Integral transforms of the Hilfer-type fractional derivatives

Original Research Article

Abstract

In this paper, some important properties concerning the Hilfer-type fractional derivative are discussed. Integral transforms for these operators are derived as particular cases of the Jafari transform, Mellin transform and Fourier transform. These integral transforms are used to derive a fractional version of the fundamental theorem of calculus. An application is get with the Jafari transform and finite Hankel transform to obtain the analytical solution to fractional radial diffusion equation in terms of the κ-Hilfer fractional derivative.

Keywords: Jafari transform, Fundamental theorem of fractional calculus, Hilfer-Type fractional derivative,

κ-Riesz fractional derivative, time-fractional radial diffusion

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1 Introduction

Fractional Calculus (FC) is considered a powerful tool to solve problems which arises from engineering, physics and other applied sciences [20, 6, 1]. Although its study has been advanced considerably in the decade of 1970, its origin dates back to the emerge of the well-known calculus of “integer” order. Many researchers nowadays consider that arbitrary order calculus plays a prominent role in modelling phenomena. In Ref. [5] a historical approach of the development of the fractional calculus is presented. In Ref. [21] the development of fractional calculus over the last decades and some recent contributions and applications are discussed. In Ref. [10], some techniques involving special functions, fractional calculus as well as their applications in partial differential equations, such as diffusion and advection phenomena, are introduced.

It is well known that fractional calculus can be studied over several approaches, such as Riemann-Liouville, Caputo, Riesz and Riesz-Feller derivatives [2, 3, 4]. In Ref. [11] a new fractional integral which generalizes the Riemann–Liouville and Hadamard fractional integrals into a single form is introduced, giving rise to a general definition of fractional derivative. Based on this latter work, in Ref. [17] a new differential operator of arbitrary order defined by means of a Caputo type modification of the generalized fractional derivative is discussed.

An even more general case is discussed in Ref.[22]. Based on a function $\Psi(x)$ which presents some specific properties, a new fractional operator is introduced and called Ψ-Hilfer fractional derivative.

Recently, a modification in the latter definition is proposed in Ref.[12], in terms of the so-called gamma-kappa function. It can be showed that all fractional derivatives cases cited in [17] and [22] can be recovered in terms of this new definition, which has been named (κ, Ψ)- Hilfer fractional derivative.

This paper is organized in the following form: In Section 1, the preliminaries concepts related to the gamma-kappa function and its integral transforms are presented. In Section 2 the fractional integrals in terms of the Ψ function and its particular case (κ, ρ)-Hilfer fractional derivative are presented.

In Section 3 the κ-Hilfer fractional derivative is proposed and some particular cases are discussed. In Section 4 some integral transforms are applied to κ fractional operators, namely the Jafari transform to κ-Hilfer fractional derivative, the Mellin transform to (κ, ρ)-Hilfer fractional derivative and the Fourier transform to κ-Riesz fractional derivative. In Section 5 the Jafari transform is used to solve the fractional radial diffusion problem, where are ploted the graphs to some numeric values. In the last one, some conclusions and observations are made.

2 Preliminaries

In order to introduce a new fractional operator, this section recovers the concepts of κ-gamma function and k-beta function as well as their properties.

Definition 2.1 ([7] k-GAMMA FUNCTION). Let $z \in \mathbb{C}$, $\mathrm{Re}(z) > 0$ and $k > 0$. The k-gamma function is defined as

$$\Gamma_k(z) = \int_0^\infty t^{z-1} e^{-\frac{t^k}{k}} dt. \quad (2.1)$$

In the limit $k \to 1$, the well-know gamma function is recovered. In Ref. [23], some interesting expressions are derived, such as:

- Relation between κ-gamma and gamma functions:

$$\Gamma_k(z) = k^{\frac{z}{k}-1}\Gamma\left(\frac{z}{k}\right). \quad (2.2)$$

Proof. Taking the change of variables $u = t^k/k$, we obtain:

$$\Gamma_k(z) = k^{\frac{z-k}{k}} \int_0^\infty u^{\frac{z}{k}-1} e^{-u} du = k^{\frac{z}{k}-1}\Gamma\left(\frac{z}{k}\right).$$

- k-gamma function of $z = k$:

$$\Gamma_k(k) = 1. \quad (2.3)$$

Proof. Using the expression in Eq.(2.2), we obtain

$$\Gamma_k(k) = k^{\frac{k}{k}-1}\Gamma(1) = 1.$$

- Recursion Formula:

$$\Gamma_k(z+k) = z\,\Gamma_k(z). \quad (2.4)$$

Proof. Using the expression in Eq.(2.2) and the gamma function recursion formula, we obtain

$$\Gamma_k(z+k) = k^{\frac{z+k}{k}-1}\,\Gamma\left(\frac{z}{k}+1\right) = zk^{\frac{z}{k}-1}\,\Gamma\left(\frac{z}{k}\right).$$

- Reflection formula:

$$\Gamma_k(z)\,\Gamma_k(k-z) = \frac{\pi}{k\sin\left(\frac{\pi z}{k}\right)}. \quad (2.5)$$

Proof. Using the expression in Eq.(2.2) and the gamma function reflection formula, we obtain

$$\Gamma_k(z)\,\Gamma_k(k-z) = \frac{1}{k}\,\Gamma\left(\frac{z}{k}\right)\Gamma\left(1-\frac{z}{k}\right) = \frac{\pi}{k\sin\left(\frac{\pi z}{k}\right)}.$$

Definition 2.2 ([7] k-Pochhammer symbol). Let $z \in \mathbb{C}$, $k \in \mathbb{R}$ and $m \in \mathbb{N}_+$. The k-Pochhammer symbol is

$$(z)_{m,k} = z(z+k)(z+2k)\dots(z+(m-1)k). \quad (2.6)$$

Using the property in Eq.(2.4), the k-Pochhamer symbol in Eq.(2.6) may be rewritten in terms of the k-gamma function, that is,

$$(z)_{m,k} = \frac{\Gamma_k(z+mk)}{\Gamma_k(z)}. \quad (2.7)$$

• The k-gamma function may be expressed in terms of k-Pochhamer symbols as it follows:

$$\Gamma_k(z) = \lim_{n\to\infty} \frac{n!k^n(nk)^{\frac{z}{k}-1}}{(z)_{n,k}}. \quad (2.8)$$

Proof. The product formula definition is given by [14]

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!n^z}{z(z+1)\dots(z+n)}.$$

Taking the change of variables $z \to z-1$ and using the gamma function reflection formula, this expression can be rewritten as

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!n^{z-1}}{z\dots(z+n-1)}.$$

Replacing the latter in Eq.(2.2), we obtain

$$\Gamma_k(z) = k^{\frac{z}{k}-1} \lim_{n\to\infty} \frac{n!n^{\frac{z}{k}-1}}{\frac{z}{k}(\frac{z}{k}+1)\dots(\frac{z}{k}+n-1)} \quad (2.9)$$

$$= \lim_{n\to\infty} \frac{k^{\frac{z}{k}+n-1}n!n^{\frac{z}{k}-1}}{}$$

$z(z+k)\ldots(z+(n-1)k)$

$$= \lim_{n\to\infty} \frac{n!k^n(nk)^{\frac{z}{k}-1}}{(z)_{n,k}}. \quad (2.10)$$

Definition 2.3 ([7] k-BETA FUNCTION). Let $z, y \in \mathbb{C}$, $Re(z) > 0$, $Re(y) > 0$, and $k > 0$. The k-beta function is given by

$$B_k(y,z) = \frac{1}{k}\int_0^1 u^{\frac{y}{k}-1}(1-u)^{\frac{z}{k}-1}du. \quad (2.11)$$

It follows directly from the definition that the relation between the k-beta and beta functions are given by

$$B_\kappa(x,y) = \frac{1}{\kappa} B\left(\frac{x}{\kappa}, \frac{y}{\kappa}\right). \quad (2.12)$$

The well-known expression for the beta function in terms of gamma functions is also valid for the the k-beta function, that is,

$$B_k(y,z) = \frac{\Gamma_k(y)\,\Gamma_k(z)}{\Gamma_k(y+z)}. \quad (2.13)$$

The integral in Eq.(2.13) can be identified as a beta function. It follow from the relation in Eq.(2.2) that

$$B_k(y,z) = \frac{1}{\ldots}$$

$$B_k\left(\frac{y}{k},\frac{z}{k}\right) = \frac{1}{k}\frac{\Gamma\left(\frac{y}{k}\right)\Gamma\left(\frac{z}{k}\right)}{\Gamma\left(\frac{y+z}{k}\right)} = \frac{k^{2-y-z}\,\Gamma_k(y)\,\Gamma_k(z)}{k^{2-y-z}\,\Gamma_k(y+z)} = \frac{\Gamma_k(y)\,\Gamma_k(z)}{\Gamma_k(y+z)}.$$

Definition 2.4 ([8] Mittag-Leffler). Let $z \in \mathbb{C}$, $Re(\alpha) > 0$, $Re(\sigma) > 0$ and $Re(\gamma) > 0$:

$$E^{\gamma}_{\alpha,\sigma}(z) = \sum_{m=0}^{\infty} \frac{(\gamma)_m}{\Gamma(\alpha k + \sigma)} \frac{z^m}{m!} \tag{2.14}$$

where $(\gamma)_m$ is Pochhammer symbol.

2.1 Integral transforms

Definition 2.5 ([16] Mellin transforms). Let $f(x)$ be a real-valued function defined on the interval $(0,\infty)$. The Mellin transform $F(s)$ of the function $f(x)$ is denoted by $M[f(x)](s)$, and defined as

$$M[f(x)](s) = \int_0^{+\infty} f(x)x^{s-1}\,dx, \tag{2.15}$$

where $k_1, k_2 \in \mathbb{R}$ and $s \in \mathbb{C}$ such that $k_1 < \Re(s) < k_2$.

The inverse of the Mellin transform is denoted by $M^{-1}[F(s)](s) = f(x)$, and defined as

$$M^{-1}[F(s)](x) = \frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} F(s)x^{-s}\,ds, \quad (2.16)$$

where k is a real constant such that $k_1 < k < k_2$.

Definition 2.6 ([9] JAFARI TRANSFORM)**.** Let $h(t)$ be a integrable function defined for $t \geq 0$ and $p(s)$, $q(s)$ positive real functions. The general integral transform T of the $h(t)$ defined by

$$T[h(t)](s) = p(s)\int_{0}^{+\infty} h(t)e^{-q(s)t}\,dt, \quad (2.17)$$

generalizes the classical integral transform like Sumudu and Laplace transform. Since it has been introduced by H. Jafari, it will be referred as Jafari transform in this work. The interested reader can find more details in Ref. [9].

Theorem 2.1 ([9] JAFARI TRANSFORM CONVOLUTION)**.** Consider the functions $h_1(t)$, $h_2(t)$ and their Jafari transforms $H_1(s)$, $H_2(s)$, respectively. The Jafari transform of the convolution product of h_1 and h_2 are given by

$$T[h_1 \star h_2](s) = T\left[\int_{0}^{\infty} h_1(\tau)h_2(t-\tau)\,d\tau\right](s) = \frac{1}{p(s)}H_1(s)H_2(s). \quad (2.18)$$

Theorem 2.2 ([9] JAFARI TRANSFORM OF THE DERIVATIVE)**.** If $f(t)$ and its first n derivatives are differentiable then

$$T\left[f^{(n)}(t)\right](s) = q^{n}T[f(t)](s) - p(s)\sum_{i=0}^{n-1} q^{n-1-i}(s)f^{(i)}(0). \quad (2.19)$$

Proposition 2.1 ([15] Jafari transform of the Mittag-Leffler function)**.** Let $0 < \alpha < 1$ and $\lambda \in \mathbb{R}$ such that $q(s) < |\lambda|^{\frac{1}{\alpha}}$

$T\big[$

$$\mathcal{L}\left\{t^{\sigma-1}E^{\gamma}_{\alpha,\sigma}(\lambda t^{\alpha})\right\} = \frac{p(s)}{q(s)^{\sigma}}\left(1-\frac{\lambda}{q(s)^{\alpha}}\right)^{-\gamma} \quad (2.20)$$

Definition 2.7 ([1] Fourier transforms). The Fourier transform of the function $g(x)$, denoted by $F[g(x)](\omega) = G(\omega)$, with $\omega \in \mathbb{R}$, is defined by

$$F[g(x)](\omega) = G(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} g(x)e^{i\omega x}dx. \quad (2.21)$$

The inverse Fourier transform, denoted by $F^{-1}[G(\omega)] = g(x)$ is given by

$$F^{-1}[G(\omega)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} G(\omega)e^{-i\omega x}d\omega. \quad (2.22)$$

Remark 2.1. The definition of the Fourier transform may present slight variations depending on the source, but provided that their normalisation constants are chosen in a way that the original function is recovered by the inverse transform, this is not substantial.

2.2 Fractional operators

In this subsection, the (k, Ψ)-fractional integrals [12] is introduced by means of a k-fractional integral
which generalizes the classical Riemann-Liouville fractional integral.

Definition 2.8 ([13](k, Ψ)-Riemann-Liouville fractional integrals). For a real number $k > 0$ and a function $\Psi(x)$, increasing and positive monotone on $[0,\infty)$ having a continuous derivative $\Psi'(x)$ on $(0,\infty)$, the left- and right-sided (k, Ψ)-Riemann-Liouville fractional integral of order $\gamma > 0$ of
a function $f \in L_1[a, b]$ are defined by

$$\left({}_kJ^{\gamma}_{a+;\Psi}f\right)(x) := \frac{1}{k\Gamma_k(\gamma)}\int_a^x \Psi'(\xi)f(\xi)$$

$$[\Psi(x) - \Psi(\xi)]^{1-\frac{\gamma}{k}} \, d\xi, \quad x > a, \quad (2.23)$$

and

$${}_kJ^{\gamma}_{b-;\Psi} f(x) := \frac{1}{k\Gamma_k(\gamma)} \int_x^b \frac{\Psi'(\xi) f(\xi)}{[\Psi(\xi) - \Psi(x)]^{1-\frac{\gamma}{k}}} \, d\xi, \quad x < b, \quad (2.24)$$

respectively.

Proposition 2.2 ([12] Semigroup property). Let $\mu_1, \mu_2, \kappa \in \mathbb{R}_+$. Then,

$${}_kJ^{\mu_1}_{a+;\Psi} \, {}_kJ^{\mu_2}_{a+;\Psi} = {}_kJ^{\mu_1+\mu_2}_{a+;\Psi} \,. \quad (2.25)$$

Definition 2.9 ((κ, ρ)-Weyl Integrals). Taking $\Psi(x) = x^\rho/\rho$ with $\rho > 0$ and $x \geq 0$ in the expressions given in Definition 2.8 and rearranging we obtain the left- and right-sided k-fractional integrals

$${}^{\rho}_kJ^{\gamma}_{a+}\varphi(x) = \frac{\rho^{1-\frac{\gamma}{k}}}{k\Gamma_k(\gamma)} \int_a^x \frac{t^{\rho-1}\varphi(t)}{(x^\rho - t^\rho)^{1-\frac{\gamma}{k}}} \, dt, \quad (2.26)$$

and

$${}^{\rho}_kJ^{\gamma}_{b-}\varphi(x) =$$

$$\frac{\rho^{1-\frac{\gamma}{k}}}{k\,\Gamma_k(\gamma)}\int_x^b \frac{t^{\rho-1}\varphi(t)}{(t^\rho - x^\rho)^{1-\frac{\gamma}{k}}}\,dt, \quad (2.27)$$

respectively. Taking $a \to -\infty$ and $b \to \infty$ in Eq.(2.26) and Eq.(2.27), respectively, we obtain interesting left- and right-sided k-fractional integrals, given by

$$\left({}^{\rho}_{k}J^{\gamma}_{+}\varphi\right)(x) = \frac{\rho^{1-\frac{\gamma}{k}}}{k\,\Gamma_k(\gamma)}\int_{-\infty}^x \frac{t^{\rho-1}\varphi(t)}{(x^\rho - t^\rho)^{1-\frac{\gamma}{k}}}\,dt, \quad (2.28)$$

and

$$\left({}^{\rho}_{k}J^{\gamma}_{-}\varphi\right)(x) = \frac{\rho^{1-\frac{\gamma}{k}}}{k\,\Gamma_k(\gamma)}\int_{x}^{\infty} \frac{t^{\rho-1}\varphi(t)}{(t^\rho - x^\rho)^{1-\frac{\gamma}{k}}}\,dt. \quad (2.29)$$

Given the similarities with the Weyl fractional integrals (Teodoro et al., 2019), we propose that Eq.(2.28) and Eq.(2.29) be called left- and right-sided (κ, ρ)-Weyl Integrals, respectively.

Definition 2.10 ([18] Hilfer-Type fractional derivative)**.** Let γ denote the order of the fractional derivative and $n \in \mathbb{Z}_+$ such that $n - 1 < \gamma \leq n$, and β denote its type, with $0 \leq \beta \leq 1$. The (left- and right-sided) fractional derivatives of φ with respect to x, for $\rho > 0$ and $\kappa > 0$, are given by

$$\left({}^{\rho}_{k}D^{\gamma,\beta}$$

$$\left({}^{\rho}_{\kappa}D^{\gamma,\beta}_{a+}\varphi\right)(x) = \left({}^{\rho}_{\kappa}J^{\beta(n\kappa-\gamma)}_{a+}\left(\kappa x^{1-\rho}\frac{d}{dx}\right)^{n}\left({}^{\rho}_{\kappa}J^{(1-\beta)(n\kappa-\gamma)}_{a+}\varphi\right)\right)(x). \quad (2.30)$$

for functions such that the expression on the right hand side exists.

The (κ, ρ)-fractional derivative generalizes several classical fractional operators [18]. In particular, taking $(k, \rho) \to (1, 1)$ we recover Riemann-Liouville and Caputo operators (associated with translations) and Hadamard operator (associated with dilations). A Particular cases are given to follow:

- The κ-Hilfer fractional derivatives are introduced by taking $\rho \to 1$ in Eq.(2.30), obtaining

$$\left({}_{\kappa}D^{\gamma,\beta}_{a+}\varphi\right)(x) = \left({}_{\kappa}J^{\beta(nk-\gamma)}_{a+}\left(\kappa\frac{d}{dx}\right)^{n}\left({}_{k}J^{(1-\beta)(nk-\gamma)}_{a+}\varphi\right)\right)(x). \quad (2.31)$$

- The κ-Riemann-Liouville fractional derivatives is given for $\beta \to 0$ the κ-Hilfer fractional derivatives are given by

$$\left({}^{RL}_{\kappa}D^{\gamma}_{a+}\varphi\right)(x) = \left(\kappa\frac{d}{}\right.$$

$$\left(\frac{d}{dx}\right)^n \left({}_\kappa J^{(n\kappa-\gamma)}_{a+}\varphi\right)(x). \quad (2.32)$$

- The κ-Caputo fractional derivatives is given for for $\beta \to 1$ the κ-Hilfer fractional derivative reduces to

$$\left({}^{C}_{\kappa}D^{\gamma}_{a+}\varphi\right)(x) = \left({}_\kappa J^{(n\kappa-\gamma)}_{a+}\left(\kappa \frac{d}{dx}\right)^n \varphi\right)(x). \quad (2.33)$$

- κ-Weyl fractional derivatives For $\beta \to 0$ and $a_+ \to -\infty$ the κ-Hilfer fractional derivative reduces to

$$\left({}^{W}_{\kappa}D^{\gamma}_{+}\varphi\right)(x) = \left(\kappa \frac{d}{dx}\right)^n \left({}_\kappa J^{(n\kappa-\gamma)}_{+}\varphi\right)(x). \quad (2.34)$$

3 Main results

3.1 Integral transforms to κ fractional operators

Theorem 3.1 (JAFARI TRANSFORM OF THE κ-HILFER). The Jafari transform of the κ-Hilfer fractional derivative, order $n-1<\gamma<n$ and $0<\beta<1$ is given by:

$$\mathrm{T}\left[{}_{\kappa}D^{\gamma,\beta}_{0^+}\varphi(x)\right](s) = (\kappa q(s))^{\frac{\gamma}{\kappa}}\,\mathrm{T}[\varphi(x)](s) - p(s)(\kappa q(s))^{\frac{-\beta(n\kappa-\gamma)}{k}} \quad (3.1)$$

$$\times \sum_{i=0}^{n-1} \kappa^{n} q(s)^{n-i-1}\left(\frac{d}{dt}\right)^{i} {}_{\kappa}J^{(1-\beta)(n\kappa-\gamma)}\varphi(0^+) \quad (3.2)$$

Proposition 3.1 (MELLIN TRANSFORM OF (κ,ρ)-RIEMANN-LIOUVILLE FRACTIONAL INTEGRALS). Let $\gamma>0$, and $\rho>0$. The Mellin transform of the (κ,ρ)-Riemann-Liouville fractional integrals Eq.(2.26) and Eq.(2.27) are given by:

$$\mathcal{M}\left[{}^{\rho}_{k}J^{\gamma}_{a^+}\varphi\right](x)\Big](s) = \frac{\rho^{-\frac{\gamma}{k}}\,\Gamma_k\left(k-\frac{ks}{\rho}-\gamma\right)}{\Gamma_k\left(k-\frac{ks}{\rho}\right)}\mathcal{M}[\varphi]\left(s+\frac{\rho\gamma}{k}\right)$$

. (3.3)

Proof. Applying Mellin tranform to Eq.(2.23), we have

$$\mathcal{M}\left[{}^{\rho}_{\kappa}J^{\gamma}_{a+}\varphi \,(x)\right](s) = \frac{\rho^{1-\gamma/k}}{\kappa\,\Gamma_{\kappa}} \int_0^{\infty} x^{s-1} \int_a^{x} \frac{\xi^{\rho-1}\varphi(\xi)}{(x^{\rho}-\xi^{\rho})^{1-\gamma\kappa}}\, d\xi\, dx$$

$$= \frac{\rho^{1-\gamma/k}}{\kappa\,\Gamma_{\kappa}} \int_a^{\infty} \xi^{\rho-1}\varphi(\xi) \int_{\xi}^{\infty} \frac{x^{s-1}}{(x^{\rho}-\xi^{\rho})^{1-\gamma\kappa}}\, dx\, d\xi. \quad (3.4)$$

Taking the change of variables $u = (\xi/x)^{\rho}$ in respect to the variable x, we obtain

$$\mathcal{M}\left[{}^{\rho}_{\kappa}J^{\gamma}_{a+}\varphi \,(x)\right](s) = \frac{\rho^{1-\gamma/k}}{\kappa\,\Gamma_{\kappa}} \int_a^{\infty} \frac{\xi^{s-\rho-\rho\gamma/\kappa}}{\rho} \int_0^{1} u^{1-s/\rho-\gamma/\kappa-1}(1-u)^{\gamma/\kappa-1}\, du\, d\xi$$

$$= \rho^{-\gamma/\kappa}$$

$$\frac{1}{\kappa\Gamma_\kappa(\gamma)} B\left(1-\frac{s}{\rho}-\frac{\gamma}{\kappa},\frac{\gamma}{\kappa}\right)\int_a^\infty \varphi(\xi)\xi^{s+\rho\gamma/\kappa-1}\,d\xi. \quad (3.5)$$

Since the Riemann-Liouville fractional integral given by Eq.(2.23) is defined for $x > a$, it follows that the function $\varphi(x)$ is defined in the domain $[a,\infty]$. This means that we could consider the function $H(x-a)\varphi(x)$ in the domain $[0,\infty]$, where $H(x)$ is the Heaviside step function. Hence the Mellin transform integration can be considered for $x > a$ without loss of generality.

At last, with the help of the representation in terms of gamma functions for the beta function, we get to the final expression, given by

$$M\left[\left({}^{\rho}_{\kappa}J^{\gamma}_{a+}\varphi\right)(x)\right](s) = \rho^{-\gamma/\kappa}\,\frac{\Gamma_\kappa(\kappa-\kappa s/\rho-\gamma)}{\Gamma_\kappa(\kappa-\kappa s/\rho)}\,M[\varphi(x)]\left(s+\frac{\rho\gamma}{\kappa}\right). \quad (3.6)$$

Corollary 3.2. Corollary Taking $\gamma \to 0$ in Eq.(3.3) implies

$$\left({}^{\rho}_{\kappa}J^{0}_{a+}\varphi\right)(x) = \varphi(x).$$

Theorem 3.3. Let $n-1 < \gamma \leq n, n \in \mathbb{N}$, $\kappa > 0$, $0 < \beta < 1$ and $\rho > 0$, the Mellin transform , with parameter s, of the $(\kappa,\rho)-$Hilfer fractional derivative is given by:

M

$$M\left[\left({}^{\rho}_{\kappa}D^{\gamma,\beta}\varphi\right)(x)\right] = \rho^{\frac{\gamma}{\kappa}}\,\frac{\Gamma_{\kappa}\left(\kappa - \frac{\kappa s}{\rho} + \gamma\right)}{\Gamma_{\kappa}\left(\kappa - \frac{\kappa s}{\rho}\right)}\, M[\varphi(x)]\left(s - \frac{\rho\gamma}{\kappa}\right). \quad (3.7)$$

Proof.

$$M\left[\left({}_{\kappa}D^{\gamma,\beta}_{a+}\varphi\right)(x)\right](s) = M\left[\left({}^{\rho}_{\kappa}J^{\beta(n\kappa-\gamma)}_{a+}\left(\kappa x^{1-\rho}\frac{d}{dx}\right)^{n} J^{(1-\beta)(n\kappa-\gamma)}_{a+}\varphi\right)(x)\right.$$

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Proposition 3.2. The Fourier transform of the

•

$_\kappa J_\gamma$

${}_\kappa J^{\gamma}_{\mp}\varphi(x)$, κ-Weyl fractional integral, given by Eq.(2.28) and Eq.(2.29), for $\rho = 1$ and $\gamma > 0$, it is given by:

$$\mathcal{F}\left[{}_\kappa J^{\gamma}_{\mp}\varphi(x)\right](\omega) = (\mp i\kappa\omega)^{-\gamma/\kappa}\,{}_\kappa F[\varphi(x)](\omega). \quad (3.10)$$

Proof. Let us start calculating the the Fourier transform of ${}_\kappa J^{\gamma}_{-}\varphi(x)$.

$$\mathcal{F}\left[{}_\kappa J^{\gamma}_{-}\varphi(x)\right](\omega) = \frac{1}{\kappa\Gamma_\kappa(\gamma)}\int_{-\infty}^{+\infty} e^{i\omega x}\int_x^{+\infty}\frac{\varphi(t)}{(t-x)^{1-\frac{\gamma}{\kappa}}}\,dt\,dx = \frac{1}{\kappa\Gamma_\kappa(\gamma)}\int_{-\infty}^{+\infty}\varphi(t)\int_{-\infty}^{t}\frac{e^{i\omega x}}{(t-x)^{1-\frac{\gamma}{\kappa}}}\,dx\,dt.$$

Using the new variable $u = t - x$ we get

$$F\left[{}_{\kappa}J^{\gamma}_{-}\varphi(x)\right](\omega) = \frac{1}{\kappa\Gamma_{\kappa}(\gamma)}\int_{-\infty}^{+\infty}\varphi(t)\int_{0}^{+\infty}\frac{e^{i\omega(t-u)}}{u^{1-\frac{\gamma}{\kappa}}}\,du\,dt. \quad (3.11)$$

In order to evaluate the integral in respect to u, we should take the change of variables $z = i\omega u$ and consider the following contour in the complex plane: a disk of radius $R \to \infty$ in the first quadrant. It follows that

$$F\left[{}_{\kappa}J^{\gamma}_{-}\varphi(x)\right](\omega) = \frac{1}{\kappa\Gamma_{\kappa}(\gamma)}F[\varphi(x)](\omega)\frac{\Gamma\left(\frac{\gamma}{\kappa}\right)}{(i\omega)^{\frac{\gamma}{\kappa}}} = (i\kappa\omega)^{-\frac{\gamma}{\kappa}}F[\varphi(x)](\omega). \quad (3.12)$$

For ${}_{\kappa}J^{\gamma}_{+}\varphi(x)$, the calculations follow the same course, so we opted to omit it.

From the expression in Eq.(3.10) the following can be derived:

$$\mathcal{F}\left[\left({}_{\kappa}J^{\gamma}_{+} + {}_{\kappa}J^{\gamma}_{-}\right)\varphi(x)\right](\omega) = 2\cos\left(\frac{\gamma\pi}{2\kappa}\right)|\kappa\omega|^{-\frac{\gamma}{\kappa}}\mathcal{F}[\varphi(x)](\omega). \quad (3.13)$$

Theorem 3.4. The Fourier transform of the κ-Weyl fractional derivative Eq.(2.34) is given by

$$\mathcal{F}\left[{}^{W}_{\kappa}D^{\gamma}_{\mp}\varphi(x)\right](\omega) = (\mp i\kappa\omega)^{\frac{\gamma}{\kappa}}\mathcal{F}[\varphi](\omega). \quad (3.14)$$

Proof.

$$\mathcal{F}\left[{}^{W}_{\kappa}D^{\gamma}_{\mp}\varphi(x)\right](\omega) = \mathcal{F}\left[\left(\mp\kappa\frac{d}{dx}\right)^{n}\left({}_{\kappa}J^{n\kappa-\gamma}_{\mp}\varphi\right)(x)\right](\omega)$$

$$= (\mp\kappa)^{n}(-i\omega)^{n}\mathcal{F}\left[\left({}_{\kappa}J^{n\kappa-\gamma}_{\mp}\varphi\right)(x)\right]$$

(ω)

$= (\mp i\kappa\omega)^n (\mp i\kappa\omega)^{-(n\kappa-\gamma)}$

${}_\kappa F\,[\varphi(x)]\,(\omega)$

$= (\mp i\kappa\omega)$

γ

${}_\kappa F\,[\varphi(x).]\,(\omega)$ (3.15)

Remark 3.1. It is extremely important to notice that integral transforms for the left-sided operators like

the (κ, ρ)-Riemann-Liouville fractional integral

$\bullet\rho$

${}_kJ\ \gamma$

$b-\varphi$

_

(x) given by eq. (2.27) are not yet known for

$b < \infty$. This problem arises when we observe that the double integrals involved in these calculations present a finite region in its domain of integration, which we do not yet know how to deal with in the case of fractional-type integrals.

3.2 Fundamental theorem of calculus to κ-fractional operators

Definition 3.1 (κ-Riesz fractional integral). The κ-Riesz fractional integral is introduced via

combination of the Weyl integrals of order $\gamma > 0$, as it follows:

$({}_\kappa J\ \gamma$

${}_0\,)\ \varphi(x) =$

_

${}_\kappa J\ \gamma$

${}_+\varphi(x) + {}_\kappa J\ \gamma$

${}_-\varphi(x)$

_

$2\cos$

$\bullet\ \gamma\pi$

2κ

_ $= g(x) * \varphi(x)$, (3.16)

where $*$ represents Fourier convolution product and the function g is given by

$g(x) =$

$|\ x\ |$

γ

κ

-1

$2\kappa\,\Gamma_\kappa(\gamma)\cos$

$\bullet\ \gamma\pi$

2κ

_ . (3.17)

Proposition 3.3. The Fourier transform of the κ-Riesz fractional integral of γ order, with $\gamma > 0$, it is given by

$F[({}_\kappa J\ \gamma$

${}_0\,)\ \varphi(x)](\omega) =|\ \kappa\omega\ |^{-\gamma}$

${}_\kappa.F[\varphi(x)](\omega)$ (3.18)

Proof. It follows directly from Eq.(3.10).

Definition 3.2 (κ-Riesz fractional derivative). The κ-Riesz fractional order γ with $1 < \gamma < 2$ is defined as

$$ {}_{\kappa}\Delta^{\frac{\gamma}{2}}\varphi(x) = -\frac{w_\kappa D^{\gamma}_{+}\varphi(x) + w_\kappa D^{\gamma}_{-}\varphi(x)}{2\cos\left(\frac{\gamma\pi}{2\kappa}\right)} = h(x) * \varphi(x), \quad (3.19) $$

with

$$ h(x) = \Gamma_\kappa(\kappa+\gamma)\sin\left(\frac{\gamma\pi}{2\kappa}\right)\frac{|x|^{-\frac{\gamma}{\kappa}-1}}{\pi}. \quad (3.20) $$

Proposition 3.4. The Fourier transform of the κ-Riesz fractional integral of γ order, with $\gamma > 0$, it is given by:

$$ F[{}_{\kappa}\Delta^{\frac{\gamma}{2}}\varphi(x)](\omega) = |\kappa\omega|^{\frac{\gamma}{\kappa}} F[\varphi(x)](\omega) \quad (3.21) $$

Theorem 3.5. Let $0 < \gamma < 2$ the fundamental theorem associated to κ-Riesz derivative and κ-Resz integral of γ order is:

$$ \left({}_{\kappa}\Delta^{\frac{\gamma}{2}} J^{\gamma}_{0}\varphi\right)(x) = \varphi(x) \quad (3.22) $$

Proof. Using the Fourier transform,

$$ F\left[{}_{\kappa}\Delta^{\frac{\gamma}{2}}\left(J^{\gamma}_{0}\varphi\right)(x)\right] $$

(ω) =| κω |

γ

κF [(κJ γ

0 φ) (x)] (ω)

=| κω |−γ

κ| κω |

γ

κF[φ(x)](ω) = F[φ(x)](ω) (3.23)

Therefore, the relation introduced by Eq.(3.22) is true.

Theorem 3.6. The fundamental theorem of calculus associated (κ, ρ)-Hilfer fractional derivative and (κ, ρ)-Riemann fractional integral with γ > 0 is:

_

ρ

κDγ,β

a

ρ

κJ γ

a φ

_

(x) = φ(x). (3.24)

Proof.

M

h_

ρ

κDγ,β

a+

ρ

κJ γ

a+φ

_

(x)

i

(s) =

ρ

γ

κ Γ

_

κ − κs

ρ + γ

_

Γ

_

κ − κs

ρ

_ M

_ ρ

κJ γ

a+φ

_ _

s −

ργ

κ

—
=

ρ

γ
κ Γ

—
κ − κs

ρ + γ

—
Γ

—
κ − κs
ρ

—
ρ−γ
κ Γ

—
κ − κ
ρ
·
s − ργ
κ

—
− γ

—
Γ

—
κ − κ
ρ
·
s − ργ
κ

—
× M[φ(x)]

—
s +

ργ

κ

—

ργ

κ

—
=
Γ

—
κ − κs

ρ + γ

—
Γ

—

κ − κs
ρ
_
Γ
_
κ − κs
ρ
_
Γ
_
κ − κs
ρ + γ
_M[φ(x)](s). (3.25)

4 Application: Fractional diffusion in a cylinder

The section introduces the κ-fractional diffusion problem in a infinitely long circular cylinder of radius $a > 0$, considering κ-Hilfer fractional derivative in time variable.

_
$_{\kappa}D^{\gamma,\beta}_{0+} u(r, t)$
_
(t) = c
_
∂2
∂r2 +
1
r
∂
∂r
_
u(r, t) + Q(r, t)
$u(a, t) = 0$ for $t > 0$
$u(r, 0) = 0$ for $0 < r \leq a$. (4.1)

with $0 < \gamma < 1$, $0 < \beta < 1$ and $Q(r, t)$ represents one heat source inside the cylinder.

To solve this problem,we use the integral transform methodology, that is, the application of the Jafari transform and finite Hankel transform in the diffusion problem. Firstly, appling Jafari transform in Eq.(4.1) gets

(κq(s))
γ
κ ‾‾u(r, s) = c
_
∂2
∂r2 +
1

$$r\frac{\partial}{\partial r}\ \overline{u}(r,s) + \overline{Q}(r,s) \quad (4.2)$$

where $\overline{u}(r,s) = T[u(r,t)](s)$ and $\overline{Q}(r,s) = T[Q(r,t)](s)$. Applying the finite hankel transform order 0 and their properties:

$$\frac{(\kappa q(s))^{\gamma}}{\kappa}\, \overline{u}_b(k_i,s) = -c k_i^2\, \overline{u}_b(k_i,s) + \overline{Q}_b(k_i,s)$$

$$\overline{u}_b(k_i,s) = \frac{\overline{Q}_b(k_i,s)}{\frac{(\kappa q(s))^{\gamma}}{\kappa} + c k_i^2}$$

$$\overline{u}_b(k_i,s) = \frac{1}{p(s)} \left(\frac{p(s)}{\frac{(\kappa q(s))^{\gamma}}{\kappa} + c k_i^2} \right) \overline{Q}_b(k_i,s)$$

$$\overline{u}_b(k_i,s) = \frac{1}{p(s)} \cdot \frac{p(s)}{\frac{(\kappa q(s))^{\gamma}}{\kappa}\left(1 + \frac{c k_i^2}{\frac{(q(s))^{\gamma}}{\kappa}}\right)}\, \overline{Q}_b(k_i,s) \quad (4.3)$$

Where k_i are the zeros of the zero order Bessel function. Applying the inverse Jafari transform and the inverse finite Hankel transform, we get:

$$u(r,t) = \frac{2\kappa^{-\frac{\gamma}{\kappa}}}{a^2}\sum_{i=1}^{\infty}\frac{J_0(rk_i)}{J_1^2(ak_i)}\int_0^{\infty}\tau^{\frac{\gamma}{\kappa}-1}E_{\gamma/\kappa,\gamma/\kappa}\left(-ck_i^2(t-\tau)^{\gamma/\kappa}\right)\bar{Q}(k_i,t)\,d\tau. \quad (4.4)$$

• To consider $Q(r,t) = Q_0$ where Q_0 is arbitrary number. The Eq.(4.4) gets:

$$u(r,t) = \frac{\kappa^{-\frac{\gamma}{\kappa}}Q_0}{4c}(a^2 - r^2) - \frac{2\kappa^{-\frac{\gamma}{\kappa}}Q_0}{ac}\sum_{i=1}^{\infty}\frac{J_0(rk_i)}{k_i^2 J_1(ak_i)}E_{\gamma/\kappa}\left(-ck_i^2 t^{\gamma/\kappa}\right). \quad (4.5)$$

• To consider $Q(r,t) = Q_0\frac{\delta(r)}{r}f(t)$, we obtain

$$u(r,t) = 2\kappa^{-\gamma}$$

$$\frac{\kappa Q_0}{a^2}\sum_{i=1}^{\infty}\frac{J_0(rk_i)}{J_1^2(ak_i)}\int_0^{\infty} f(\tau)\,\tau^{\frac{\gamma}{\kappa}-1}E_{\gamma/\kappa,\gamma/\kappa}\left(-ck_i^2(t-\tau)^{\gamma/\kappa}\right)d\tau. \quad (4.6)$$

Note that $\kappa \to 1, \gamma \to 1$ recovers the the same result found in [5].

The graphs to equation Eq.(4.5) are simulated to follow via, considering some values to variables Q_0, c, a, r and the parameters γ and $\kappa = 1$.

Figure 1: Graph to $u(r,t)$ in Eq. (4.5) with $Q_0 = c = 200$, $a = 1$, $\gamma \to 1$ and $\kappa = 1$.

Figure 2: Graph to $u(r,t)$ in Eq. (4.5) with $Q_0 = c = 200$, $a = 1$, $\gamma \to 0.8$ and $\kappa = 1$.

Figure 3: Graph to $u(r,t)$ in Eq. (4.5) with $Q_0 = c = 200$, $a = 1$, $\gamma \to 0.7$ and $\kappa = 1$.

Figure 4: Graph to $u(r,t)$ in Eq. (4.5) with $Q_0 = c = 200$, $a = 1$, $\gamma \to 0.66$ and $\kappa = 1$.

Concluding remarks

The results obtained in this work shows the importance of κ-Hilfer fractional derivative, since this operator generalizes several well-known definitions of fractional derivatives. The most important results consolidates the so-called κ fractional operators in terms of integral transforms. In particular,

we highlighted the calculation of the Jafari transform to κ-Hilfer fractional derivative as well as the introduction of κ-Riesz fractional operators. At last, the fundamental theorem of the calculus to κ-fractional operators, and Hilfer-Type fractional derivative are presented. It is important to notice that the prelude of the theory to κ-fractional operators present some mistakes, that has been repaired over time, that confirms the value of this article. The Jafari transform is applied in a problem radial diffusion with κ−Hilfer fractional derivative in the time, obtaining the explicit solution to equation. Finally, The graphs to Eq. (4.5) are shown by figures Fig.1, Fig.2, Fig.3 and Fig.4 building for Python language in its version 3.6, using the Pycharm IDE. The graphs are surfaces that deforms in agree the value of the parameter γ, which possibilities the indication for the special physics situation to adjust the solution to obtain the better results. The natural continuation of the paper is the generalization of the Jafari transform, that is, Ψ-Jafari transform that generalize Ψ-Laplace and Ψ-Shehu transforms. The integral transforms are associated with Linear fractional equations in terms Ψ-Hilfer fractional derivative

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Appendix

A Mellin Transform of the derivative as in Eq. (3.8)

In Eq. (3.8), we need to deal with the Mellin transform of a nth order derivative in the following form:

$$\mathcal{M}\left[\left(x^{1-\rho}\frac{d}{dx}\right)^n r(x)\right](s) = (-1)^n\rho^n \frac{\Gamma\left(\frac{s}{\rho}\right)}{\Gamma\left(\frac{s}{\rho}-n\right)} \mathcal{M}[r(x)]\,(s-n\rho). \quad (A.1)$$

Proof. Let us use finite induction. Starting with $n = 1$, we have

$$\mathcal{M}\left[x^{1-\rho}\frac{dr(x)}{dx}\right] = \int_0^\infty x^{s-\rho}\frac{dr(x)}{dx}\,dx. \quad (A.2)$$

Using integration by parts, we obtain

M [

$$\left(x^{1-\rho}\frac{dr(x)}{dx}\right) = \left. x^{s-\rho} r(x)\right|_0^\infty - (s-\rho)\int_0^\infty x^{s-\rho-1} r(x)\,dx. \quad (A.3)$$

Here we must observe that r(x) must be a function such that $x^{-\rho} r(x)$ vanishes both for $x = 0$ and $x \to \infty$. Under these conditions, we have that

$$M\left(x^{1-\rho}\frac{dr(x)}{dx}\right) = -(r-\rho) M[r(x)]\,(s-\rho)\,. \quad (A.4)$$

Let us now perform the induction step. Assuming Eq. (A.1) is valid for some n, we have:

$$M\left[\left(x^{1-\rho}\frac{d}{dx}\right)^{n+1} r(x)\right] = M\left[\left(x^{1-\rho}\frac{d}{dx}\right)^{n}\left(x^{1-\rho}\frac{dr(x)}{dx}\right)\right] \quad (A.5)$$

$$= (-1)^n \rho^n \frac{\Gamma\left(\frac{s}{\rho}\right)}{\Gamma\left(\frac{s}{\ldots}\right.}$$

$$\left.\frac{}{\rho} - n\right) \mathcal{M}\left[x^{1-\rho} \frac{dr(x)}{dx} \right](s - n\rho) \quad (A.6)$$

$$= (-1)^n \rho^n \frac{\Gamma\left(\frac{s}{\rho}\right)}{\Gamma\left(\frac{s}{\rho} - n\right)} \int_0^\infty x^{s-(n+1)\rho} \frac{dr(x)}{dx} dx \quad (A.7)$$

$$= (-1)^n \rho^n \frac{\Gamma\left(\frac{s}{\rho}\right)}{\Gamma\left(\frac{s}{\rho} - n\right)} \left(\left[x^{s-(n+1)\rho} \frac{dr(x)}{dx} \right]_0^\infty \right. \quad (A.8)$$

$$\left. - [s - (n+1)\rho] \int_0^\infty x^{s-(n+1)\rho-1} r(x) dx \right) . \quad (A.9)$$

Once more, we must assume that $x^{s-(n+1)\rho} \frac{dr(x)}{dx}$ vanishes for $x \to 0$ and $x \to \infty$ and thus we obtain.

$\mathcal{M}$ "

$$\left(x^{1-\rho}\frac{d}{dx}\right)^{n+1} r(x)\Bigg] = (-1)^{n+1}\rho^{n+1}\frac{\Gamma\left(\frac{s}{\rho}\right)}{\Gamma\left(\frac{s}{\rho}-n\right)\left(\frac{s}{\rho}-(n+1)\right)} \times M[r(x)]\,(s-(n+1)\rho)\,. \quad (A.10)$$

At last, using Gamma's reflection formula, we obtain the desired result.

Examining this proof, we notice that we must use this result with caution, since it is valid only for functions $r(x)$ such that $x^{-\rho}r(x)$ and $x^{-\rho}dr(x)/dx$ vanishes for $x \to 0$ and $x \to \infty$. For such cases, we have the interesting result that the derivative both in the sense of Riemann Liouville and Caputo have the same Mellin transform.

B Integral transforms type Laplace transform

The following diagram presents some particular cases of the Jafari transform. These results can be found in reference [9](Jafari, 2021).

Figure 5: The diagram some particular cases Jafari transform

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