃

+

$$36_2$$

—
t

a₁

+

$$72t_2$$

a₂

+

$$36_2a_2t_3$$

a₂₁

a₃

—
_2

a_3
 $+$
 $2 \cdot 363a_2t_4$
 a_{21}
 a_4
 $+362$
 $-$
 a_2t_3
 a_{21}
 a_3
 $+$
 $722a_4t_5$
 a_{22}
 a_5
 $+$
 $364(a_2)2a_6t_7$
 a_{41}
 a_{23}
 a_7
 $+$
 $144a_3t_4$
 $a_1a_2a_4$
 $+$
 $2 \cdot 362a_2a_4t_5$
 a_{31}
 a_3a_5
 $+$
 $4 \cdot 363a_5t_6$
 a_{21}
 a_3a_6
 $-$
 u_{41}
 $(x; t) = E_{\square 1}$
 $-$
 s_E
 $-$
 Lu_{40}
 $(x; t)$
 $-$
 $= 0$
 u_{42}
 $(x; t) = E_{\square 1}$
 $-$
 s_E
 $-$
 Lu_{41}
 $(x; t)$
 $-$
 $= 0$
 \dots
 u_4
 $n(x; t) = E_{\square 1}$
 $-$
 s_E
 $-$
 Lu_4

$$n_{n-1}(x; t)$$

$$= 0; 8 n_{n-1} \quad (108)$$

So

$$u_4(x; t) =$$

X

n_0

u_4

$$n(x; t) = u_4$$

(x; t)

$$= 6x + 216x$$

t_

a_1

+

72t_2

a_2

+

4_36t_3

a_3

+

2_36a_2t_4

a_21

a_4

+362

a_2t_3

a_21

a_3

+

72a_4t_5

a_22

a_5

+

364(a_2)2a_6t_7

a_41a_23

a_7

+

144a_3t_4

a_1a_2a_4

+

2_36a_2a_4t_5

a_31

a_3a_5

+

4_36a_5t_6

a_21

a_3a_6

:

(109)

is approximate solution of the problem (90)-(91) in step 4.

Numerical analysis

When $_ = 1$; the exact solution of the problem (90)-(91) is given by $u_{ex}(x; t) =$

6x

1 □ 36t

: We

will compare this exact solution with the approximate solution $u_{ap}(x; t) = u_4(x; t)$ for $\alpha = 1$:

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If $\alpha = 1$; then we have:

$$u_{ap}(x; t) = u_4(x; t) = 6x + 216x$$

$$t + 36t^2 + 864t^3 + 7776t^4 \quad (110)$$

—

t₃

3

+

1296t₅

5

+

186624t₇

7

+ 18t₄ +

864t₅

5

+ 5184t₆

—

or again

$$u_{ap}(x; t) = 6x + 216x$$

$$t + 36t^2 + 1296t^3 + 31104t^4 + 559872t^5 + 6718464t^6 + 241864624t^7$$

—

(111)

The following comparison table (Table 2) gives the deviation between the exact solution and the approximated solution for values of x and t between 0 and 1; and between 0 and 0:01; respectively, for $\alpha = 1$: We represent graphically the exact solution and the approximate solution for $\alpha = 1$ in the following figure (Figure 2)

$x \quad t \quad u_{ex}(x; t) \quad u_{ap}(x; t) \quad |u_{ex}(x; t) - u_{ap}(x; t)|$

0 0 0 0 0

0:1 0:001 0:6224 0:6224 3:6132 $\times 10^{-7}$

0:2 0:001 1:2448 1:2448 7:2264 $\times 10^{-7}$

0:3 0:001 1:8672 1:8672 1:0840 $\times 10^{-6}$

0:4 0:004 2:8037 2:8033 4:6563 $\times 10^{-4}$

0:5 0:004 3:5047 3:5041 5:8203 $\times 10^{-4}$

0:6 0:004 4:2056 4:2049 6:9844 $\times 10^{-4}$

0:7 0:007 5:6150 5:6052 0:0098

0:8 0:007 6:4171 6:4059 0:0112

0:9 0:007 7:2193 7:2066 0:0126

1 0:01 9:3750 9:2983 0:0767

Table 2 : Comparison of the exact solution with the approximate solution of the KDV problem (90) □ (91) for $\alpha = 1$:

Figure 2: Exact solution and approximate solution of the KDV problem (90)-(91) for $\alpha = 1$:

5 Conclusion

In this work, we have given a new technique that allows to find the exact solution or an approximate solution of ordinary or fractional nonlinear differential equations with given initial conditions. This technique consists in coupling the Elzaky transform and the SBA method. The

results obtained in the resolution of some nonlinear fractional differential equations prove the efficiency and simplicity of this new technique.

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