Coupling the SBA method and the Elzaki transformation to solve nonlinear fractional differential equations

Abstract

In this paper, we propose a new technique well adapted to the solution of nonlinear fractional differential equations. This technique combines the Elzaki transorm and the Some Blaise-Abbo (SBA) method. It allows to find the exact solution or an acceptable approximate solution of the equation.

Keywords: SBA method, Elzaki transform, SBATEM, Fractional differential equation, Caputo fractional derivative.

1 Introduction

Fractional derivatives are used in the modeling of many physical phenomena, such as heat diffusion through a semi-infinite solid, flow in oil reservoirs, rheological properties of solids etc. In general, it is difficult to find the exact solution of a nonlinear fractional differential equation. Many numerical methods are used to find an approximate solution. Commonly used numerical methods are the variational iteration method (VIM) [18, 19], the Adomian decompositional method (ADM) [4, 5, 7], and the generalized differential transformation method (GDTM) [3, 6] .Recently, the SBA method [9, 20, 21] which is a combination of the Adomian method, the method of successive approximations [14] and the Picard principle, is also used. The nonlinear fractional differential equations are also solved with techniques combining numerical methods with integral transformations, such as the Homotopy perturbation method combined with the Elzaki transformation (EHTPM) [12, 13], the Homotopy perturbation method combined with the Sumudu transformation (HPSTM) [16], the Adomian decomposition method combined with the Elzaki transformation (EADM) [17]. In this paper, we propose a new technique to find the exact solution or an approximate solution of nonlinear fractional differential equations. This technique is a combination of the SBA method and the Elzaki transform (SBATEM). After having recalled some notions on fractional calculus and on the Elzaki transform, we will give the principle of this new technique, then we will apply it on some examples of nonlinear fractional differential equations.

2 Definitions and basic properties

In this section, we recall some definitions and properties of fractional calculus and the Elzaki transformation.

2.1 Gamma function and Mittag-Leffler function

Gamma function The Euler Gamma function [1, 11] is defined on the half-plane $P = \{z \in \mathbb{C}/\ Re(z) > 0\}$ by

$$\Gamma(z) = \int_0^{+\infty} t^{z-1} \mathrm{e}^{-t} dt \tag{1}$$

For any natural number $n : \Gamma(n+1) = n!$.

Mittag-Leffler function For any complex number z, we define the one-parameter Mittag-Leffler function [2, 11] by

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)};\ \alpha \in \mathbb{C},\ Re(\alpha) > 0. \tag{2}$$

In particular, when $\alpha = 1$, this function coincides with the exponential function:

$$E_1(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty} \frac{z^k}{k!} = \mathrm{e}^z \tag{3}$$

2.2 Caputo fractional derivative

Definition 1 Let $[a, b]$ be a finite interval of R and $f \in L^1([a, b])$. The fractional Riemann-Liouville left-handed integral of order $\alpha > 0$ of the function f is defined by [2]

$$I^{\alpha}_{a,x} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt \tag{4}$$

Definition 2 The fractional Caputo left derivative of order $\alpha > 0$ of the function $f(x)$, $x \in [a, b]$ is defined by [15]

$$\begin{aligned} {}_C D^{\alpha}_{a,x} f(x) &= I^{m-\alpha}_{a,x}\left(f^{(m)}(x)\right) \\ &= \frac{1}{\Gamma(m-\alpha)} \int_a^x \frac{f^{(m)}(t)}{(x-t)^{\alpha-m+1}} dt \end{aligned} \tag{5}$$

Where $m = [\alpha] + 1$ if $\alpha \notin \mathbb{N}$ and $m = \alpha$ if $\alpha \in \mathbb{N}$.

2.3 Elzaki Transform

Consider the following set of functions of exponential order

$$A = \left\{ f(t) : \exists M, k_1, k_2 > 0, |f(t)| < M\mathrm{e}^{|t|/k_j}, \mathrm{si}\, t \in (-1)^j \times [0, \infty) \right\} \tag{6}$$

Definition For $f \in A$,the Elzaki transform of f is given by the following formula [8]

$$E[f(t)] = T(s) = s \int_0^{\infty} \mathrm{e}^{-\frac{t}{s}} f(t) dt,\ k_1 \le s \le k_2 \tag{7}$$

From the formula (7), we obtain the following Elzaki transforms:

$$E[1] = s^2,\ E[t] = s^3\ E[t^{\alpha}] = \Gamma(\alpha+1) s^{\alpha+2},\ \alpha > 0 \tag{8}$$

The Elzaki transform verifies the linearity property: $\forall\, f, g \in A$ and $\forall\, a, b \in R$, we have

$$E\left[af(t) + bg(t)\right] = aE\left[f(t)\right] + bE\left[g(t)\right] \tag{9}$$

Theorem 1 The Elzaki transform of the fractional Caputo derivative is [10]:

$$E\left[D_t^\alpha f(t)\right] = s^{-\alpha} E\left[f(t)\right] - \sum_{k=0}^{m-1} s^{2-\alpha+k} f^{(k)}(0),\; m-1 < \alpha \leq m \tag{10}$$

Theorem 2 [8] Let $T(u)$ be the Elzaki transform of $f(t)$ such that
(i) $sT\left(\frac{1}{s}\right)$ is a meromorphic function, with singularities having $Re(s) < \alpha$, and
(ii) There exists a circular region $\lceil$ with radius R and positive constants, M and K with

$$\left| sT\left(\frac{1}{s}\right) \right| < MR^{-K} \tag{11}$$

Then the function $f(t)$ is given by

$$E^{-1}\left[T(s)\right] = \frac{1}{2\pi i} \int_{\alpha - i.\infty}^{\alpha + i.\infty} \mathrm{e}^{st} sT\left(\frac{1}{s}\right) ds = \sum \text{resudues of } \left[\mathrm{e}^{st} sT\left(\frac{1}{s}\right)\right] \tag{12}$$

3 Description of the SBATEM technique

Consider the following nonlinear and inhomogeneous fractional differential equation

$$D_t^\alpha u(x,t) = Lu(x,t) + Nu(x,t) + g(x,t),\; \alpha > 0 \tag{13}$$

with the initial conditions:

$$u(x,0) = h^0(x),\; \frac{\partial^k u(x,0)}{\partial t^k} = h^k(x),\;\; k \in \{1, ..., m-1\} \tag{14}$$

where $D_t^\alpha = \dfrac{\partial^\alpha}{\partial t^\alpha}$ is the fractional derivative of Caputo with respect to t of order $\alpha > 0$; L and N are linear and nonlinear differential operators, respectively.

Applying the Elzaki transform to (13), we obtain

$$E\left[D_t^\alpha u(x,t)\right] = E\left[Lu(x,t)\right] + E\left[Nu(x,t)\right] + E\left[g(x,t)\right] \tag{15}$$

Using Theorem 1 and the initial conditions (14), we obtain

$$E\left[u(x,t)\right] = \sum_{k=0}^{m-1} s^{2+k} h^k(x) + s^\alpha E\left[g(x,t)\right] + s^\alpha E\left[Lu(x,t)\right] + s^\alpha E\left[Nu(x,t)\right] \tag{16}$$

Applying the inverse Elzaki transform to (16), we obtain

$$u(x,t) = H(x,t) + E^{-1}\left[s^\alpha E\left[Lu(x,t)\right]\right] + E^{-1}\left[s^\alpha E\left[Nu(x,t)\right]\right] \tag{17}$$

where

$$H(x,t) = E^{-1}\left[\sum_{k=0}^{m-1} s^{2+k} h^k(x)\right] + E^{-1}\left[s^\alpha E\left[g(x,t)\right]\right] \tag{18}$$

Apply the method of successive approximations to (17), we get:

$$u^k(x,t) = H(x,t) + E^{-1}\left[s^\alpha E\left[Lu^k(x,t)\right]\right] + E^{-1}\left[s^\alpha E\left[Nu^{k-1}(x,t)\right]\right] \tag{19}$$

The Adomian agorithm associated with (19) is the following:

$$\begin{cases} u_0^k(x,t) & = H(x,t) + E^{-1}\left[s^\alpha E\left[Nu^{k-1}(x,t)\right]\right],\ k \geq 1 \\ u_n^k(x,t) & = E^{-1}\left[s^\alpha E\left[Lu_{n-1}^k(x,t)\right]\right],\ n \geq 1 \end{cases} \tag{20}$$

We will call the above algorithm the SBATEM algorithm.
Let us apply Picard's principle to (20): we choose $u^0 \in V$ any root of the equation $Nu = 0$.
Step 1 for $k = 1$, we compute u^1 using the following algorithm

$$\begin{cases} u_0^1(x,t) & = H(x,t) \\ u_n^1(x,t) & = E^{-1}\left[s^\alpha E\left[Lu_{n-1}^1(x,t)\right]\right],\ n \geq 1 \end{cases} \tag{21}$$

If the series $\sum_{n\geq 0} u_n^1$ is convergent, then we get:

$$u^1 = \sum_{n\geq 0} u_n^1 \tag{22}$$

approximate solution of the problem (13)-(14) in step 1.
Step 2 for $k = 2$, we compute u^2 using the following algorithm:

$$\begin{cases} u_0^2(x,t) & = H(x,t) + E^{-1}\left[s^\alpha E\left[Nu^1(x,t)\right]\right] \\ u_n^2(x,t) & = E^{-1}\left[s^\alpha E\left[Lu_{n-1}^2(x,t)\right]\right],\ n \geq 1 \end{cases} \tag{23}$$

If the series $\sum_{n\geq 0} u_n^2$ is convergent, then we get:

$$u^2 = \sum_{n\geq 0} u_n^2 \tag{24}$$

approximate solution of the problem (13)-(14) in step 2.
Step k Recursively, if the series $\sum_{n\geq 0} u_n^k$ is convergent for $k \geq 1$, then we get

$$u^k = \sum_{n\geq 0} u_n^k \tag{25}$$

approximate solution of the problem (13)-(14) in step k. The solution of the problem (13)-(14) is then:

$$u = \lim_{k\to\infty} u^k \tag{26}$$

Proposition 1
Consider the following nonlinear and inhomogeneous fractional differential equation:

$$D_t^\alpha u(x,t) = Lu(x,t) + Nu(x,t) + g(x,t),\ \alpha > 0 \tag{27}$$

with the initial conditions:

$$u(x,0) = h^0(x),\ \frac{\partial^k u(x,0)}{\partial t^k} = h^k(x),\ k \in \{1,...,m-1\} \tag{28}$$

where $D_t^\alpha = \dfrac{\partial^\alpha}{\partial t^\alpha}$ is the fractional derivative of Caputo with respect to t of order $\alpha > 0$; L is a linear operator and N a nonlinear operator defined in a suitably chosen space V; $g \in V$ and u the unknown function.

Let be the SBATEM agorithm associated to (27)-(28) :

$$\begin{cases} u_0^k(x,t) & = H(x,t) + E^{-1}\left[s^\alpha E\left[Nu^{k-1}(x,t)\right]\right],\ k \geq 1 \\ u_n^k(x,t) & = E^{-1}\left[s^\alpha E\left[Lu_{n-1}^k(x,t)\right]\right],\ n \geq 1 \end{cases} \tag{29}$$

By Picard's principle, we choose $u^0 \in V$ such that $Nu^0 = 0$.

(a) If $Nu^1 = 0$, then the problem (27)-(28) admits a unique solution $u = u^1$.

(b) If for a fixed integer p, $u^p = u^{p-1},\ p \geq 2$, then the problem (27)-(28) admits a unique solution $u = u^{p-1}$.

Proof

Existence

(a) Let u^1 be the approximate solution in step 1. Assume that $Nu^1 = 0$, so the scheme in step 2 is written:

$$\begin{cases} u_0^2(x,t) & = H(x,t) \\ u_n^2(x,t) & = E^{-1}\left[s^\alpha E\left[Lu_{n-1}^2(x,t)\right]\right],\ n \geq 1 \end{cases} \tag{30}$$

This scheme is identical to the scheme in step 1. So the approximate solution in step 2 is $u^2 = u^1$.

We have $Nu^2 = Nu^1 = 0$; therefore the scheme at step 3 is also identical to the scheme at step 2. Therefore, the solution at

step 3 is $u^3 = u^2 = u^1$.

Recursively, the approximate solution at step k ($k \geq 2$) is u^k=u^{k-1} = ...=u^1.

The solution of the problem (27)-(28) is

$$u = \lim_{k\to\infty} u^k = u^1 \tag{31}$$

(b) Suppose that for a fixed integer p, $u^p = u^{p-1},\ p \geq 2$; then we have $Nu^p = Nu^{p-1}$.

At step $p+1$, the algorithm is written:

$$\begin{cases} u_0^{p+1}(x,t) & = H(x,t) + E^{-1}\left[s^\alpha E\left[Nu^p(x,t)\right]\right] \\ u_n^{p+1}(x,t) & = E^{-1}\left[s^\alpha E\left[Lu_{n-1}^{p+1}(x,t)\right]\right],\ n \geq 1 \end{cases} \tag{32}$$

From this algorithm, we obtain:

$u_0^{p+1}(x,t) = H(x,t) + E^{-1}\left[s^\alpha E\left[Nu^p(x,t)\right]\right] = H(x,t) + E^{-1}\left[s^\alpha E\left[Nu^{p-1}(x,t)\right]\right] = u_0^p(x,t)$

$u_1^{p+1}(x,t) = E^{-1}\left[s^\alpha E\left[Lu_0^{p+1}(x,t)\right]\right] = E^{-1}\left[s^\alpha E\left[Lu_0^p(x,t)\right]\right] = u_1^p(x,t)$

$u_2^{p+1}(x,t) = E^{-1}\left[s^\alpha E\left[Lu_1^{p+1}(x,t)\right]\right] = E^{-1}\left[s^\alpha E\left[Lu_1^p(x,t)\right]\right] = u_2^p(x,t)$

...

$u_n^{p+1}(x,t) = E^{-1}\left[s^\alpha E\left[Lu_{n-1}^{p+1}(x,t)\right]\right] = E^{-1}\left[s^\alpha E\left[Lu_{n-1}^p(x,t)\right]\right] = u_n^p(x,t)$

So

$$u^{p+1} = \sum_{n\geq 0} u_n^{p+1} = \sum_{n\geq 0} u_n^p = u^p \tag{33}$$

Similarly, we show that at step $p+2$, $u^{p+2} = u^{p+1}$.

Recursively, the approximate solution at step k $(k \geq p-1)$ is $u^k = u^{k-1} = ... = u^p = u^{p-1}$.
The solution of the problem (27)-(28) is thus

$$u = \lim_{k \to \infty} u^k = u^{p-1} \tag{34}$$

Uniqueness suppose that the problem (27)-(28) admits by the SBA method two distinct solutions u and v. Let $\varphi = u - v$. Then we have:

$$\begin{cases} u_0^k(x,t) & = H(x,t) + E^{-1}\left[s^\alpha E\left[Nu^{k-1}(x,t)\right]\right], \ k \geq 1 \\ u_n^k(x,t) & = E^{-1}\left[s^\alpha E\left[Lu_{n-1}^k(x,t)\right]\right], \ n \geq 1 \end{cases} \tag{35}$$

and

$$\begin{cases} v_0^k(x,t) & = H(x,t) + E^{-1}\left[s^\alpha E\left[Nv^{k-1}(x,t)\right]\right], \ k \geq 1 \\ v_n^k(x,t) & = E^{-1}\left[s^\alpha E\left[Lv_{n-1}^k(x,t)\right]\right], \ n \geq 1 \end{cases} \tag{36}$$

Making the difference (35)-(36), we get

$$\begin{cases} \varphi_0^k = E^{-1}\left[s^\alpha E\left[Nu^{k-1}(x,t)\right]\right] - E^{-1}\left[s^\alpha E\left[Nv^{k-1}(x,t)\right]\right], \ k \geq 1 \\ \varphi_n^k = E^{-1}\left[s^\alpha E\left[L\varphi_{n-1}^k(x,t)\right]\right], \ n \geq 1 \end{cases} \tag{37}$$

where $\varphi_n^k = u_n^k - v_n^k$.
Step 1 for $k = 1$, we have

$$\begin{cases} \varphi_0^1 = 0 \\ \varphi_n^1 = E^{-1}\left[s^\alpha E\left[L\varphi_{n-1}^1(x,t)\right]\right], \ n \geq 1 \end{cases} \tag{38}$$

- for $n = 1$, we have

$$\varphi_1^1 = E^{-1}\left[s^\alpha E\left[L\varphi_0^1(x,t)\right]\right] = 0 \tag{39}$$

- for $n = 2$, we have

$$\varphi_2^1 = E^{-1}\left[s^\alpha E\left[L\varphi_1^1(x,t)\right]\right] = 0 \tag{40}$$

- We find that for all $n \geq 0$, $\varphi_n^1 = 0$. Therefore, we have:

$$\varphi^1 = \sum_{n \geq 0} \varphi_n^1 = 0 \tag{41}$$

Therefore, we obtain $u^1 = v^1$.
Step 2 for $k = 2$, we have

$$\begin{cases} \varphi_0^2 = L_t^{-1}Nu^1 - L_t^{-1}Nv^1 \\ \varphi_n^2 = L_t^{-1}R(\varphi_{n-1}^2), \ n \geq 1 \end{cases} \tag{42}$$

Since $u^1 = v^1$, then $Nu^1 = Nv^1$. As a result, the scheme (42) is written

$$\begin{cases} \varphi_0^2 = 0 \\ \varphi_n^2 = L_t^{-1}R(\varphi_{n-1}^2), \ n \geq 1 \end{cases} \tag{43}$$

This scheme is identical to the scheme in step 1; thus for all $n \geq 0$, $\varphi_n^2 = 0$. Hence

$$\varphi^2 = \sum_{n\geq 0} \varphi_n^2 = 0 \tag{44}$$

Therefore, we get $u^2 = v^2$.

Recursively, for all $k \geq 1$, $u^k = v^k$. Therefore $u = v$; which is absurd. So the problem (27)-(28) admits a unique solution.

4 Applications

In this section, we apply the SBATEM technique to solve four examples of nonlinear fractional differential equations.

Example 1 Consider the following nonlinear fractional partial differential equation

$$D_t^{\alpha} u - 3\left(u_x\right)^2 + u_{xxx} = 0,\ t > 0,\ 0 < \alpha \leq 1 \tag{45}$$

with the initial condition

$$u(x, 0) = 6x \tag{46}$$

We have: $Lu(x,t) = -u_{xxx}(x,t)$, $Nu(x,t) = 3\left(u_x(x,t)\right)^2$ and $g(x,t) = 0$.

The SBATEM algorithm associated to the problem (45)-(46) is

$$\begin{cases} u_0^k(x,t) & = H(x,t) + E^{-1}\left[s^{\alpha} E\left[Nu^{k-1}(x,t)\right]\right],\ k \geq 1 \\ u_n^k(x,t) & = E^{-1}\left[s^{\alpha} E\left[Lu_{n-1}^k(x,t)\right]\right],\ n \geq 1 \end{cases} \tag{47}$$

with

$$\begin{aligned} H(x,t) &= E^{-1}\left[\sum_{k=0}^{m-1} s^{2+k} h^k(x)\right] + E^{-1}\left[s^{\alpha} E\left[g(x,t)\right]\right] \\ &= E^{-1}\left[s^2 h^0(x)\right] = h^0(x) = 6x \end{aligned} \tag{48}$$

The algorithm (47) is again written

$$\begin{cases} u_0^k(x,t) & = 6x + E^{-1}\left[s^{\alpha} E\left[Nu^{k-1}(x,t)\right]\right],\ k \geq 1 \\ u_n^k(x,t) & = E^{-1}\left[s^{\alpha} E\left[Lu_{n-1}^k(x,t)\right]\right],\ n \geq 1 \end{cases} \tag{49}$$

Let us apply to (49) Picard's principle: we take $u^0 = 0$, then $Nu^0 = 0$.

Step 1 for $k = 1$, we compute u^1 using the following algorithm

$$\begin{cases} u_0^1(x,t) & = 6x \\ u_n^1(x,t) & = E^{-1}\left[s^{\alpha} E\left[Lu_{n-1}^1(x,t)\right]\right],\ n \geq 1 \end{cases} \tag{50}$$

We have

$$\begin{cases} u_1^1(x,t) & = E^{-1}\left[s^{\alpha} E\left[Lu_0^1(x,t)\right]\right] = 0 \\ u_2^1(x,t) & = E^{-1}\left[s^{\alpha} E\left[Lu_1^1(x,t)\right]\right] = 0 \\ \vdots & \\ u_n^1(x,t) & = E^{-1}\left[s^{\alpha} E\left[Lu_{n-1}^1(x,t)\right]\right] = 0,\ \forall\, n \geq 1 \end{cases} \tag{51}$$

So

$$u^1(x,t) = \sum_{n\geq 0} u_n^1(x,t) = u_0^1(x,t) = 6x \tag{52}$$

is approximate solution of the problem (45)-(46) in step 1.
Step 2 for $k=2$, we compute u^2 using the following algorithm:

$$\begin{cases} u_0^2(x,t) & = 6x + E^{-1}\left[s^\alpha E\left[Nu^1(x,t)\right]\right] \\ u_n^2(x,t) & = E^{-1}\left[s^\alpha E\left[Lu_{n-1}^2(x,t)\right]\right],\ n\geq 1 \end{cases} \tag{53}$$

We have

$$Nu^1(x,t) = 3(6)^2 = 108 \tag{54}$$

and

$$\begin{cases} u_0^2(x,t) & = 6x + E^{-1}\left[s^\alpha E\left[Nu^1(x,t)\right]\right] = 6x + \dfrac{108t^\alpha}{\Gamma(\alpha+1)} \\ u_1^2(x,t) & = E^{-1}\left[s^\alpha E\left[Lu_0^2(x,t)\right]\right] = 0 \\ u_2^2(x,t) & = E^{-1}\left[s^\alpha E\left[Lu_1^2(x,t)\right]\right] = 0 \\ \vdots \\ u_n^2(x,t) & = E^{-1}\left[s^\alpha E\left[Lu_{n-1}^2(x,t)\right]\right] = 0,\ \forall\, n\geq 1 \end{cases} \tag{55}$$

So

$$u^2(x,t) = \sum_{n\geq 0} u_n^2(x,t) = u_0^2(x,t) = 6x + \frac{108t^\alpha}{\Gamma(\alpha+1)} \tag{56}$$

is approximate solution of the problem (45)-(46) in step 2.
Step 3 for $k=3$, we compute u^3 using the following algorithm:

$$\begin{cases} u_0^3(x,t) & = 6x + E^{-1}\left[s^\alpha E\left[Nu^2(x,t)\right]\right] \\ u_n^3(x,t) & = E^{-1}\left[s^\alpha E\left[Lu_{n-1}^3(x,t)\right]\right],\ n\geq 1 \end{cases} \tag{57}$$

We have

$$Nu^2(x,t) = 3(6)^2 = 108 \tag{58}$$

and

$$\begin{cases} u_0^3(x,t) & = 6x + E^{-1}\left[s^\alpha E\left[Nu^2(x,t)\right]\right] = 6x + \dfrac{108t^\alpha}{\Gamma(\alpha+1)} \\ u_1^3(x,t) & = E^{-1}\left[s^\alpha E\left[Lu_0^3(x,t)\right]\right] = 0 \\ u_2^3(x,t) & = E^{-1}\left[s^\alpha E\left[Lu_1^3(x,t)\right]\right] = 0 \\ \vdots \\ u_n^3(x,t) & = E^{-1}\left[s^\alpha E\left[Lu_{n-1}^3(x,t)\right]\right] = 0,\ \forall\, n\geq 1 \end{cases} \tag{59}$$

So

$$u^3(x,t) = \sum_{n\geq 0} u_n^3(x,t) = u_0^3(x,t) = 6x + \frac{108t^\alpha}{\Gamma(\alpha+1)} \tag{60}$$

We have $u^3 = u^2$; so by Proposition 1 (b), the exact solution of the problem (45)-(46) is:

$$u(x,t) = u^2(x,t) = 6x + \frac{108t^\alpha}{\Gamma(\alpha+1)} \tag{61}$$

Example 2 Consider the following nonlinear diffusion problem:

$$\begin{cases} D_t^\alpha u = ku_{xx} + u^3 + (u_{xx})^3 \\ u(x,0) = \sin x \end{cases} \tag{62}$$

where $0 < \alpha \leq 1$, $x \in R$ and $t > 0$.
We have: $Lu(x,t) = ku_{xx}(x,t)$, $Nu(x,t) = (u(x,t))^3 + (u_{xx}(x,t))^3$ and $g(x,t) = 0$.
The SBATEM algorithm associated to the problem (62) is

$$\begin{cases} u_0^k(x,t) & = H(x,t) + E^{-1}\left[s^\alpha E\left[Nu^{k-1}(x,t)\right]\right],\ k \geq 1 \\ u_n^k(x,t) & = E^{-1}\left[s^\alpha E\left[Lu_{n-1}^k(x,t)\right]\right],\ n \geq 1 \end{cases} \tag{63}$$

with

$$\begin{aligned} H(x,t) &= E^{-1}\left[\sum_{k=0}^{m-1} s^{2+k} h^k(x)\right] + E^{-1}\left[s^\alpha E\left[g(x,t)\right]\right] \\ &= E^{-1}\left[s^2 h^0(x)\right] = h^0(x) = \sin x \end{aligned} \tag{64}$$

The algorithm (64) is again written

$$\begin{cases} u_0^k(x,t) & = \sin x + E^{-1}\left[s^\alpha E\left[Nu^{k-1}(x,t)\right]\right],\ k \geq 1 \\ u_n^k(x,t) & = E^{-1}\left[s^\alpha E\left[Lu_{n-1}^k(x,t)\right]\right],\ n \geq 1 \end{cases} \tag{65}$$

Let us apply to (65) Picard's principle: we take $u^0 = 0$, then $Nu^0 = 0$.
Step 1 for $k = 1$, we compute u^1 using the following algorithm

$$\begin{cases} u_0^1(x,t) & = \sin x \\ u_n^1(x,t) & = E^{-1}\left[s^\alpha E\left[Lu_{n-1}^1(x,t)\right]\right],\ n \geq 1 \end{cases} \tag{66}$$

We have

$$\begin{cases} u_1^1(x,t) & = E^{-1}\left[s^\alpha E\left[Lu_0^1(x,t)\right]\right] = E^{-1}\left[s^\alpha E\left[-k\sin x\right]\right] = \dfrac{-k\sin x t^\alpha}{\Gamma(\alpha+1)} \\ u_2^1(x,t) & = E^{-1}\left[s^\alpha E\left[Lu_1^1(x,t)\right]\right] = E^{-1}\left[s^\alpha E\left[\dfrac{k^2\sin x t^\alpha}{\Gamma(\alpha+1)}\right]\right] = \dfrac{k^2\sin x t^{2\alpha}}{\Gamma(2\alpha+1)} \\ u_3^1(x,t) & = E^{-1}\left[s^\alpha E\left[Lu_2^1(x,t)\right]\right] = E^{-1}\left[s^\alpha E\left[\dfrac{-k^3\sin x t^{2\alpha}}{\Gamma(2\alpha+1)}\right]\right] = \dfrac{-k^3\sin x t^{3\alpha}}{\Gamma(3\alpha+1)} \\ \vdots \\ u_n^1(x,t) & = E^{-1}\left[s^\alpha E\left[Lu_{n-1}^1(x,t)\right]\right] = \dfrac{\sin x(-kt^\alpha)^n}{\Gamma(n\alpha+1)},\ \forall n \geq 1 \end{cases} \tag{67}$$

So

$$u^1(x,t) = \sum_{n\geq 0} u_n^1(x,t) = \text{sin}x \sum_{n\geq 0} \frac{(-kt)^{n\alpha}}{\Gamma(n\alpha+1)} = \text{sin}x E_\alpha(-kt^\alpha) \tag{68}$$

is approximate solution of the problem (62) in step 1.
Step 2 for $k=2$, we compute u^2 using the following algorithm:

$$\begin{cases} u_0^2(x,t) & = \text{sin}x + E^{-1}\left[s^\alpha E\left[Nu^1(x,t)\right]\right] \\ u_n^2(x,t) & = E^{-1}\left[s^\alpha E\left[Lu_{n-1}^2(x,t)\right]\right],\ n\geq 1 \end{cases} \tag{69}$$

We have

$$Nu^1(x,t) = \left(u^1(x,t)\right)^3 + \left(u_{xx}^1(x,t)\right)^3 = (\text{sin}x E_\alpha(-kt^\alpha))^3 + (-\text{sin}x E_\alpha(-kt^\alpha))^3 = 0 \tag{70}$$

so by Proposition 1 (a), the exact solution of the problem (45)-(46) is:

$$u(x,t) = \text{sin}x E_\alpha(-kt^\alpha) \tag{71}$$

Example 3 Consider the following fractional Riccati differential equation:

$$\frac{d^\alpha y(t)}{dt^\alpha} = 2y(t) - y^2(t) + 1,\ 0 < \alpha \leq 1 \tag{72}$$

with the following initial condition

$$y(0) = 0 \tag{73}$$

We have: $Ly(t) = 2y(t)$, $Ny(t) = -y^2(t)$ and $g(t) = 1$.
The SBATEM algorithm associated to the problem (72)-(73) is

$$\begin{cases} y_0^k(t) & = H(t) + E^{-1}\left[s^\alpha E\left[Ny^{k-1}(t)\right]\right],\ k\geq 1 \\ & \\ y_n^k(t) & = E^{-1}\left[s^\alpha E\left[Ly_{n-1}^k(t)\right]\right],\ n\geq 1 \end{cases} \tag{74}$$

with

$$\begin{aligned} H(t) &= E^{-1}\left[\sum_{k=0}^{m-1} s^{2+k} h^k(t)\right] + E^{-1}\left[s^\alpha E\left[g(t)\right]\right] \\ &= E^{-1}\left[s^2 h^0(t)\right] + E^{-1}\left[s^\alpha E\left[1\right]\right] = h^0(t) + \frac{t^\alpha}{\Gamma(\alpha+1)} = \frac{t^\alpha}{\Gamma(\alpha+1)} \end{aligned} \tag{75}$$

The algorithm (74) is again written

$$\begin{cases} y_0^k(t) & = \dfrac{t^\alpha}{\Gamma(\alpha+1)} + E^{-1}\left[s^\alpha E\left[Ny^{k-1}(t)\right]\right],\ k\geq 1 \\ & \\ y_n^k(t) & = E^{-1}\left[s^\alpha E\left[Ly_{n-1}^k(t)\right]\right],\ n\geq 1 \end{cases} \tag{76}$$

Let us apply to (76) Picard's principle: we take $y^0 = 0$, then $Ny^0 = 0$.

Step 1 for $k=1$, we compute y^1 using the following algorithm

$$\begin{cases} y_0^1(t) &= \dfrac{t^\alpha}{\Gamma(\alpha+1)} \\ \\ y_n^1(t) &= E^{-1}\left[s^\alpha E\left[Ly_{n-1}^1(t)\right]\right], \; n\geq 1 \end{cases} \tag{77}$$

We have

$$\begin{cases} y_1^1(t) &= E^{-1}\left[s^\alpha E\left[Ly_0^1(t)\right]\right] = E^{-1}\left[s^\alpha E\left[2\dfrac{t^\alpha}{\Gamma(\alpha+1)}\right]\right] = \dfrac{1}{2}\dfrac{(2t^\alpha)^2}{\Gamma(2\alpha+1)} \\ y_2^1(t) &= E^{-1}\left[s^\alpha E\left[Ly_1^1(t)\right]\right] = E^{-1}\left[s^\alpha E\left[4\dfrac{t^{2\alpha}}{\Gamma(2\alpha+1)}\right]\right] = \dfrac{1}{2}\dfrac{(2t^\alpha)^3}{\Gamma(3\alpha+1)} \\ y_3^1(t) &= E^{-1}\left[s^\alpha E\left[Ly_2^1(t)\right]\right] = E^{-1}\left[s^\alpha E\left[8\dfrac{t^{3\alpha}}{\Gamma(3\alpha+1)}\right]\right] = \dfrac{1}{2}\dfrac{(2t^\alpha)^4}{\Gamma(4\alpha+1)} \\ \vdots \\ y_n^1(t) &= E^{-1}\left[s^\alpha E\left[Ly_{n-1}^1(t)\right]\right] = \dfrac{1}{2}\dfrac{(2t^\alpha)^{n+1}}{\Gamma((n+1)\alpha+1)}, \; n\geq 1 \end{cases} \tag{78}$$

So

$$y^1(t) = \sum_{n\geq 0} y_n^1(t) = \frac{1}{2}\sum_{n\geq 0}\frac{(2t^\alpha)^{n+1}}{\Gamma((n+1)\alpha+1)} = \frac{1}{2}\sum_{n\geq 1}\frac{(2t^\alpha)^n}{\Gamma(n\alpha+1)} = \frac{1}{2}E_\alpha(2t^\alpha) - \frac{t^\alpha}{\Gamma(\alpha+1)} \tag{79}$$

is approximate solution of the problem (72)-(73) in step 1.
Step 2 for $k=2$, we compute y^2 using the following algorithm:

$$\begin{cases} y_0^2(t) &= \dfrac{t^\alpha}{\Gamma(\alpha+1)} + E^{-1}\left[s^\alpha E\left[Ny^1(t)\right]\right], \; k\geq 1 \\ \\ y_n^2(t) &= E^{-1}\left[s^\alpha E\left[Ly_{n-1}^2(t)\right]\right], \; n\geq 1 \end{cases} \tag{80}$$

We have

$$\begin{aligned} Ny^1(t) &= -\left(\frac{1}{2}E_\alpha(2t^\alpha) - \frac{t^\alpha}{\Gamma(\alpha+1)}\right)^2 \\ &= -\left(\frac{t^\alpha}{a_1} + 2\frac{t^{2\alpha}}{a_2} + 4\frac{t^{3\alpha}}{a_3} + 8\frac{t^{4\alpha}}{a_4} + 16\frac{t^{5\alpha}}{a_5} + 32\frac{t^{3\alpha}}{a_6} + 64\frac{t^{7\alpha}}{a_7}...\right)^2 \end{aligned} \tag{81}$$

For $t \ll 1$, we approximate $Ny^1(t)$ by:

$$\begin{aligned} Ny^1(t) \simeq& -\frac{t^{2\alpha}}{a_1^2} - 4\frac{t^{3\alpha}}{a_1a_2} - 4\left(\frac{1}{a_2^2} + \frac{2}{a_1a_3}\right)t^{4\alpha} - \left(\frac{16}{a_1a_4} + \frac{16}{a_2a_3}\right)t^{5\alpha} - \left(\frac{32}{a_1a_5} + \frac{32}{a_2a_4}\right)t^{6\alpha} \\ &- \left(\frac{64}{a_1a_6} + \frac{64}{a_2a_5}\right)t^{7\alpha} - \left(\frac{128}{a_1a_7} + \frac{128}{a_2a_6}\right)t^{8\alpha} \end{aligned} \tag{82}$$

- Calculation of y_0^2:

$$y_0^2(t) = \frac{t^\alpha}{a_1} + E^{-1}\left[s^\alpha E\left[Ny^1(t)\right]\right] = \frac{t^\alpha}{a_1} - \frac{a_2 t^{3\alpha}}{a_1^2 a_3} - \frac{4a_3 t^{4\alpha}}{a_1 a_2 a_4}$$
$$- 4\left(\frac{1}{a_2^2} + \frac{2}{a_1 a_3}\right)\frac{a_4}{a_5}t^{5\alpha} - \left(\frac{16}{a_1 a_4} + \frac{16}{a_2 a_3}\right)\frac{a_5}{a_6}t^{6\alpha} - \left(\frac{32}{a_1 a_5} + \frac{32}{a_2 a_4}\right)\frac{a_6}{a_7}t^{7\alpha}$$
$$- \left(\frac{64}{a_1 a_6} + \frac{64}{a_2 a_5}\right)\frac{a_7}{a_8}t^{8\alpha} - \left(\frac{128}{a_1 a_7} + \frac{128}{a_2 a_6}\right)\frac{a_8}{a_9}t^{9\alpha} \quad (83)$$

-Calculation of y_1^2:

$$y_1^2(t) = E^{-1}\left[s^\alpha E\left[Ly_0^2(t)\right]\right] = \frac{2t^{2\alpha}}{a_2} - \frac{2a_2 t^{4\alpha}}{a_1^2 a_4} - \frac{8a_3 t^{5\alpha}}{a_1 a_2 a_5}$$
$$- 8\left(\frac{1}{a_2^2} + \frac{2}{a_1 a_3}\right)\frac{a_4}{a_6}t^{6\alpha} - 2\left(\frac{16}{a_1 a_4} + \frac{16}{a_2 a_3}\right)\frac{a_5}{a_7}t^{7\alpha} - 2\left(\frac{32}{a_1 a_5} + \frac{32}{a_2 a_4}\right)\frac{a_6}{a_8}t^{8\alpha}$$
$$- 2\left(\frac{64}{a_1 a_6} + \frac{64}{a_2 a_5}\right)\frac{a_7}{a_9}t^{9\alpha} - 2\left(\frac{128}{a_1 a_7} + \frac{128}{a_2 a_6}\right)\frac{a_8}{a_{10}}t^{10\alpha} \quad (84)$$

- Calculation of y_2^2:

$$y_2^2(t) = E^{-1}\left[s^\alpha E\left[Ly_1^2(t)\right]\right] = \frac{4t^{3\alpha}}{a_3} - \frac{4a_2 t^{5\alpha}}{a_1^2 a_5} - \frac{16a_3 t^{6\alpha}}{a_1 a_2 a_6}$$
$$- 16\left(\frac{1}{a_2^2} + \frac{2}{a_1 a_3}\right)\frac{a_4}{a_7}t^{7\alpha} - 4\left(\frac{16}{a_1 a_4} + \frac{16}{a_2 a_3}\right)\frac{a_5}{a_8}t^{8\alpha} - 4\left(\frac{32}{a_1 a_5} + \frac{32}{a_2 a_4}\right)\frac{a_6}{a_9}t^{9\alpha}$$
$$- 4\left(\frac{64}{a_1 a_6} + \frac{64}{a_2 a_5}\right)\frac{a_7}{a_{10}}t^{10\alpha} - 4\left(\frac{128}{a_1 a_7} + \frac{128}{a_2 a_6}\right)\frac{a_8}{a_{11}}t^{11\alpha} \quad (85)$$

- Calculation of y_3^2:

$$y_3^2(t) = E^{-1}\left[s^\alpha E\left[Ly_2^2(t)\right]\right] = \frac{8t^{4\alpha}}{a_4} - \frac{8a_2 t^{6\alpha}}{a_1^2 a_6} - \frac{32a_3 t^{7\alpha}}{a_1 a_2 a_7}$$
$$- 32\left(\frac{1}{a_2^2} + \frac{2}{a_1 a_3}\right)\frac{a_4}{a_8}t^{8\alpha} - 8\left(\frac{16}{a_1 a_4} + \frac{16}{a_2 a_3}\right)\frac{a_5}{a_9}t^{9\alpha} - 8\left(\frac{32}{a_1 a_5} + \frac{32}{a_2 a_4}\right)\frac{a_6}{a_{10}}t^{10\alpha}$$
$$- 8\left(\frac{64}{a_1 a_6} + \frac{64}{a_2 a_5}\right)\frac{a_7}{a_{11}}t^{11\alpha} - 8\left(\frac{128}{a_1 a_7} + \frac{128}{a_2 a_6}\right)\frac{a_8}{a_{12}}t^{12\alpha} \quad (86)$$

We find that for any $n \geq 1$:

$$y_n^2(t) = E^{-1}\left[s^\alpha E\left[Ly_{n-1}^2(t)\right]\right] = \frac{2^{n+1}t^{(n+1)\alpha}}{2a_{n+1}} - \frac{2^{n+3}a_2 t^{(n+3)\alpha}}{8a_1^2 a_{n+3}} - \frac{2^{n+4}a_3 t^{(n+4)\alpha}}{4a_1 a_2 a_{n+4}}$$
$$- \left(\frac{1}{a_2^2} + \frac{2}{a_1 a_3}\right)\frac{2^{n+5}a_4 t^{(n+5)\alpha}}{8a_{n+5}} - \left(\frac{1}{a_1 a_4} + \frac{1}{a_2 a_3}\right)\frac{2^{n+6}a_5}{4a_{(n+6)}}t^{(n+6)\alpha} - \left(\frac{1}{a_1 a_5} + \frac{1}{a_2 a_4}\right)\frac{2^{n+6}a_6}{4a_{n+7}}t^{(n+7)\alpha}$$
$$- \left(\frac{1}{a_1 a_6} + \frac{1}{a_2 a_5}\right)\frac{2^{n+8}a_7}{4a_{n+8}}t^{(n+8)\alpha} - \left(\frac{1}{a_1 a_7} + \frac{1}{a_2 a_6}\right)\frac{2^{n+9}a_8}{4a_{n+9}}t^{(n+9)\alpha} \quad (87)$$

So

$$y^2(t) = \sum_{n\geq 0} y_n^2(t)$$

is an approximate solution of the problem (72)-(73) in step 2.

Numerical analysis

When $\alpha = 1$,the exact solution of the problem (72)-(73) is given by $y_{ex}(t) = 1+\sqrt{2}\tanh(\sqrt{2}t+(1/2)\log((\sqrt{2}-1)/(\sqrt{2}+1)))$. We will compare this exact solution with the approximate solution $y_{ap}(t) = y^2(t) = \sum_{n\geq 0} y_n^2(t)$ for $\alpha = 1$.

If $\alpha = 1$,then we have:

$$y_{ap}(t) = y^2(t) = \sum_{n\geq 0}\frac{(2t)^{(n+1)}}{2\Gamma(n+2)} - \sum_{n\geq 0}\frac{2(2t)^{(n+3)}}{8\Gamma(n+4)} - \sum_{n\geq 0}\frac{3(2t)^{(n+4)}}{4\Gamma(n+5)} - \sum_{n\geq 0}\frac{7(2t)^{(n+5)}}{4\Gamma(n+6)} - \sum_{n\geq 0}\frac{15(2t)^{(n+6)}}{4a_{n+6}} - \sum_{n\geq 0}\frac{21(2t)^{(n+7)}}{4a_{n+7}} - \sum_{n\geq 0}\frac{28(2t)^{(n+8)}}{4a_{n+8}} - \sum_{n\geq 0}\frac{36(2t)^{(n+9)}}{4a_{n+9}} \quad (88)$$

By simplifying (88), we obtain

$$y_{ap}(t) = y^2(t) = -\frac{109e^{2t}}{4} + \frac{109}{4} + \frac{111t}{2} + \frac{111t^2}{2} + \frac{110t^3}{3} + \frac{107t^4}{6} + \frac{20t^5}{3} + \frac{17t^6}{9} + \frac{128t^7}{315} + \frac{2t^8}{35} \quad (89)$$

The following comparison table (Table 1) gives the deviation between the exact solution and the approximated solution for values of t between 0 and 0.5 for $\alpha = 1$. We represent graphically the exact solution and the approximate solution for $\alpha = 1$ in the following figure (Figure 1)

t	$y_{ex}(t)$	$y_{ap}(t)$	$\lvert y_{ex}(t) - y_{ap}(t)\rvert$
0	0	0	0
0.10	0.1103	0.1103	1.7600×10^{-6}
0.15	0.1734	0.1734	1.5295×10^{-5}
0.20	0.2420	0.2419	7.3574×10^{-5}
0.25	0.3159	0.3157	2.5567×10^{-4}
0.30	0.3951	0.3944	7.2254×10^{-4}
0.35	0.4792	0.4774	0.0018
0.40	0.5678	0.5639	0.0039
0.45	0.6603	0.6524	0.0079
0.50	0.7560	0.7410	0.0150

Table 1 : Comparison of the exact solution with the approximate solution of the Riccati problem $(72)-(73)$ for $\alpha = 1$.

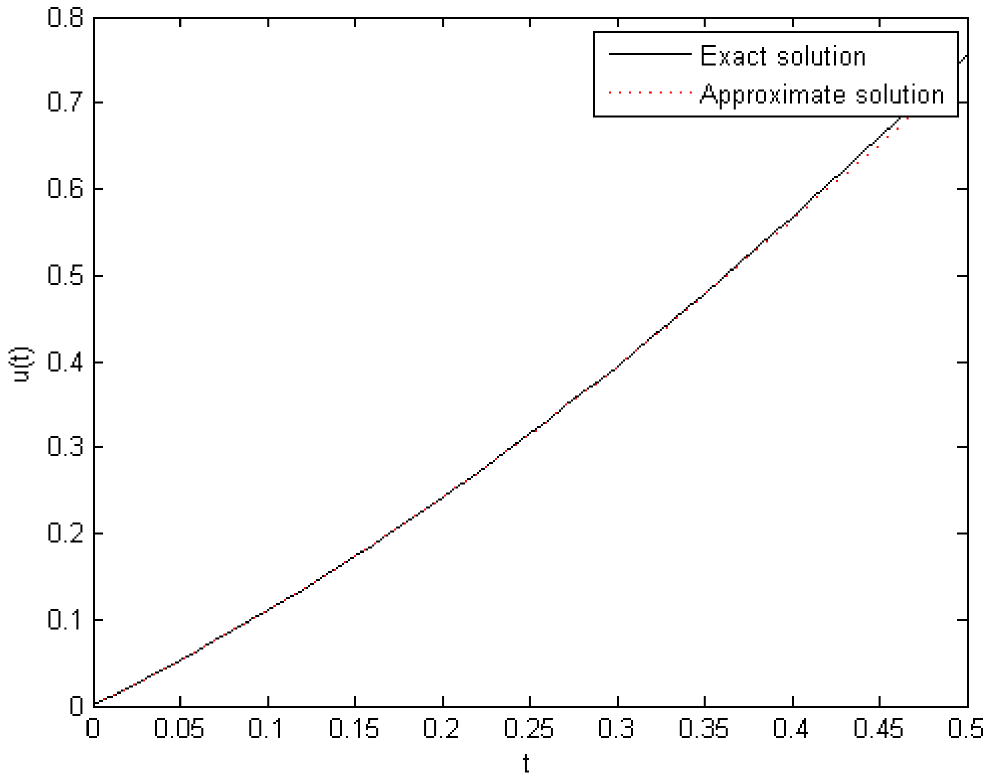

Figure 1: Exact solution and approximate solution of the Riccati problem (72)-(73) for $\alpha = 1$.

Example 4 Consider the following fractional Korteweg-de Vries (KDV) equation

$$D_t^{\alpha} u - 3\left(u^2\right)_x + u_{xxx} = 0,\ t > 0,\ 0 < \alpha \leq 1 \tag{90}$$

with the initial condition

$$u(x, 0) = 6x. \tag{91}$$

We have: $Lu(x,t) = -u_{xxx}(x,t)$, $Nu(x,t) = 3\left(u^2(x,t)\right)_x$ and $g(x,t) = 0$.
The SBATEM algorithm associated to the problem (90)-(91) is

$$\begin{cases} u_0^k(x,t) & = H(x,t) + E^{-1}\left[s^{\alpha} E\left[Nu^{k-1}(x,t)\right]\right],\ k \geq 1 \\ u_n^k(x,t) & = E^{-1}\left[s^{\alpha} E\left[Lu_{n-1}^k(x,t)\right]\right],\ n \geq 1 \end{cases} \tag{92}$$

with

$$\begin{aligned} H(x,t) &= E^{-1}\left[\sum_{k=0}^{m-1} s^{2+k} h^k(x)\right] + E^{-1}\left[s^{\alpha} E\left[g(x,t)\right]\right] \\ &= E^{-1}\left[s^2 h^0(x)\right] = h^0(x) = 6x \end{aligned} \tag{93}$$

The algorithm (92) is again written

$$\begin{cases} u_0^k(x,t) & = 6x + E^{-1}\left[s^\alpha E\left[Nu^{k-1}(x,t)\right]\right],\ k \geq 1 \\ \\ u_n^k(x,t) & = E^{-1}\left[s^\alpha E\left[Lu_{n-1}^k(x,t)\right]\right],\ n \geq 1 \end{cases} \tag{94}$$

Let us apply to (94) Picard's principle: we take $u^0 = 0$, then $Nu^0 = 0$.
Step 1 for $k = 1$, we compute u^1 using the following algorithm

$$\begin{cases} u_0^1(x,t) & = 6x \\ \\ u_n^1(x,t) & = E^{-1}\left[s^\alpha E\left[Lu_{n-1}^1(x,t)\right]\right],\ n \geq 1 \end{cases} \tag{95}$$

We have

$$\begin{cases} u_1^1(x,t) & = E^{-1}\left[s^\alpha E\left[Lu_0^1(x,t)\right]\right] = 0 \\ u_2^1(x,t) & = E^{-1}\left[s^\alpha E\left[Lu_1^1(x,t)\right]\right] = 0 \\ \vdots \\ u_n^1(x,t) & = E^{-1}\left[s^\alpha E\left[Lu_{n-1}^1(x,t)\right]\right] = 0,\ \forall\, n \geq 1 \end{cases} \tag{96}$$

So

$$u^1(x,t) = \sum_{n\geq 0} u_n^1(x,t) = u_0^1(x,t) = 6x \tag{97}$$

is approximate solution of the problem (90)-(91) in step 1.
Step 2 for $k = 2$, we compute u^2 using the following algorithm:

$$\begin{cases} u_0^2(x,t) & = 6x + E^{-1}\left[s^\alpha E\left[Nu^1(x,t)\right]\right] \\ \\ u_n^2(x,t) & = E^{-1}\left[s^\alpha E\left[Lu_{n-1}^2(x,t)\right]\right],\ n \geq 1 \end{cases} \tag{98}$$

We have

$$Nu^1(x,t) = 3(36x^2)_x = 216x \tag{99}$$

and

$$\begin{cases} u_0^2(x,t) & = 6x + E^{-1}\left[s^\alpha E\left[Nu^1(x,t)\right]\right] = 6x + \dfrac{216xt^\alpha}{\Gamma(\alpha+1)} \\ u_1^2(x,t) & = E^{-1}\left[s^\alpha E\left[Lu_0^2(x,t)\right]\right] = 0 \\ u_2^2(x,t) & = E^{-1}\left[s^\alpha E\left[Lu_1^2(x,t)\right]\right] = 0 \\ \vdots \\ u_n^2(x,t) & = E^{-1}\left[s^\alpha E\left[Lu_{n-1}^2(x,t)\right]\right] = 0,\ \forall\, n \geq 1 \end{cases} \tag{100}$$

So

$$u^2(x,t) = \sum_{n\geq 0} u_n^2(x,t) = u_0^2(x,t) = 6x + \frac{216xt^\alpha}{\Gamma(\alpha+1)} \tag{101}$$

is approximate solution of the problem (90)-(91) in step 2.

Step 3 for $k = 3$, we compute u^3 using the following algorithm:

$$\begin{cases} u_0^3(x,t) &= 6x + E^{-1}\left[s^\alpha E\left[Nu^2(x,t)\right]\right] \\ \\ u_n^3(x,t) &= E^{-1}\left[s^\alpha E\left[Lu_{n-1}^3(x,t)\right]\right], \; n \geq 1 \end{cases} \tag{102}$$

We have

$$\begin{aligned} Nu^2(x,t) = 3\left[\left(6x + \frac{216xt^\alpha}{\Gamma(\alpha+1)}\right)^2\right]_x &= 6x\left(6 + \frac{216t^\alpha}{\Gamma(\alpha+1)}\right)^2 \\ = 216x\left(1 + \frac{36t^\alpha}{\Gamma(\alpha+1)}\right)^2 &= 216x\left[1 + \frac{72t^\alpha}{\Gamma(\alpha+1)} + \frac{36^2t^{2\alpha}}{(\Gamma(\alpha+1))^2}\right] \end{aligned} \tag{103}$$

and

$$\begin{cases} u_0^3(x,t) &= 6x + E^{-1}\left[s^\alpha E\left[Nu^2(x,t)\right]\right] = 6x + \dfrac{108t^\alpha}{\Gamma(\alpha+1)} \\ &= 6x + 216xE^{-1}\left[s^\alpha E\left[1 + \dfrac{72t^\alpha}{\Gamma(\alpha+1)} + \dfrac{36^2t^{2\alpha}}{(\Gamma(\alpha+1))^2}\right]\right] \\ &= 6x + 216x\left(\dfrac{t^\alpha}{\Gamma(\alpha+1)} + \dfrac{72t^{2\alpha}}{\Gamma(2\alpha+1)} + \dfrac{36^2\Gamma(2\alpha+1)t^{3\alpha}}{(\Gamma(\alpha+1))^2\Gamma(3\alpha+1)}\right) \\ u_1^3(x,t) &= E^{-1}\left[s^\alpha E\left[Lu_0^3(x,t)\right]\right] = 0 \\ u_2^3(x,t) &= E^{-1}\left[s^\alpha E\left[Lu_1^3(x,t)\right]\right] = 0 \\ \vdots \\ u_n^3(x,t) &= E^{-1}\left[s^\alpha E\left[Lu_{n-1}^3(x,t)\right]\right] = 0, \; \forall n \geq 1 \end{cases} \tag{104}$$

So

$$\begin{aligned} u^3(x,t) = \sum_{n\geq 0} u_n^3(x,t) &= u_0^3(x,t) \\ &= 6x + 216x\left(\frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{72t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{36^2\Gamma(2\alpha+1)t^{3\alpha}}{(\Gamma(\alpha+1))^2\Gamma(3\alpha+1)}\right) \end{aligned} \tag{105}$$

is approximate solution of the problem (90)-(91) in step 3.

Step 4 for $k = 4$, we compute u^4 using the following algorithm:

$$\begin{cases} u_0^4(x,t) &= 6x + E^{-1}\left[s^\alpha E\left[Nu^3(x,t)\right]\right] \\ \\ u_n^4(x,t) &= E^{-1}\left[s^\alpha E\left[Lu_{n-1}^4(x,t)\right]\right], \; n \geq 1 \end{cases} \tag{106}$$

To simplify the expressions, let's put a=c $a_n = a_n(\alpha) = \Gamma(n\alpha + 1)$. We have

$$
\begin{aligned}
Nu^3(x,t) &= 3\left[\left(6x + 216x\left(\frac{t^\alpha}{a_1} + \frac{72t^{2\alpha}}{a_2} + \frac{36^2 a_2 t^{3\alpha}}{a_1^2 a_3}\right)\right)^2\right]_x \\
&= 216x\left[\left(1 + 36\left(\frac{t^\alpha}{a_1} + \frac{72t^{2\alpha}}{a_2} + \frac{36^2 a_2 t^{3\alpha}}{a_1^2 a_3}\right)\right)^2\right] \\
&= 216x\left[1 + \frac{72t^\alpha}{a_1} + \frac{4\times 36^2 t^{2\alpha}}{a_2} + \frac{2\times 36^3 a_2 t^{3\alpha}}{a_1^2 a_3} + 36^2\left(\frac{t^\alpha}{a_1} + \frac{72t^{2\alpha}}{a_2} + \frac{36^2 a_2 t^{3\alpha}}{a_1^2 a_3}\right)^2\right] \\
&= 216x\left[1 + \frac{72t^\alpha}{a_1} + \frac{4\times 36^2 t^{2\alpha}}{a_2} + \frac{2\times 36^3 a_2 t^{3\alpha}}{a_1^2 a_3}\right. \\
&\left. +36^2\left(\frac{t^{2\alpha}}{a_1^2} + \frac{72^2 t^{4\alpha}}{a_2^2} + \frac{36^4 (a_2)^2 t^{6\alpha}}{a_1^4 a_3^2} + \frac{144 t^{3\alpha}}{a_1 a_2} + \frac{2\times 36^2 a_2 t^{4\alpha}}{a_1^3 a_3} + \frac{4\times 36^3 t^{5\alpha}}{a_1^2 a_3}\right)\right]
\end{aligned} \tag{107}
$$

and

$$
\begin{cases}
u_0^4(x,t) &= 6x + E^{-1}\left[s^\alpha E\left[Nu^3(x,t)\right]\right] \\
&= 6x + 216x\left[\frac{t^\alpha}{a_1} + \frac{72t^{2\alpha}}{a_2} + \frac{4\times 36^2 t^{3\alpha}}{a_3} + \frac{2\times 36^3 a_2 t^{4\alpha}}{a_1^2 a_4}\right. \\
&\left. +36^2\left(\frac{a_2 t^{3\alpha}}{a_1^2 a_3} + \frac{72^2 a_4 t^{5\alpha}}{a_2^2 a_5} + \frac{36^4 (a_2)^2 a_6 t^{7\alpha}}{a_1^4 a_3^2 a_7} + \frac{144 a_3 t^{4\alpha}}{a_1 a_2 a_4} + \frac{2\times 36^2 a_2 a_4 t^{5\alpha}}{a_1^3 a_3 a_5} + \frac{4\times 36^3 a_5 t^{6\alpha}}{a_1^2 a_3 a_6}\right)\right] \\
u_1^4(x,t) &= E^{-1}\left[s^\alpha E\left[Lu_0^4(x,t)\right]\right] = 0 \\
u_2^4(x,t) &= E^{-1}\left[s^\alpha E\left[Lu_1^4(x,t)\right]\right] = 0 \\
\vdots & \\
u_n^4(x,t) &= E^{-1}\left[s^\alpha E\left[Lu_{n-1}^4(x,t)\right]\right] = 0,\ \forall\, n \geq 1
\end{cases} \tag{108}
$$

So

$$
\begin{aligned}
u^4(x,t) &= \sum_{n\geq 0} u_n^4(x,t) = u_0^4(x,t) \\
&= 6x + 216x\left[\frac{t^\alpha}{a_1} + \frac{72t^{2\alpha}}{a_2} + \frac{4\times 36^2 t^{3\alpha}}{a_3} + \frac{2\times 36^3 a_2 t^{4\alpha}}{a_1^2 a_4}\right. \\
&\left. +36^2\left(\frac{a_2 t^{3\alpha}}{a_1^2 a_3} + \frac{72^2 a_4 t^{5\alpha}}{a_2^2 a_5} + \frac{36^4 (a_2)^2 a_6 t^{7\alpha}}{a_1^4 a_3^2 a_7} + \frac{144 a_3 t^{4\alpha}}{a_1 a_2 a_4} + \frac{2\times 36^2 a_2 a_4 t^{5\alpha}}{a_1^3 a_3 a_5} + \frac{4\times 36^3 a_5 t^{6\alpha}}{a_1^2 a_3 a_6}\right)\right]
\end{aligned} \tag{109}
$$

is approximate solution of the problem (90)-(91) in step 4.

Numerical analysis

When $\alpha = 1$,the exact solution of the problem (90)-(91) is given by $u_{ex}(x,t) = \dfrac{6x}{1-36t}$. We will compare this exact solution with the approximate solution $u_{ap}(x,t) = u^4(x,t)$ for $\alpha = 1$.

If $\alpha = 1$,then we have:

$$u_{ap}(x,t) = u^4(x,t) = 6x + 216x \left[t + 36t^2 + 864t^3 + 7776t^4 + 36^2\left(\frac{t^3}{3} + \frac{1296t^5}{5} + \frac{186624t^7}{7} + 18t^4 + \frac{864t^5}{5} + 5184t^6\right)\right] \quad (110)$$

or again

$$u_{ap}(x,t) = 6x + 216x \left[t + 36t^2 + 1296t^3 + 31104t^4 + 559872t^5 + 6718464t^6 + \frac{241864624t^7}{7}\right] \quad (111)$$

The following comparison table (Table 2) gives the deviation between the exact solution and the approximated solution for values of x and t between 0 and 1, and between 0 and 0.01, respectively, for $\alpha = 1$.We represent graphically the exact solution and the approximate solution for $\alpha = 1$ in the following figure (Figure 2)

x	t	$u_{ex}(x,t)$	$u_{ap}(x,t)$	$\lvert u_{ex}(x,t) - u_{ap}(x,t)\rvert$
0	0	0	0	0
0.1	0.001	0.6224	0.6224	3.6132×10^{-7}
0.2	0.001	1.2448	1.2448	7.2264×10^{-7}
0.3	0.001	1.8672	1.8672	1.0840×10^{-6}
0.4	0.004	2.8037	2.8033	4.6563×10^{-4}
0.5	0.004	3.5047	3.5041	5.8203×10^{-4}
0.6	0.004	4.2056	4.2049	6.9844×10^{-4}
0.7	0.007	5.6150	5.6052	0.0098
0.8	0.007	6.4171	6.4059	0.0112
0.9	0.007	7.2193	7.2066	0.0126
1	0.01	9.3750	9.2983	0.0767

Table 2 : Comparison of the exact solution with the approximate solution of the KDV problem (90) – (91) for $\alpha = 1$.

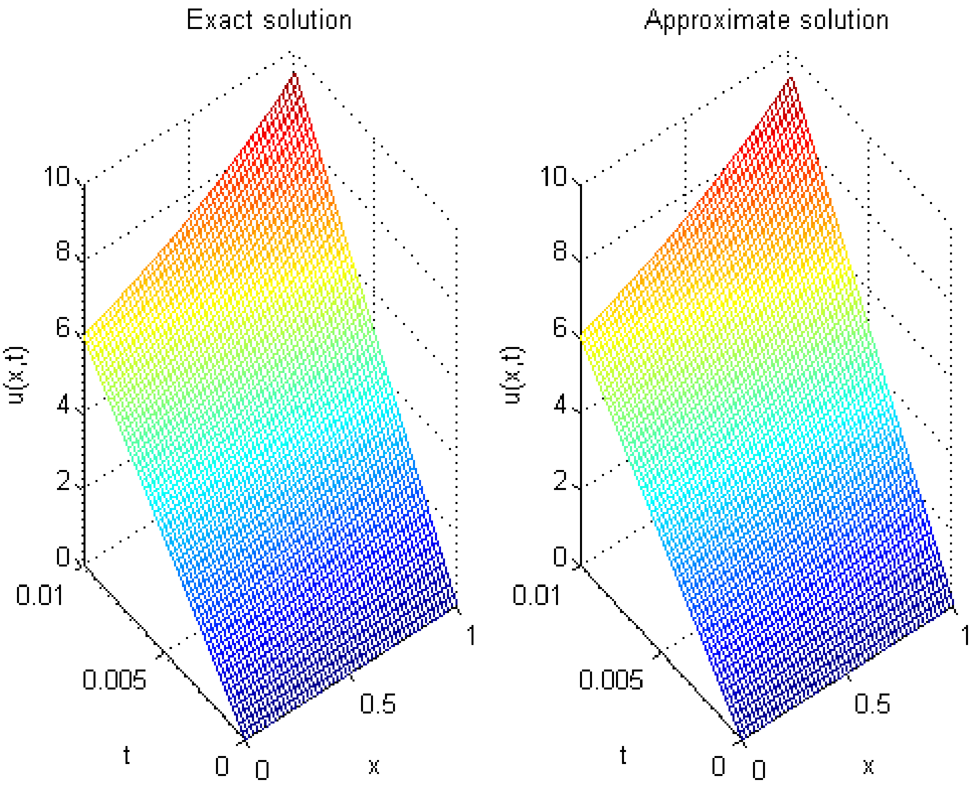

Figure 2: Exact solution and approximate solution of the KDV problem (90)-(91) for $\alpha = 1$.

5 Conclusion

In this work, we have given a new technique that allows to find the exact solution or an approximate solution of ordinary or fractional nonlinear differential equations with given initial conditions. This technique consists in coupling the Elzaky transform and the SBA method. The results obtained in the resolution of some nonlinear fractional differential equations prove the efficiency and simplicity of this new technique.

References

[1] Podlubny, I., & Kenneth, V. T. (1999). Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications (1st ed.). Academic Press.

[2] Hilfer, R. (2000). Applications of fractional calculus in physics. World scientific.

[3] Yavuz, M., Ozdemir, N., & Okur, Y. Y. (2016, July). Generalized differential transform method for fractional partial differential equation from finance. In Proceedings, International

Conference on Fractional Differentiation and its Applications, Novi Sad, Serbia (Vol. 778, p. 785).

[4] Duan, J. S., Rach, R., Baleanu, D., & Wazwaz, A. M. (2012). A review of the Adomian decomposition method and its applications to fractional differential equations. Communications in Fractional Calculus, 3(2), 73-99.

[5] Song, L., & Wang, W. (2013). A new improved Adomian decomposition method and its application to fractional differential equations. Applied Mathematical Modelling, 37(3), 1590-1598.

[6] Cetinkaya, A., Kiymaz, O., & Camli, J. (2011). Solutions of nonlinear PDE's of fractional order with generalized differential transform method. In International Mathematical Forum (Vol. 6, No. 1, pp. 39-47).

[7] Jafari, H., & Daftardar-Gejji, V. (2006). Solving a system of nonlinear fractional differential equations using Adomian decomposition. Journal of Computational and Applied Mathematics, 196(2), 644-651.

[8] Elzaki, T. M. (2012). On The New Integral Transform"Elzaki Transform"Fundamental Properties Investigations and Applications. Global Journal of Mathematical Sciences: Theory and Practical, 4(1), 1-13.

[9] wassiha NEBIE, A., Frederic, B. E. R. E., Bakari, A. B. B. O., & Youssouf, P. A. R. E. (2021). Solving Some Derivative Equations Fractional Order Nonlinear Partials Using the Some Blaise Abbo Method. Journal of Mathematics Research, 13(2), 101-101.

[10] Sedeeg, A. K. H. (2016). A coupling Elzaki transform and homotopy perturbation method for solving nonlinear fractional heat-like equations. Am. J. Math. Comput. Model, 1, 15-20.

[11] Kilbas, A. A., Srivastava, H. M., & Trujillo, J. J. (2006). Theory and Applications of Fractional Differential Equations.

[12] Loyinmi, A. C., & Akinfe, T. K. (2020). Exact solutions to the family of Fisher's reaction-diffusion equation using Elzaki homotopy transformation perturbation method. Engineering Reports, 2(2), e12084.

[13] Elzaki, T. M., Hilal, E. M., Arabia, J. S., & Arabia, J. S. (2012). Homotopy perturbation and Elzaki transform for solving nonlinear partial differential equations. Mathematical Theory and Modeling, 2(3), 33-42.

[14] Picard, E. (1893). Sur l'application des méthodes d'approximations successives à l'étude de certaines équations différentielles. Journal de mathématiques pures et appliquées 4e série, tome 9, p. 217-272.

[15] Almeida, R., Tavares, D., Torres, D. F. M. (2019). The Variable-Order Fractional Calculus of Variations.SpringerBriefs in Applied Sciences and Technology.

[16] Singh, J., Kumar, D., & Sushila, D. (2011). Homotopy perturbation Sumudu transform method for nonlinear equations. Adv. Theor. Appl. Mech, 4(4), 165-175.

[17] Dosunmu, D. A., Edeki, S. O., Achudume, C., & Udjor, V. O. (2022, February). Elzaki Adomian decomposition method applied to Logistic differential model. In Journal of Physics: Conference Series (Vol. 2199, No. 1, p. 012019). IOP Publishing.

[18] Molliq, Y., Noorani, M. S. M., & Hashim, I. (2009). Variational iteration method for fractional heat-and wave-like equations. Nonlinear Analysis: Real World Applications, 10(3), 1854-1869.

[19] Wu, G. C., & Lee, E. W. M. (2010). Fractional variational iteration method and its application. Physics Letters A, 374(25), 2506-2509.

[20] Kamate, A., Bationo, J. Y., Djibet M., & Pare, Y. (2022). Analytical Solutions of Classical and Fractional Navier-Stokes Equations by the SBA Method. Journal of Mathematics Research; Vol. 14, No. 4.

[21] Kamate, A., Djibet, M., Bationo, J. Y., Abbo, B. , & Pare, Y. (2022). Analytical solution of some nonlinear fractional integro-differential equations of the Fredholm second kind by a new approximation technique of the numerical sba method. Int. J. Numer. Methods Appl. 21, No. 1, 37-58.