Domination Defect in the Edge Corona of Graphs

Original Research Article

Abstract

A nonempty set $S \subseteq V(G)$ of fixed cardinality $\gamma(G) - k$ is called a $\zeta_k - set$ of G, where $1 \leq k \leq \gamma(G) - 1$, if S gives the minimum cardinality $|V(G) \setminus N_G[S]|$ for all the possible subsets of $V(G)$, each of which has $\gamma(G) - k$ elements. This is the number of vertices in G which are left undominated by S. In this paper, we characterize the *k-domination defect* sets of graphs resulting from the edge corona $G \diamond H$ of a connected graph G and any graph H. As a consequence, the *k-domination defect* $\zeta_k(G \diamond H)$ of $G \diamond H$ is obtained.

Keywords: k-domination defect; minimum dominating set; edge corona

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1 Introduction

Let $G = (V(G), E(G))$ be a graph. For each $x \in V(G)$, the set $N_G(x) = \{y \in V(G)|\ xy \in E(G)\}$ is called the *open neigborhood* of x in G, while the set $N_G[x] = N_G(x) \cup \{x\}$ is called the *closed neigborhood* of x in G. For a nonempty set $S \subseteq V(G)$, the *open neigborhood* of S in G and the *closed neigborhood* of S in G are given by the sets $N_G(S) = \cup_{x \in S} N_G(x)$ and $N_G[S] = N_G(S) \cup S$, respectively.

A *dominating set* of a graph G is a nonempty set $S \subseteq V(G)$ that produces $N_G[S] = V(G)$. The domination number of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set in G. Due to the minimality of $\gamma(G)$, if a set W of vertices in G has cardinality $|W| < \gamma(G)$, then there is at least 1 vertex in G which is not dominated by W. It is in this notion that Das and Desormeaux [1] introduced in 2018 the concept of k-domination defect of a graph, which was explored by [2] in 2021 on graphs resulting from the join and vertex corona products of two graphs. Further, the domination defect of some parameterized families of graphs was also investigated by [3] in 2022.

Let G be a specific graph of order n with $\gamma(G) \geq 2$ and let $1 \leq k < \gamma(G)$. Let $S \subseteq V(G)$ with $|S| = \gamma(G) - k$. The k-defect of S is $\zeta_k(S) = |V(G) \setminus N_G[S]| = n - |N_G[S]|$. The $k-domination$ $defect$ of G, denoted by $\zeta_k(G)$, is the minimum cardinality of the set $V(G) \setminus N_G[W]$ for such a set $W \subseteq V(G)$ with $|W| = \gamma(G) - k$. A set $S \subseteq V(G)$ of cardinality $\gamma(G) - k$ for which $|V(G) \setminus N_G[S]| = \zeta_k(G)$ is called a $\zeta_k - set$ of G. We emphasize without explicitly saying that if G is a graph with $\gamma(G) \geq 2$ and $S \subseteq V(G)$ is a ζ_k-set of G, where $1 \leq k < \gamma(G)$, then $|S| = \gamma(G) - k$ such that $|N_G[S]| = max\{|N_G[W]| : W \subseteq V(G), |W| = \gamma(G) - k\}$.

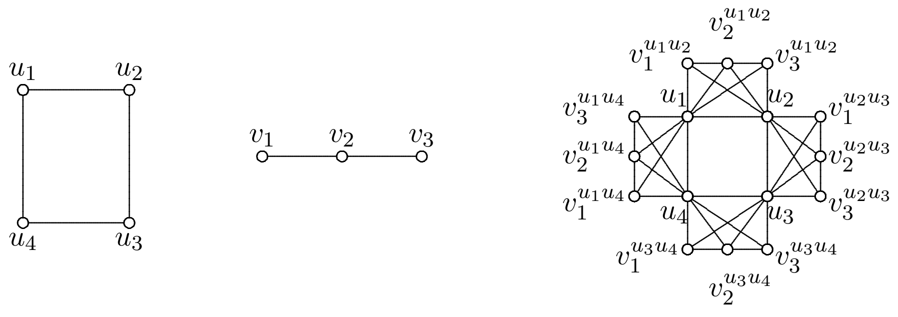
As discussed in [1] and [2], the concept of *k-domination defect* of a graph allows us to study the vulnerability of a facility if it would be guarded with fewer than the minimum number of necessary guards. In this paper, we extend our investigation of the concept to the binary operation *edge corona* $G \diamond H$ of a connected graph G and any graph H. It is our present goal to characterize the domination defect sets of these resulting graph, similar to the works in [4] and [5]. As a consequence, the domination defect number of said graphs will be obtained, reminiscent of the studies in [6] and [7].

The binary operation considered here is the edge corona of two graphs. The *edge corona*, a variation of the corona product, was introduced in 2010 by Hou and Shiu [8] where the spectrum and the number of spanning trees were studied. This graph product is non-commutative in nature. All graphs considered here are in the context of being finite, undirected, and simple graphs. For other graph theoretic terminologies not defined in this paper, the appropriate definitions in the book of Chartrand, Lesniak, and Zhang [10] are used.

2 Main Results

The *edge corona* of two graphs G and H on disjoint sets of m and n vertices, p and q edges, respectively, is defined in [8] as the graph obtained by taking one copy of G and p copies of H, and then joining two end-vertices of the $i-th$ edge of G to every vertex in the $i-th$ copy of H. This binary operation is denoted by $G \diamond H$. The edge corona of G and H has $m + pn$ vertices and $p + 2pn + pq$ edges. As discussed in [4], if $ab \in E(G)$, then the copy H whose vertices are connected one by one to both a and b in $G \diamond H$ is called the $ab-copy$ of H and is denoted by H^{ab}. If $V(H) = \{v_1, v_2, ..., v_n\}$, then the vertices of H^{ab} may be denoted by $v_1^{ab}, v_2^{ab}, ..., v_n^{ab}$. Observe that $\langle\{a, b\} \cup V(H^{ab})\rangle_{G \diamond H} \cong P_2 + H$. Figure 1 provides an illustration of the edge corona operation $G \diamond H$, where G is the cycle graph C_4 and H is the path P_3.

From the above definition, it is straightforward to see that if a graph G is a star and H is any graph, then $\gamma(G \diamond H) = 1$. To avoid triviality, we only consider connected graph G which is *not a star* for the edge corona $G \diamond H$.



C_4 P_3 $C_4 \diamond P_3$

Figure 1: The cycle C_4, path P_3, and the edge corona $C_4 \diamond P_3$

In this section, we recall the concept of a *vertex cover* of a graph G, together with some known results on the γ-set and domination number of $G \diamond H$, where a $\gamma - set$ of a graph is a particular subset of the vertex set of that graph whose cardinality is equal to the domination number of the graph. These are briefly presented below.

A *vertex cover* of a graph G is a set S of vertices of G such that each edge of G is incident to at least one vertex in S. The minimum cardinality of such a set is the *vertex covering number* of G and is denoted by $\beta(G)$. Any vertex cover of G of cardinality equal to $\beta(G)$ is called a $\beta - set$ of G. Note that if a set S is a $\beta - set$ of G, then every vertex $v \in G \setminus S$ is adjacent to at least one vertex in S. We present this simple observation in the following lemma.

Lemma 2.1. *Let G be a connected nontrivial graph which is not a spanning star. If a set $S \subseteq V(G)$ is a vertex covering of G, then S is also a dominating set of G. As a consequence, $\gamma(G) \leq \beta(G)$.*

Theorem 2.2. [9] *Suppose G is a connected graph with m edges and H be any graph. Then $D \subseteq V(G \diamond H)$ is a dominating set of $V(G \diamond H)$ if and only if $V(e_i + H) \cap D$ is a dominating set of $e_i + H$ for every $e_i \in E(G)$.*

Theorem 2.3. [9] *Let G be a connected graph and H an arbitrary graph. Then $\gamma(G \diamond H) = \beta(G)$.*

We note that if $D = V(G)$, then D is a dominating set of $G \diamond H$. A slightly stronger claim is presented in the lemma that follows.

Lemma 2.4. *Let G be a connected graph of order $m \geq 3$ and H be any graph of order n. Then $G \diamond H$ contains a $\gamma - set$ D such that $D \subsetneq V(G)$.*

Proof. Let D be a $\beta - set$ of G. Clearly, $D \subsetneq V(G)$. To show that D is a dominating set of $G \diamond H$, let $x \in V(G \diamond H) \setminus D$.
Case 1. Suppose $x \in V(G) \setminus D$. Since D is a vertex cover of G, where G is connected and nontrivial, it follows that x is adjacent to at least one vertex $y \in D$.
Case 2. Let $x \in V(H^{ab})$ for some $ab \in E(G)$. Since D is a $\beta - set$ of G, it follows that either $a \in D$ or $b \in D$. Since x is adjacent to both a and b in $G \diamond H$, it follows also that x is adjacent to at least one vertex of D.

Combining the two cases results to the conlusion that D is a dominating set of $G \diamond H$. Using Theorem 2.3, we can see that D is now a $\gamma - set$ of $G \diamond H$.

□

Theorem 2.5. *Let G be a graph of order $m \geq 3$ and let H be any graph of order n. If $k = 1, 2, ..., \beta(G) - 1\}$, then $\zeta_k(G \diamond H) \geq kn$.*

Proof. Let $S \subseteq V(G \diamond H)$ with $|S| = \gamma(G \diamond H) - k = \beta(G) - k$, $k = 1, 2, ..., \beta(G) - 1$, be a $\zeta_k - set$ of $G \diamond H$. From the definition, $\zeta_k(G \diamond H) = |V(G \diamond H) \setminus N_{G\diamond H}[S]$. Since G is connected, G has at least $m - 1$ edges. Further, every vertex $v \in S$ has at least one neighbor in G and has at least n neighbors in a copy of H. Hence, it follows that

$$\begin{aligned}\zeta_k(G \diamond H) = |V(G \diamond H) \setminus N_{G\diamond H}[S]| &\geq m + (m - 1)n - |N_{G\diamond H}[S]| \\ &\geq m + mn - n - (2|S| + n|S|) \\ &\geq m + mn - n - (2 + n)(\beta(G) - k) \\ &\geq m + mn - n - 2\beta(G) + 2k - n\beta(G) + kn \\ &\geq m + 2k - 2\beta(G) + n(m - 1 - \beta(G)) + kn\end{aligned}$$

Since $k > 0$ and $\beta(G) \leq \lceil \frac{m}{2} \rceil$, we have $m + 2k - 2\beta(G) \geq 0$. Moreover, $\beta(G)$ cannot exceed $m - 1$; thus, $n(m - 1 - \beta(G)) \geq 0$. Therefore, $\zeta_k(G \diamond H) \geq kn$. □

Lemma 2.6. *Let G be a connected graph of order $m \geq 3$ and let H be any graph of order n. Let $k = 1, 2, ..., \beta(G) - 1$. Then $G \diamond H$ contains a $\zeta_k - set$ S such that $S \subseteq V(G)$.*

Proof. The reasoning of the proof here is very similar to that of Lemma 2.4. □

The characterization of the *k-domination defect sets* of the *edge corona* of two graphs G and H follows below. Here, we define $E_S(G)$ to be the set of edges in G which are covered by $S \subseteq V(G)$. The maximum cardinality of such set is denoted by p'.

Theorem 2.7. *Let G be a connected graph of order $m \geq 3$, with $|E(G)| = p$, and let H be any graph of order n with $|E(H)| = q$. For $k = 1, 2, ...\beta(G) - 1$, let $S \subseteq V(G)$ be a ζ_k-set of $G \diamond H$. If $|E_S(G)| = p'$, then the maximum cardinality of the closed neighborhood for such a set S in $G \diamond H$ is given by:*

$$|N_{G\diamond H}[S]| = \begin{cases} m + p'n, & \text{if } |S| = \gamma(G), \gamma(G) + 1, ..., \beta(G) - 1; \\ m - \zeta_r(G) + p'n, & \text{where } r = \gamma(G) - (\beta(G) - k), \text{ if } |S| = 1, 2, ..., \gamma(G) - 1. \end{cases}$$

Proof. Let G and H be graphs with orders and sizes as described above. Let $S \subseteq V(G)$ be a $\zeta_k - set$ of $(G \diamond H)$. Then $|N_{(G\diamond H}[S]| = max\{|N_{G\diamond H}[W]| : W \subseteq V(G), |W| = \beta(G) - k\}$. We consider the following 2 cases for the value of $|S|$:

Case 1. Suppose $|S| = \gamma(G), \gamma(G) + 1, ..., \beta(G) - 1$. In this case, $\beta(G) \geq \gamma(G)$. Now, $|N_{G\diamond H}[S]| = |N_G[S] \bigcup_{ab\in E(G)} V(H^{ab})|$ for some $ab \in E(G)$. Clearly, $N_G[S]$ and $V(H^{ab})$ are disjoint sets. Noting that $|S| \geq \gamma(G)$, $|V(H^{ab})| = n$ and $|E_S(G)| = p'$ is the maximum number of edges in G which are covered by S, it follows that

$$\begin{aligned}|N_{G\diamond H}[S] &= |N_G[S] \cup_{ab\in E(G)} V(H^{ab})| \\ &= |V(G)| + \Sigma_{ab\in E(G)}|V(H^{ab})| \\ &= m + p'n.\end{aligned}$$

Case 2. Suppose $|S| = 1, 2, ..., \gamma(G) - 1$. Then $N_G[S] \subsetneq V(G)$. Since $S \subseteq V(G)$ is a $\zeta_k - set$ of $G \diamond H$, it follows that $|N_G[S]|$ is maximum among the subsets $Z \subseteq V(G)$, $|Z| = \gamma(G) - r$, where

$r = 1, 2, ..., \gamma(G) - 1$. Further, recall that $\zeta_r(G) = m - |N_G[S]|$; hence we have

$$\begin{aligned}|N_{G\diamond H}[S] &= |N_G[S] \cup_{ab\in E(G)} V(H^{ab})| \\ &= |N_G[S]| + \Sigma_{ab\in E(G)}|V(H^{ab})| \\ &= |N_G[S]| + p'n \\ &= m - \zeta_r(G) + p'n.\end{aligned}$$

□

Theorem 2.8. *Let G be a connected graph of order $m \geq 3$ and let H be any graph of order n. A set $S \subseteq V(G)$ of cardinality $\beta(G) - k$, where $k = 1, 2, ..., \beta(G) - 1$, is a ζ_k-set of $G \diamond H$ if and only if $|E_S(G)| = p'$ and exactly one of the following holds:*

(i) S is a dominating set of G, where $|S| = \gamma(G), \gamma(G) + 1, ..., \beta(G) - 1$;

(ii) S is a ζ_r − set of G, $r = \gamma(G) - (\beta(G) - k)$, where $|S| = 1, 2, ..., \gamma(G) - 1$.

Proof. Let $S \subseteq V(G)$ of cardinality $\beta(G) - k$, where $k = 1, 2, ..., \beta(G) - 1$, be a ζ_k-set of $G \diamond H$. This means that $|N_{G\diamond H}[S]|$ is maximum among the $W \subseteq V(G \diamond H)$, $|W| = 1, 2, ..., \beta(G) - 1$, where $k = 1, 2, ..., \beta(G) - 1$. We consider two cases for the value of $|S|$:

Case 1. Suppose $|S| = \gamma(G), \gamma(G) + 1, ..., \beta(G) - 1$. By assumption, S is a ζ_k − *set* of $G \diamond H$. The maximality of $|N_{G\diamond H}[S]|$ implies that $|E_S(G)| = p'$ is also maximum in G. Moreover, the fact that $S \subseteq V(G)$ with $|S| \geq \gamma(G)$, it follows that $N_G[S] = V(G)$. Hence, S is a dominating set of G. The converse is straightforward noting that $|N_G[S]| = |V(G)| = m$ and $|E_S(G)| = p'$ is the maximum number of edges in G which are covered by S.

Case 2. Suppose $|S| = 1, 2, ..., \gamma(G) - 1$. Since $S \subseteq V(G)$ is a ζ_k − *set* of $G \diamond H$, then $|E_S(G)| = p'$ and $N_S[G]$ is maximum among the subsets $Z \subseteq V(G)$, $|Z| = \gamma(G) - r$, where $r = 1, 2, ..., \gamma(G) - 1$. Hence, S is a ζ_r − *set* of G. The converse is also straightforward.

Corollary 2.9. *Let G be a graph of order $m \geq 3$ and let H be any graph of order n. Then,*

$$\zeta_k(G \diamond H) = \begin{cases} n(p - p'), & \text{if } k \in \{1, 2, ..., \beta(G) - \gamma(G)\}; \\ n(p - p') + \zeta_r(G), & \text{if } k \in \{\beta(G) - \gamma(G) + 1, \beta(G) - \gamma(G) + 2, ..., \beta(G) - 1\}, \text{ with} \\ & r = \gamma(G) - (\beta(G) - k). \end{cases}$$

Proof. Suppose $S \subseteq V(G)$ is a ζ_k − *set* of $G \diamond H$ and let $|E_S(G)| = p'$. By definition, $\zeta_k(G \diamond H) = |V(G \diamond H)| - |N_{G\diamond H}[S]| = (m + pn) - |N_G[S] \cup_{ab\in E(G)} V(H^{ab})| = (m + pn) - |N_G[S] - p'n|$. We consider the following cases:

Case 1. Suppose $k \in \{1, 2, ..., \beta(G) - \gamma(G)\}$. Then $|S| = \gamma(G), \gamma(G) + 1, ..., \beta(G) - 1$. By Theorem 2.7, $|N_{G\diamond H}[S]| = m + p'n$. Hence,

$$\begin{aligned}\zeta_k(G \diamond H) &= (m + pn) - |N_{G\diamond H}[S]| \\ &= (m + pn) - (m + p'n) \\ &= n(p - p').\end{aligned}$$

Case 2. Suppose $k \in \{\beta(G) - \gamma(G) + 1, \beta(G) - \gamma(G) + 2, ..., \beta(G) - 1\}$. Then, $|S| = 1, 2, ..., \gamma(G) - 1$. By Theorem 2.7, $|N_{G\diamond H}[S]| = m - \zeta_r(G) + p'n$. In this case,

$$\begin{aligned}\zeta_k(G \diamond H) &= (m + pn) - |N_{G\diamond H}[S]| \\ &= (m + pn) - (m - \zeta_r(G) + p'n) \\ &= n(p - p') + \zeta_r(G).\end{aligned}$$

□

3 Conclusion

In this paper, the k-domination defect of graphs resulting from the binary operation edge corona is investigated. The $\zeta_k - sets$ of the edge corona $G \diamond H$ are characterized and as a direct consequence, the corresponding k-domination defect $\zeta_k(G \diamond H)$ is then determined. It is our hope that the generated results here can be of use to others when dealing with more complex graphs and other graph operations.

References

[1] A. Das and W.J. Desormeaux , Domination defect in graphs: guarding with fewer guards, *Indian J. Pure Appl. Math.*, 49(2) (2018): 349-364, https://doi: 10.1007/s13226-018-0273-8

[2] A.T. Miranda and R.G. Eballe, Domination defect for the join and corona of graphs, *Applied Mathematical Sciences*, 15 (12) (2021), 615 - 623. https://doi.org/10.12988/ams.2021.914597

[3] A.T. Miranda and R.G. Eballe, Domination defect of some parameterized families of graphs, *Communications in Mathematics and Applications* (Submitted)

[4] M.P. Militante and R.G. Eballe, Exploring the Vertex and Edge Corona of Graphs for their Weakly Connected 2-Domination, *International Journal of Contemporary Mathematical Sciences*, 16(4)(2021), 161-172.

[5] R.G. Eballe, R. Aldema, E.M. Paluga, R.F. Rulete, F.P. Jamil (2012). Global Defensive Alliances in the Join, Corona and Composition of Graphs. Ars Comb., 107, 225-245.

[6] C.B. Balandra, (2017). Liar's domination in graphs under some operations. Tamkang Journal of Mathematics, 48(1), 61-71.

[7] M.P. Militante and R.G. Eballe, (2022). Weakly Connected 2-Domination in the Lexicographic Product of Graphs. International Journal of Mathematical Analysis, 16(3), 125-132.

[8] Y. Hou and W. Shiu, The Spectrum of the Edge Corona of Two Graphs, Electronic Journal of Linear Algebra, 20 (2010), 586-594. https://doi.org/10.13001/1081-3810.1395

[9] Rezaee Abdolhosseinzadeh, I., and Rahbarnia, F. (2017). On Generalized Edge Corona Product of Graphs. arXiv e-prints, arXiv-1712.

[10] G. Chartrand and P. Zhang, (2009) *Chromatic Graph Theory*, CRC Press, Taylor and Francis Group,USA (2009).

[11] F. Harary (2009) *Graph Theory. Reading*, MA: Addison-Wesley.

[12] C.L. Armada, S.R. Canoy, and C.E. Go (2018), Forcing domination numbers of graphs under some binary operations, *Advances and Applications in Discrete Mathematics*, 19 (3) (2018), 213-228. http://dx.doi.org/10.17654/DM019030213

[13] E. J. Cockayne, R. M. Dawes, and S. T. Hedetniemi, Total domination in graphs, Networks 10 (1980), no. 3, 211–219. https://doi.org/10.1002/net.3230100304