Original Research Article

TOPOLOGICAL NEAR GROUPS

Abstract: The concept of topological group is a simple combination of the concepts of abstract group and topological space. The purpose of this paper is to combine the concepts of topological space and near groups to define topological near groups on an nearness approximation space and study its properties.

Keyword: Near sets, near group, sub near group, topological group, topological near group.

Mathematical Subject Classification: 03E99, 20A05, 20E99

1 Introduction

The theory of rough sets, proposed by Pawlak [11] in 1982 is a powerful mathematical tool for uncertain data while modelling the problems in computer science, medical science, data analysis and many other diverse fields [13,17,20]. An algebraic approach to rough sets has been studied by Iwinski [5]. The notion of rough subgroup was introduced by Biswas and Nanda [1]. On the other hand, Kuroki and Wang [6] gave some properties of the lower and upper approximation with respect to normal subgroup and Davvaz [2] introduced the notion of rough subring (respectively ideal) with respect to an ideal of ring.

In 2002, J.F.Peters developed the near set theory as a generalization of rough set theory. Peters utilized the features of object to develop the nearness of objects [16] and consequently, the classified our universal, set with respect to the object information available. The near set approach leads to partitions of ensembles of sample objects with measurable information content and an approach to feature selection. A probe function is a real valued function representing a feature of physical objects such as images or behaviours of individual biological organisms.

The main objective of this paper is to introduce topological near groups, which extends the notion of a topological group to include the algebraic structures of near groups. In section 2, some basic notions of near sets, near groups and topological groups are given. In Section 3, new definition of topological near group is introduced and its properties are given; and some examples for topological near groups are also discussed. In section 4, topological near normal subgroup is defined and some properties of it are proved.

2 Preliminaries

In this section, some definitions and results about near sets, near groups and topological groups used in this paper are given

Definition 2.1 [14] Let $X, X' \subseteq \mathcal{O}, \mathcal{B} \subseteq \mathcal{F}$. Set X is near X' if and only if there exists $x \in X, x' \in X', \psi_i \in \mathcal{B}$ such that $x \sim_{\psi_i} x'$.

Remark 2.2 [14] If X is near X', then X is a near set relative to X' and X' is a near set relative to X.

Definition 2.3 [14] Let $X \subseteq \mathcal{O}$ and $x, x' \in X$. If x is near x', then X is called a near set relative to itself or the reflexive of X.

Definition 2.4 [14] Let $\mathcal{B} \subseteq \mathcal{F}$ be a set of functions representing features of objects $x, x' \in \mathcal{O}$. Objects x, x' are called minimally near each other if there exists $\psi_i \in \mathcal{B}$ such that $x \sim_{\{\psi_i\}} x'$,$\Delta_{\psi_i} = 0$.

Definition 2.5 [14] Let $x, x' \epsilon\, \mathcal{O}, \mathcal{B} \subseteq \mathcal{F}$. Then

$$\sim_{\mathcal{B}} = \{(x, x') \,\epsilon\, \mathcal{O} \times \mathcal{O} \,|\, \forall \psi_i \,\epsilon\, \mathcal{B}, \Delta_{\psi_i} = 0\}$$

is called the indiscernibility relation $\mathcal{O}$, where the description length $i \le |\psi|$.

Theorem 2.6 [14] The objects in a class $[x]_{\mathcal{B}} \in \xi_{\mathcal{B}}$ are near objects.

Definition 2.7 [14] Let $(\mathcal{O}, \mathcal{F}, \sim_{\mathcal{B}_r}, \mathcal{N}_r, \mathcal{V}_{\mathcal{N}_r})$ be a nearness approximation space and let "." be a binary operation defined on $\mathcal{O}$. Let $X \subseteq \mathcal{O}$ and $\mathcal{B}_r \subseteq \mathcal{F}, r \le |\mathcal{B}|$. An indiscernibility relation $\sim_{\mathcal{B}_r}$ on $\mathcal{O}$ is called a complete indiscernibility relation $\sim_{\mathcal{B}_r}$ on perceptual objects $\mathcal{O}$, if $[x]_{\mathcal{B}_r}[y]_{\mathcal{B}_r} = [xy]_{\mathcal{B}_r}$ for all $x, y \,\epsilon\, X$.

A nearness approximation space (NAS) is a tuple NAS= $(\mathcal{O}, \mathcal{F}, \sim_{\mathcal{B}_r}, \mathcal{N}_r, \mathcal{V}_{\mathcal{N}_r})$ where the approximation space NAS is defined with a set of perceived objects $\mathcal{O}$, set of probe functions $\mathcal{F}$ representing object features, indiscernibility relation $\sim_{\mathcal{B}_r}$, defined relative to $\mathcal{B}_r \subseteq \mathcal{B} \subseteq \mathcal{F}$, collection of partitions (families of neighbourhoods$\mathcal{N}_r(\mathcal{B})$, and overlap function $\mathcal{V}_{\mathcal{N}_r}$.

Theorem 2.8 [3] Let $(\mathcal{O}, \mathcal{F}, \sim_{\mathcal{B}_r}, \mathcal{N}_r, \mathcal{V}_{\mathcal{N}_r})$ be a nearness approximation space and $X, Y \subset \mathcal{O}$, then the following statement hold;

(1) $\mathcal{N}_r(\mathcal{B})_*(X) \subseteq X \subseteq \mathcal{N}_r(\mathcal{B})^*(X)$

(2) $\mathcal{N}_r(\mathcal{B})^*(X \cup Y) = \mathcal{N}_r(\mathcal{B})^*(X) \cup \mathcal{N}_r(\mathcal{B})^*(Y)$

(3) $\mathcal{N}_r(\mathcal{B})_*(X \cap Y) = \mathcal{N}_r(\mathcal{B})_*(X) \cap \mathcal{N}_r(\mathcal{B})_*(Y)$

(4) $X \subseteq Y \ implies\ \mathcal{N}_r(\mathcal{B})_*(X) \subseteq \mathcal{N}_r(\mathcal{B})_*(Y)$

(5) $X \subseteq Y \ implies \mathcal{N}_r(\mathcal{B})^*(X) \subseteq \mathcal{N}_r(\mathcal{B})^*(Y)$

(6) $\mathcal{N}_r(\mathcal{B})_*(X \cup Y) \supseteq \ \mathcal{N}_r(\mathcal{B})_*(X) \cup \mathcal{N}_r(\mathcal{B})_*(Y)$

(7) $\mathcal{N}_r(\mathcal{B})^*(X \cap Y) \subseteq \mathcal{N}_r(\mathcal{B})^*(X) \cap \mathcal{N}_r(\mathcal{B})^*(Y)$

Definition 2.9 [18] A topological group is a group $(G,*)$ together with a topology on G that satisfies the following two properties:

(1) The mapping $f: G \times G \to G$ defined by $f(x, y) = xy$ is continuous when G is endowed with the product topology

(2) The inverse mapping $g: G \to G$ defined by $g(x) = x^{-1}$ is continuous

We remark that item (1) is equivalent to the statement that , whenever $W \subseteq G$ is open, and $W \in \mathcal{N}(x_1x_2)$, then there exists open sets V_1 ad V_2 such that $V_1 \in \mathcal{N}(x_1)$; $V_2 \in \mathcal{N}(x_2)$ and $V_1V_2 = \{x_1x_2 \,|x_1 \in V_1; x_2 \in V_2\} \subseteq W$. Also, item (2) is equivalent to showing that whenever $V \subseteq G$ is open, then $V^{-1} = \{x^{-1}|x \in V\} \in \mathcal{N}(x^{-1})$ is open.

Let G be a topological group and let H be a subgroup of G. Then H becomes a topological group when endowed with the topology induced by G.

Definition 2.10 [3] Let $NAS = \left(\mathcal{O}, \mathcal{F}, \sim_{\mathcal{B}_r}, \mathcal{N}_r, \mathcal{V}_{\mathcal{N}_r}\right)$, be a nearness approximation space and let $\cdot$ be a binary operation defined on $\mathcal{O}$. A subset G of perceptual objects $\mathcal{O}$ is called a near group if the following properties are satisfied

(1) $\forall x, y \in G, x \cdot y \in \mathcal{N}_r(\mathcal{B})^*$G

(2) $\forall x, y, z \in G, (x \cdot y) \cdot z = x \cdot (y \cdot z)$ property holds in $\mathcal{N}_r(\mathcal{B})^*$G.

(3) $\exists e \in \mathcal{N}_r(\mathcal{B})^*$G such that $\forall x \in G, x \cdot e = e \cdot x = x, e$ is called the near identity elements of the near group G.

(4) $\forall x \in G, \exists y \in G$ such that , $x \cdot y =, y \cdot x = e, y$ is called the near inverse element of x in G.

Proposition 2.11 [3] Let G be a near group

(1) $\forall x, y \in$ H, $x \cdot y \in \mathcal{N}_r(\mathcal{B})^*$H

(2) $\forall x \in$ H, $x^{-1} \in$ H

(3) There is one and only one identity element in near group G.

(4) $\forall x \in$ G, there is only one y such that $x \cdot y = y \cdot x = e$; we denote it by x^{-1}

(5) $(x^{-1})^{-1} = x$.

(6) $(x \cdot y)^{-1} = y^{-1} \cdot x^{-1}$.

Proposition 2.12 [3] Let G be a near group. For all $a, x, x', y, y' \in G$

(1) If $a \cdot x = a \cdot x^{-1}$ $then$ $x = x'$.

(2) If $y \cdot a = a \cdot y^{-1}$ $then$ $y = y'$.

Definition 2.13 [3] A nonempty subset H of a near group G is called its sub neargroup, if it is itself a near group with respect to the operation $(\cdot)$.

The only guaranteed trivial sub neargroup of near group G is G itself. A necessary and sufficient condition for $\{e\}$ to be a trivial sub neargroup of near group G is $e \in G$.

Definition 2.14 [3] A necessary and sufficient condition for a subset H of a near group G to be a sub neargroup is that:

(i) $\forall x, y \in H, xy \in \mathcal{N}_r(\mathcal{B})^*$H,

(ii) $\forall x \in H, x^{-1} \in H$.

Definition 2.15 [3] A sub neargroup $\mathcal{N}$ of a near group G is called a normal sub neargroup if $a \cdot \mathcal{N} = \mathcal{N} \cdot a$ for all $a \in G$.

3 Topological Near Group

Definition 3.1 A topological near group is a near group $(G,*)$ together with a topology τ on $N_r(B)^*G$ satisfying the following two properties,

(a) $f: G \times G \to N_r(B)^*G$ defined by $f(a,b) = ab$ is continuous with respect to product topology on $G \times G$ and the topology τ_G on G induced by τ.

(b) $g: G \to G$ defined by $g(a) = a^{-1}$ is continuous with respect to the topology τ_G on G induced by τ.

(a) is equivalent to the statement that $T \subseteq N_r(B)^*G$ is open and $T \in N(a_1, a_2)$, $\exists$ open sets $E_1 \subseteq G$ and $E_2 \subseteq G$ such that $E_1 \in N(a_1), E_2 \in N(a_2)$ and $E_1E_2 = \{a_1a_2 \mid a_1 \in E_1; a_2 \in E_2\} \subseteq T$.

(b) is equivalent to the statement that whenever $E \subseteq G$ is open then $E^{-1} = \{a^{-1} | a \in E\} \in N(a^{-1})$ is open.

Example 3.2 Let $X = \{0,1,2,3\}$ be a set of perceptual objects, $B = \{\psi_1, \psi_2, \psi_3\}$ be a set of functions with respect to addition modulo 4 and $(*)$ be binary operation. Sample values of the probe functions ψ_i are defined as

The function ψ_i defined as,

$$\psi_1: \ X \to V_1 \ defined\ by\ \ \psi_1(n) = n(n-1) \qquad \forall\, n \in X$$

$$\psi_2: \ X \to V_2 \ defined\ by\ \ \psi_2(n) = n(n-1)(n-2) \qquad \forall\, n \in X$$

$$\psi_3: \ X \to V_3 \ defined\ by\ \psi_3(n) = n(n-1)(n-2)(n-3) \quad \forall\, n \in X$$

	0	1	2	3
ψ_1	0	0	2	6
ψ_2	0	0	0	6
ψ_3	0	0	0	0

Let us construct the equivalence classes for each combination, these equivalence classes are defined as

$[0]_{\{\psi_1\}} = \{x' \in X | \psi_1(x') = 0\}$,

$= \{0,1\}$

$[2]_{\{\psi_1\}} = \{2\}$

$[3]_{\{\psi_1\}} = \{3\}$

Hence we have $\xi_{\{\psi_1\}}= \{[0]_{\{\psi_1\}}, [2]_{\{\psi_1\}}, [3]_{\{\psi_1\}}\}$

$[0]_{\{\psi_2\}} = \{x' \in X | \psi_2(x') = 0\}$,

$= \{0,1,2\}$

$[3]_{\{\psi_2\}} = \{3\}$

Thus, $\xi_{\{\psi_2\}}= \{[0]_{\{\psi_2\}}, [3]_{\{\psi_2\}}\}$

$[0]_{\{\psi_3\}} = \{x' \in X | \psi_3(x') = 0\}$,

$= \{0,1,2,3\}$

Therefore as $\xi_{\{\psi_3\}}= \{[0]_{\{\psi_3\}}\}$

Therefore, for $r = 1$ a classification of X is $\mathcal{N}_1(\mathcal{B}) = \{\{\xi_{\{\psi_1\}}\}, \{\xi_{\{\psi_2\}}\}, \{\xi_{\{\psi_3\}}\}\}$.

Then, $\mathcal{N}_r(\mathcal{B})^*G \quad = \bigcup_{x:[x]_{\{\psi_i\}} \cap G \neq \emptyset} [x]_{\{\psi_i\}}$

$= \{0,1,2,3\}$

From Definition

(1) $\forall a, b \in G, \quad ab \in \mathcal{N}_r(\mathcal{B})^*(G)$

(2) The property $\forall a, b, c \in G, \quad (a \cdot b) \cdot c = a \cdot (b \cdot c)$ holds in $\mathcal{N}_r(\mathcal{B})^*(G)$

(3) $\exists\ 0 \in \mathcal{N}_r(\mathcal{B})^*(G)$ such that $\forall a \in G, a \cdot 0 = 0. a = a$

(4) $\forall a \in G, \exists b \in G$ such that $a \cdot b = b \cdot a = e$ (b is called a near inverse of a in G)

G is a near group

Let $\tau = \{\emptyset, \mathcal{N}_r(\mathcal{B})^*(G), \{1\}, \{2\}, \{3\}, \{2,3\}, \{1,3\}, \{1,2\}, \{1,2,3\}\}$ on $\mathcal{N}_r(\mathcal{B})^*(G)$

Then, $\tau_G = \{\emptyset, G, \{1\}, \{2\}, \{3\}, \{2,3\}, \{1,3\}, \{1.2\}\}$ is the relative topology on G

From Def (3.1)

(a) $1 * 1 = 2,\ for\ T \in N(2) \subseteq \tau,$ there exist open set $U = \{1\} \in N(1) \subseteq \tau_G$, such that $UU \subseteq T$

$2 * 2 = 0,\ for\ T \in N(0) \subseteq \tau$, there exist open set $U = \{2\} \in N(2) \subseteq \tau_G$, such that $UU \subseteq T$

$3 * 3 = 2,\ for\ T \in N(2) \subseteq \tau$, there exist open set $U = \{3\} \in N(3) \subseteq \tau_G$, such that $UU \subseteq T$

$2 * 3 = 1,\ for\ T \in N(1) \subseteq \tau$, there exist open set $U = \{2\} \in N(2) \subseteq \tau_G$, and $V = \{3\} \in N(3) \subseteq \tau_G$, such that $UV \subseteq T$

$1 * 3 = 0,\ for\ T \in N(0) \subseteq \tau$, there exist open set $U = \{1\} \in N(1) \subseteq \tau_G$, and $V = \{3\} \in N(3) \subseteq \tau_G$, such that $UV \subseteq T$

$1 * 2 = 3,\ for\ T \in N(3) \subseteq \tau$, there exist open set $U = \{1\} \in N(1) \subseteq \tau_G$, and $V = \{2\} \in N(2) \subseteq \tau_G$, such that $UV \subseteq T$

$1 * 2 * 3 = 2,\ for\ T \in N(2) \subseteq \tau$, there exist open set $U = \{1\} \in N(1) \subseteq \tau_G$, $V = \{2\} \in N(2) \subseteq \tau_G$, and $W = \{3\} \in N(3) \subseteq \tau_G$, such that $UVW \subseteq T$

(b) $\{1\}^{-1} = \{3\}$ is open

$\{2\}^{-1} = \{2\}$ is open

$\{3\}^{-1} = \{1\}$ is open

$\{2,3\}^{-1} = \{2,1\}$ is open

$\{1,3\}^{-1} = \{3,1\}$ is open

$\{1,2\}^{-1} = \{3,2\}$ is open

Therefore G is a topological near group.

Proposition 3.3 Let G be a topological near group. If $G = \mathcal{N}_r(\mathcal{B})^* G$, then G is topological group.

Proof: Let G be a topological near group and $G = \mathcal{N}_r(\mathcal{B})^* G$

(1) We have $\forall a, b \in G, \quad a \cdot b \in \mathcal{N}_r(\mathcal{B})^* G = G$

(2) $\forall a, b, c \in G;\ (ab)c = a(bc)$, association property holds in $\mathcal{N}_r(\mathcal{B})^* G = G$

(3) $\exists 0 \in \mathcal{N}_r(\mathcal{B})^* G$ such that $\forall a \in G,\ a \cdot 0 = 0 \cdot a = a$ (0 is the near identity element of the near group G. As $\in \mathcal{N}_r(\mathcal{B})^* G = G, 0 \in G$

(4) $\forall a \in G, \exists b \in G$ such that $a \cdot b = b \cdot a = e$ (b is called a near inverse of a in G)

Therefore G is a group

Since G is a topological near group and $G = \mathcal{N}_r(\mathcal{B})^* G$, we have the maps

(a) $f: G \times G \rightarrow G$ defined by $G \;\; f(a, b) = ab$ is continuous

(b) $g: G \rightarrow G,$ defined by $\;a = a^{-1}$is continuous

Hence G is a topological group.

Definition 3.4 Let G be a topological near group. For a fixed element a in G, we define

(i) A mapping $L_a: G \rightarrow \mathcal{N}_r(\mathcal{B})^* G$ which is defined by $L_a(x) = ax$ is called left transformation from G into $\mathcal{N}_r(\mathcal{B})^* G$.

(ii) A mapping $R_a: G \rightarrow \mathcal{N}_r(\mathcal{B})^* G$ which is defined by $R_a(x) = xa$ is called right transformation from G into $\mathcal{N}_r(\mathcal{B})^* G$.

Proposition 3.5 Let G be a topological near group and fix $a \in G$. Then,

(i) The map $L_a: G \rightarrow \mathcal{N}_r(\mathcal{B})^* G$ defined by $L_a(x) = ax$ is one-to-one and continuous, for every $x \in G$.

(ii) The map $R_a: G \rightarrow \mathcal{N}_r(\mathcal{B})^* G$ defined by $R_a(x) = xa$ is one-to-one and continuous, for every $x \in G$.

(iii) The map $f: G \rightarrow G$ defined by $f(a) = a^{-1}$ is a homeomorphism for every $a \in G$.

Proof:

(i) For every $x_1, x_2 \in G$,

$$L_a(x_1) = L_a(x_2)$$
$$\Rightarrow \; ax_1 = ax_2$$
$$\Rightarrow \; a^{-1}(ax_1) = a^{-1}(ax_2)$$
$$\Rightarrow \; (a^{-1}a)x_1 = (a^{-1}a)x_2$$
$$\Rightarrow \; x_1 = x_2$$

Hence L_a is one – to – one.

Let $T \in N(ax) \subseteq \mathcal{N}_r(\mathcal{B})^* G$. By definition of topological near group $\exists U \in N(a) \subseteq \tau_G$ and $V \in N(x) \subseteq \tau_G$ such that $UV \subseteq T$. Since $aV \subseteq UV \subseteq T$. $L_a(V) = aV \subseteq T$. Therefore L_a is continuous on x. Since x is an arbitrary element of G, then L_a is continuous on G.

(ii) For every $x_1, x_2 \in G$,

$$R_a(x_1) = R_a(x_2)$$
$$\Rightarrow \; x_1 a = x_2 a$$
$$\Rightarrow \; (x_1 a)a^{-1} = (x_2 a)a^{-1}$$
$$\Rightarrow x_1(a\, a^{-1}) = x_2(a\, a^{-1})$$
$$\Rightarrow \; x_1 = x_2$$

Hence R_a is one – to – one.

Let $T \in N(xa) \subseteq \mathcal{N}_r(\mathcal{B})^*G$. By definition of topological near group $\exists U \in N(a) \subseteq \tau_G$ and $V \in N(x) \subseteq \tau_G$ such that $VU \subseteq T$. Since $Va \subseteq VU \subseteq T$, $R_a(V) = Va \subseteq T$. Therefore R_a is continuous on x. Since x is an arbitrary element of G, then R_a is continuous on G.

(iii) The map $f: G \to G$ defined by $f(a) = a^{-1}$ is one- to- one and onto, since inverse element is unique in a group G. By definition of topological near group f is continuous. The continuity of the inverse mapping $f^{-1}: G \to G$ defined by, $f^{-1}(a) = a^{-1}$ also holds good. Therefore f is a homeomorphism.

Remark 3.6 The map L_a and R_a defined previously are not onto. Furthermore, the maps L_a and R_a are not open. Let $G = \{1,2,3\}$,

$\tau = \{\emptyset, \mathcal{N}_r(\mathcal{B})^*(G), \{1\}, \{2\}, \{3\}, \{1,3\}, \{1,2\}, \{2,3\}, \{1,2,3\}\}$ and,

$\tau_G = \{\emptyset, G, \{1\}, \{2\}, \{3\}, \{1,3\}, \{1,2\}, \{2,3\}\}$ from the Example 2, $2 \in G$ and $V \in \{2\} \subseteq \tau_G$. Thus $V \subseteq \tau_G$ is open, but then $L_2(\{2\}) = 2 + 2 = \{0\}$ is not open in τ.

Proposition 3.7 Let G be a topological near group and $V \subseteq G$. Then V is open (Closed)$\Leftrightarrow V^{-1}$ is open (Closed).

Proof: Let V be open in G. By Proposition 3.5, $f: G \to G$ defined by $f(a) = a^{-1}$ is a homeomorphism. (i.e) $f: G \to G$ and $f^{-1}: G \to G$ are continuous.$f(V) = V^{-1}$ is open in G. Similarly, if V^{-1} is open $\Rightarrow f^{-1}(V^{-1}) = V$ is open in G. Hence V is open in $G \Leftrightarrow V^{-1}$ is open in G.

Proposition 3.8 Let G be a topological near group. Let $G, \{e\} \in \mathcal{N}_r(\mathcal{B})^*G$ be an open subsets of $\mathcal{N}_r(\mathcal{B})^*G$.

(i) If $e \in G$, $\{e\} \cap G = \{e\} \subseteq \tau_{G.}$ Therefore $\{e\}$ is open in space G.

(ii) If $e \notin G$, then there exists an open subset $V \subseteq G$ such that $V = V^{-1}$

Proof:

(i) Let $e \in G$. Since $f: G \times G \to \mathcal{N}_r(\mathcal{B})^*G$ is continuous. $f^{-1}(\{e\})$ is open in $G \times G$ and $ee = e \in \{e\} \in \tau$. $\exists$ open sets $V_1, V_2 \in \tau_G$ with $e \in V_1, e \in V_2$ with $V_1V_2 \subseteq \{e\} \in N\{e\}$.

(ii) Let $e \notin G$. Since G is a topological near group and $\{e\} \subseteq \mathcal{N}_r(\mathcal{B})^*G$ is an open subset, then from definition of topological near group $f^{-1}(\{\mathrm{e}\})$ is open in $G \times G$.

So $\forall a, b \in G$, $ab = e$, $\exists$ open sets $V_1 \in N(a)$ and $V_2 \in N(b)$ such that $V_1V_2 \subseteq \{e\} \in N\{e\}$.

Proposition 3.9 Let G be a topological near group and $T \subseteq \mathcal{N}_r(\mathcal{B})^*G$ be an open subset with $e \in T$. Then there exists an open set V, with $e \in V$ such that $V = V^{-1}$ and $VV \subseteq T$.

Proof: Since $f: G \times G \to \mathcal{N}_r(\mathcal{B})^*G$ is continuous, $f^{-1}(T)$ is open in $G \times G$ and $ee = e \in T \in \tau$. Hence, there exists open sets $V_1, V_2 \in \tau_G$ with $e \in V_1, e \in V_2$ such that $V_1V_2 \subseteq T$. V_1^{-1}, V_2^{-1}

are open, by Proposition 3.7. hence $V = V_1 \cap V_2 \cap {V_1}^{-1} \cap {V_2}^{-1}$ is also open. Thus $e \in V, V = V^{-1}$ and $VV \subseteq V_1V_2 \subseteq T$.

4 Topological sub neargroup and topological normal sub neargroup

In this section we introduce the concept of topological sub neargroup and topological normal sub neargroups. We consider the relative topology on a sub neargroup

Definition 4.1 Let G be a topological near group and let A be a subgroup of G .Then, A is called a topological sub neargroup of G if

(1) $f_A: A \times A \rightarrow \mathcal{N}_r(\mathcal{B})^*A$ defined by $f_A(a,b) = ab$ is continuous where the topology on $\mathcal{N}_r(\mathcal{B})^*A$ is topology induced by $\mathcal{N}_r(\mathcal{B})^*G$.

(2) $g_A: A \rightarrow A$ defined by $g_A(a) = a^{-1}$ is continuous.

Proposition 4.2 Let A_1 and A_2 be two topological sub neargroups of the topological near group G. A sufficient condition for intersection of two topological sub neargroups of a topological neargroup to be a topological sub neargroup is

$\mathcal{N}_r(\mathcal{B})^*(A_1) \cap \mathcal{N}_r(\mathcal{B})^*(A_2) = \mathcal{N}_r(\mathcal{B})^*(A_1 \cap A_2)$.

Proof: Suppose $A_1$and A_2 are two topological sub neargroups of G. It is obvious that $A_1 \cap A_2 \subset G$. Let $a, b \in A_1 \cap A_2$, Because A_1 and A_2 are sub neargroups, we have $ab \in \mathcal{N}_r(\mathcal{B})^*(A_1), ab \in \mathcal{N}_r(\mathcal{B})^*(A_2)$ and $a^{-1} \in A_1$, $b^{-1} \in A_2$, i.e. $ab \in \mathcal{N}_r(\mathcal{B})^*(A_1) \cap \mathcal{N}_r(\mathcal{B})^*(A_2)$ and $a^{-1} \in A_1 \cap A_2$. Assuming $\mathcal{N}_r(\mathcal{B})^*(A_1) \cap \mathcal{N}_r(\mathcal{B})^*(A_2) = \mathcal{N}_r(\mathcal{B})^*(A_1 \cap A_2)$, we have $ab \in \mathcal{N}_r(\mathcal{B})^*(A_1 \cap A_2)$ and $a^{-1} \in A_1 \cap A_2$. Thus $A_1 \cap A_2$ is a sub neargroup of G.

Proposition 4.3 If $f: G \times G \rightarrow \mathcal{N}_r(\mathcal{B})^*G$ defined by $f(a,b) = ab$ is continuous, and A is a sub neargroup of G Then its restriction $f_A: A \times A \rightarrow \mathcal{N}_r(\mathcal{B})^*A$ defined by $f_A(a,b) = ab$ is continuous.

Proof: Let $T \subseteq \mathcal{N}_r(\mathcal{B})^*A$ be open, and $T \in N_A(a,b)$. $Then\, T = T' \cap A,\ T'$ is open in $\mathcal{N}_r(\mathcal{B})^*G$ and $T' \in N_G(a,b)$. For every $V_1' \in N_G(a)$ and $V_2' \in N_G(b) \Rightarrow V_1'V_2' \subset T'$. Therefore f is continuous. $V_1' \cap A = V_1 \in N_A(a)$ and $V_2' \cap A = V_2 \in N_A(b)$. $V_1V_2 = (V_1' \cap A)(\, V_2' \cap A) \in N_A(a,b)$. Also $V_1V_2 \subseteq \mathcal{N}_r(\mathcal{B})^*A$. $V_1V_2 = (V_1' \cap A)(\, V_2' \cap A) \subseteq V_1'V_2' \cap A \subset T' \cap A = T$. Therefore $f_A: A \times A \rightarrow \mathcal{N}_r(\mathcal{B})^*A$ is continuous.

Proposition 4.4 If $g: G \rightarrow G$ defined by $g(a) = a^{-1}$ is continuous and A is a sub neargroup of G. Then its restriction $g_A: A \rightarrow A$ defined by $g_A(a) = a^{-1}$ is continuous.

Proof: Since g is continuous, $g|_A$ is continuous by proposition 3.7

Proposition 4.5 Let G be a topological near group. Then every sub neargroup A of G with relative topology is a topological sub neargroup.

Proof: Since the restriction mapping $f: G \times G \to \mathcal{N}_r(\mathcal{B})^* G, f_A: A \times A \to \mathcal{N}_r(\mathcal{B})^* A$ and inverse mapping $g_A: A \to A$ defined by $g_A(a) = a^{-1}$ are continuous (by Proposition 4.3 and 4.4) Hence A is a topological sub neargroup of G.

Proposition 4.6 Let A_1 and A_2 are two topological sub near group of the topological near group G with $(\mathcal{N}_r(\mathcal{B})^* A_1)\,(\mathcal{N}_r(\mathcal{B})^* A_2) = \mathcal{N}_r(\mathcal{B})^*(A_1 A_2)$. Then $A_1\ A_2$ is a topological sub near group of the topological near group G iff $A_1 A_2 = A_2 A_1$ where $A_1 A_2 = \{a_1 a_2 | a_1 \in A_1, a_2 \in A_2\}$

Proof: $\Rightarrow$ Let $A_1\ A_2$ be a topological sub neargroup of G. We prove that $A_1\ A_2 = A_2\ A_1$. Suppose that $a_2 a_1 \in A_2 A_1\ Then\ (a_2 a_1)^{-1} = a_1^{-1} a_2^{-1} \in A_1 A_2$ since $A_1 A_2$ is a sub neargroup then, $(a_1^{-1} a_2^{-1})^{-1} = a_2 a_1 \in A_1 A_2$ thus $A_2\ A_1 \subseteq A_1\ A_2$. Again suppose that $\in A_1\ A_2$. Since $A_1\ A_2$ is a sub neargroup, $x^{-1} = a_1 a_2 \in A_1 A_2,$ where $a_1 \in A_1$ and $a_2 \in A_2$. Thus $(x^{-1})^{-1} = a_2^{-1} a_1^{-1} \in A_2 A_1$ so $A_1\ A_2 \subseteq A_2\ A_1$. Consequently $A_1 A_2 = A_2 A_1$.

$\Leftarrow$ Let $A_1\ A_2 = A_2\ A_1$ we prove that $A_1\ A_2$ be a topological sub neargroup of G. Suppose that $a_1\ a_2\ , b_1\ b_2\ \in A_1\ A_2$. Then

$$\begin{aligned}(a_1\ a_2\ )(\ b_1\ b_2) &= a_1(a_2 b_1) b_2 \\ &= a_1(b_1 a_2) b_2 \\ &= (a_1\ b_1\ )(\ a_2\ b_2) \\ &\in \mathcal{N}_r(\mathcal{B})^* A_1\ \mathcal{N}_r(\mathcal{B})^* A_2\end{aligned}$$

Since $\mathcal{N}_r(\mathcal{B})^* A_1\ \mathcal{N}_r(\mathcal{B})^* A_2 = \mathcal{N}_r(\mathcal{B})^* A_1 A_2$ we have $(a_1\ a_2\ )(\ b_1\ b_2) = \mathcal{N}_r(\mathcal{B})^*(A_1 A_2)$. Since associativity holds good in $A_1$and A_2, it holds good in $A_1 A_2$ also $e \in A_1, e \in A_2 \Rightarrow e \cdot e = e \in A_1 A_2$. Let $a_1 a_2 \in A_1 A_2$. Then, $(a_1 a_2)^{-1} = a_2^{-1} a_1^{-1} \in A_2 A_1$.But $A_1 A_2 = A_2 A_1$ so $(a_1 a_2)^{-1} \in A_1 A_2$. Thus $A_1\ A_2$ is a sub neargroup.

Definition 4.7 A topological sub neargroup N of a topological near group G is called a topological normal sub neargroup if, for all $a \in G,\ a \cdot N = N \cdot a$ where $'\cdot'$ is the binary operation in G.

Proposition 4.8 A necessary and sufficient condition for a topological sub neargroup N of a topological near group G to be a topological normal sub neargroup is that

$$a \cdot N \cdot a^{-1} = N\ for\ all\ a \in G$$

Proof: Suppose N is topological normal sub neargroup of G. By definition 4.7 $\forall a \in G$ we have $\cdot N = N \cdot a$.

This implies $(a \cdot N) \cdot a^{-1} = (N \cdot a) \cdot a^{-1}$

$$a \cdot N \cdot a^{-1} = N(a \cdot a^{-1})$$

$$a \cdot N \cdot a^{-1} = N$$

Suppose N is a topological sub near group of G and for all $a \in G, a \cdot N \cdot a^{-1} = N$.
Then $(a \cdot N \cdot a^{-1}) \cdot a = N \cdot a$ i.e) $a \cdot N = N \cdot a$. Thus N is a topological normal sub near group of G.

Proposition 4.9 If A_1 and A_2 are two topological normal sub near groups of the topological near group G with $(\mathcal{N}_r(\mathcal{B})^* A_1)\ (\mathcal{N}_r(\mathcal{B})^* A_2) = \mathcal{N}_r(\mathcal{B})^*(A_1 A_2)$. Let $A_1 A_2 = A_2 A_1$ Then $A_1 A_2$ is a topological normal sub near group of the topological near group G.

Proof: By Proposition 4.6, $A_1 A_2$ is a topological sub neargroup of G. Then sub neargroup $a(A_1 A_2) = a(A_1)A_2 = (A_1 a)A_2 \Rightarrow A_1(aA_2) = A_1(A_2 a) = (A_1 A_2)a, \forall a \in G$.
Hence $A_1 A_2$is a topological sub neargroup of G.

Proposition 4.10 A necessary and sufficient condition for a topological sub near group N of the topological near group G to be a topological normal sub neargroup is that
$a \cdot n \cdot a^{-1} \in N$ for all $a \in G$ and $n \in N$

Proof: Necessary that N is a topological normal sub near group of the topological near group G. We have, $a \cdot N \cdot a^{-1} = N$ for all $a \in G$ For any $\in N$, $a \cdot n \cdot a^{-1} \in N$.
Sufficiency N is a topological sub near group of the topological near group G. Suppose $a \cdot n \cdot a^{-1} \in N$ for all $a \in G$ and $n \in N$. To prove that $a \cdot N = N \cdot a$. Let $x \in a \cdot N$. Therefore $x = an$ for some $\in N$. $x = (a \cdot n \cdot a^{-1})a \in N \cdot a$, $a \cdot N \subseteq N \cdot a$. Let $x \in N \cdot a$. Therefore $x = N \cdot a$ for some $n \in N$. $x = a(a^{-1} \cdot n \cdot a) = a(a^{-1} \cdot n(a^{-1})^{-1} \in a \cdot N, N \cdot a \subseteq a \cdot N$, so $N \cdot a = a \cdot N$. Thus N is a topological normal sub neargroup of G.

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