Another Look of Rings Domination in Ladder Graph

Original Article

Abstract

For a nontrivial connected graph G with no isolated vertex, a nonempty subset $D \subseteq V(G)$ is a rings dominating set if each vertex $v \in V-D$ is adjacent to at least two vertices in $V-D$. Thus, the dominating set D of $V(G)$ is a rings dominating set if for all $v \in V-D$, $|N(v) \cap (V-D)| \geq 2$. The cardinality of minimum rings dominating set of G is the rings domination number of G, denoted by γ_{ri} whereas the cardinality of maximum rings dominating set is the upper rings domination number and is denoted by γ'_{ri}. Here, we determine how the rings dominating set is constructed in the ladder graph with the inclusion of generated conditions for this type of domination and give new approach for its parameter.

Keywords: rings dominating set, minimum cardinality of rings dominating set, rings domination number, maximum cardinality of rings dominating set, upper rings domination number.

2020 Mathematics Subject Classification: 05C35

1 Introduction

Recently, a new parameter in domination of graphs was introduced called rings domination which has almost similar characteristics with the usual domination. As stated in [10], [15], [17], it has been a great area of study in graph theory to deal with, considering its applications in computer networks, electronics, and more, which inspires the creation of this paper. Prior to this, it intends to disclose the idea of rings dominating set specified over the ladder graph L_n. It is a graph generated from the cartesian product of paths of order n and of order two that induces a finite, planar, and undirected graph which has $2n$ vertices and $3n-2$ edges.

Additionally, rings domination put a condition on the $V-D$ set in the graph. Hence, we will be studying the behavior of rings dominating set in the ladder graph focusing on its $V-D$ set. At the end of this paper, fundamental results are presented showcasing the conditions of rings dominating set and its largest cardinality for this particular graph.

2 Preliminary Notes

Some definitions of the concepts covered in this study are included below. You may refer on the remaining terms and definitions in [1], [5], [6], [11].

Definition 2.1. [6] (**Cartesian Product of Graphs**) The Cartesian product G of two graphs G_1 and G_2, commonly denoted by $G_1 \times G_2$ has vertex set $V(G) = V(G_1) \times V(G_2)$, where two distinct vertices (u, v) and (x, y) of $G_1 \times G_2$ are adjacent if either (1) $u = x$ and $vy \in E(G_2)$ or (2) $v = y$ and $ux \in E(G_1)$.

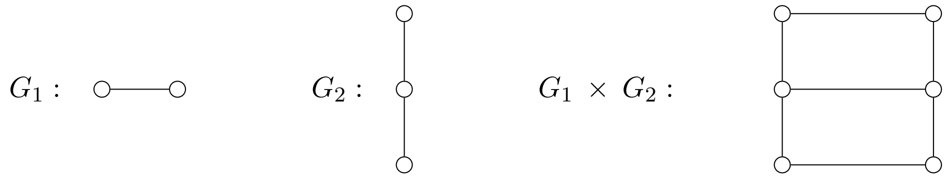

Figure 1: The cartesian product of graphs G_1 and G_2

Definition 2.2. [1] **(Rings Dominating Set, Rings Domination number)** A dominating set $D \subseteq V(G)$ is a rings dominating set if each vertex $v \in V(G) - D$ is adjacent to at least two vertices in $V(G) - D$.The minimum cardinality among the rings dominating sets of G is called the rings domination number of G and is denoted by $\gamma_{ri}(G)$.

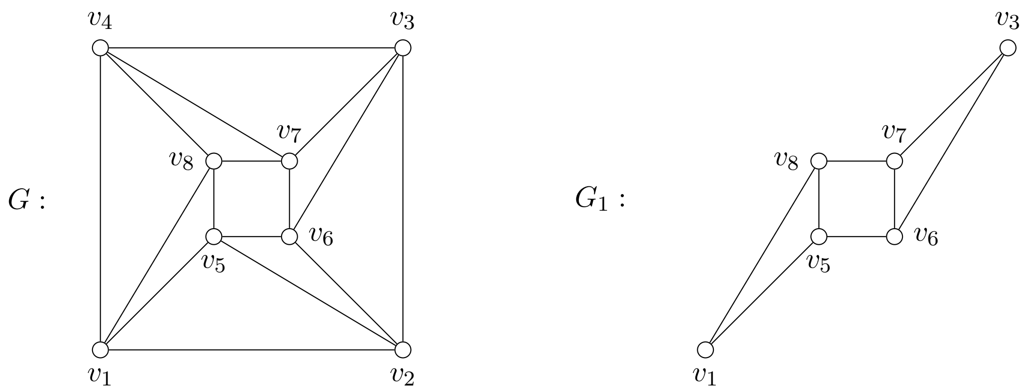

Figure 2: The graph of G and subgraph G_1

Example 2.1. Consider the graph G in Figure 2. Obviously, $D = \{v_2, v_4\}$ is a dominating set. Now see the subgraph G_1 generated upon deleting vertices v_2 and v_4. Observe that for every vertex $v \in V(G) - D$ is adjacent to at least two vertices in $V(G) - D$, that is, $deg(v_1) = deg(v_3) = 2$ and $deg(v_5) = deg(v_6) = deg(v_7) = deg(v_8) = 3$. Hence, the dominating set D is a rings dominating set. Also, note that D is the rings dominating set with the least cardinality, so $\gamma_{ri}(G) = 2$.

3 Main Results

In this section, the characteristics of rings dominating set in the ladder graph are presented. Also, the maximum cardinality of the rings dominating set for this particular graph is identified and new approach in determining the rings domination number is determined.

3.1 Characteristics of Rings Dominating Set in the Ladder Graph, L_n

For convenience, we consider the path graph G of order $n \geq 4$ with a vertex set $V(G) = \{w_0, w_1, ..., w_n\}$ and $V(H) = \{u, v\}$. The following shows the characteristics of rings dominating set in the ladder graph $L_n = P_2 \times P_n$.

Theorem 3.1. *Let $G = P_2$ where $n \geq 4$ and $H = P_n$. Suppose $D \subseteq V(P_2 \times P_n)$, then $D = \cup(\{a\} \times T_a)$ where $a \in S$ such that $S \subseteq V(P_2)$ and $T_a \subseteq V(P_n)$ is a rings dominating set of $P_2 \times P_n$ if and only if the following holds.*

1. *$S = P_2$; and*
2. *T_a is a restrained dominating set of P_n.*

Proof. Suppose $P_2 = \{u, v\}$ and $P_n = \{w_1, w_2, ..., w_n\}$ so that the vertex set of the ladder graph $V(P_2 \times P_n) = \{(u, w_1), (u, w_2), ..., (u, w_n), (v, w_1), (v, w_2), ..., (v, w_n)\}$. Let $\emptyset \neq D \subseteq V(P_2 \times P_n)$ be a rings dominating set. Then $D = \bigcup_{a \in S}(\{a\} \times T_a)$ where $S \subseteq V(P_2)$ and $T_a \subseteq V(P_n)$. Suppose $S \neq V(P_2)$, then $|S| = 1$. Without loss of generality, let $S = \{u\}$. Since D is a rings dominating set, then $D = \bigcup_{u \in S}(\{u\} \times T_u) = \{(u, w_1), (u, w_2), ..., (u, w_n)\}$. Thus, $V(P_n \times P_2) - D = \{(v, w_1), (v, w_2), ..., (v, w_n)\}$. Now, $|N(v, w_1) \cap [V(P_n \times P_2) - D]| = |N(v, w_n) \cap [V(P_2 \times P_n) - D]| = 1$. A contradiction to the assumption that D is a rings dominating set. Hence, $S = V(P_2)$.

Further, assume T_a is not a restrained dominating set. It is equivalent to say that our assumption is $V(P_n) - T_a$ has an isolated vertex or T_a is not a dominating set at the first place. Again, since $D = \bigcup_{a \in S}(\{a\} \times T_a)$, then it is a must that whenever (u, w_i) is in D, then (v, w_i) is also in D for some natural number i. Otherwise, we find a vertex (u, w) such that $|N((u, w)) \cap [V(P_2 \times P_n) - D]| \leq 1$. When $V(P_n) - T_a$ has at least one isolated vertex but T_a is a dominating set, then D is a dominating set with the induced subgraph $(P_2 \times P_n)[V(P_2 \times P_n) - D]$ that has at least two corresponding vertices of degree 1, that is, $|N((u, w)) \cap [V(P_2 \times P_n) - D]| = |N((v, w)) \cap [V(P_2 \times P_n) - D]| = 1$. On the other hand, when $V(P_n) - T_a$ has no isolated vertex but T_a is not a dominating set, then for every vertex $(u, w) \in V(P_2 \times P_n) - D$, $deg((u, w)) \geq 2$. Even so, a number of vertices of multiple of two is not dominated. In any case contradicts the assumption that D is a rings dominating set. Therefore T_a is a restrained dominating set.

Conversely, suppose conditions 1 and 2 are satisfied. Let $D = \bigcup_{a \in S}(\{a\} \times T_a)$ where $S \subseteq V(P_2)$ and $T_a \subseteq V(P_n)$. Since T_a is a restrained dominating set, then both induced subgraphs $V(P_2 \times P_n)[V(P_2 \times P_n) - T_u]$ and $V(P_2 \times P_n)[V(P_2 \times P_n) - T_v]$ where $u, v \in V(P_2)$, have no isolated vertices. Clearly, $(v, w_i) \in D$ whenever $(u, w_i) \in D$. Thus, $V(P_2 \times P_n)[V(P_2 \times P_n) - D]$ is a cluster of cycles of order 4. Obviously, for every vertex $v \in V(P_2 \times P_n) - D$, $|N(v) \cap [V(P_2 \times P_n) - D]| = 2$. Hence, D is a rings dominating set. This completes the proof. □

Example 3.2. *Let $G = P_2$ and $H = P_4$. Then the ladder graph L_4 is shown in Figure 3 below. Note that the vertex set of L_4 is $V(P_2 \times P_4) = \{(u, w_1), (u, w_2), (u, w_3), (u, w_4), (v, w_1), (v, w_2), (v, w_3), (v, w_4)\}$. Here, we let $a = (u, w_1), b = (u, w_2), c = (u, w_3), d = (u, w_4), e = (v, w_1), f = (v, w_2), g =$*

$(v, w_3), h = (v, w_4)$. *Clearly, $S = P_2$ as we have taken vertices containing both $u, v \in P_2$. Now, consider the subgraphs, say S_k for $k = 1, 2$ shown in Figure 4. For each T_{a_i} where $i = 1, 2$ is a restrained dominating set since there exists no isolated vertex in $L_n[V(S_k) - T_{a_i}]$. Lastly, note $a, e \in D$ are corresponding vertices in L_4. Similarly, $d, g \in D$ are corresponding vertices. Hence, conditions 1 and 2 of Theorem 3.1 are satisfied.*

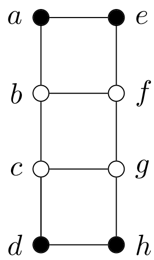

Figure 3: The rings dominating set of L_4

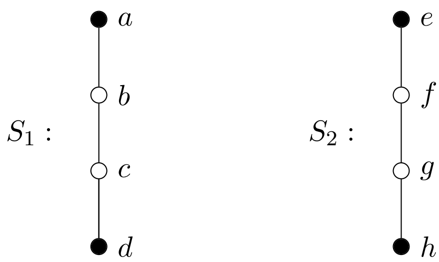

Figure 4: The subgraphs S_1 and S_2 of L_4

3.2 Rings Domination Number of a Ladder graph via Congruence Modulo

Here, we introduce a different way of finding the rings domination number of a ladder graph via congruence modulo. Consequently, this will serve as another convenient way of determining the rings domination number of a ladder graph of a specific order which is divided into three different cases shown below.

Theorem 3.3. *Let $L_n = P_2 \times P_n$ be a ladder graph of order 2n. For $n \geq 4$, the rings domination* *number of the ladder graph is*

$$\gamma_{ri}(L_n) = \begin{cases} 2(\frac{n+6}{3}), & \textit{if } n \equiv 0 (\textit{mod 3}) \\ \\ 2(\frac{n+2}{3}), & \textit{if } n \equiv 1 (\textit{mod 3}) \\ \\ 2(\frac{n+4}{3}), & \textit{if } n \equiv 2 (\textit{mod 3}). \end{cases}$$

Proof. Let $L_n = P_2 \times P_n$ be a ladder graph of order $2n$ such that $P_n = \{1, 2, 3, ..., n\}$ and $P_2 = \{a, b\}$. Suppose S is a rings dominating set of minimum cardinality, that is, S is a γ_{ri}-set, then we consider the following cases.

Case 1: $n \equiv 0(\text{mod } 3)$.

Let $K \subseteq V(L_n)$ with $K = \{(a,1),(a,4),(a,5),(a,6),(a,9),(a,12),(a,15),...,(a,n-3),(a,n)\} \cup \{(b,1),(b,4),(b,5),(b,6),(b,9),(b,12),(b,15),...,(b,n-3),(b,n)\}$. Clearly, K is a rings dominating set and $K = V(P_2 \times P)$, where $P \subseteq V(P_n)$ and $P = \{1,4,5,6,9,12,15,...,n-3,n\}$. For this circumstance, $|K| = 2(\frac{n+6}{3})$. Since S is a γ_{ri}-set, $|K| \geq |S|$. Hence $|S| \leq |K| = 2(\frac{n+6}{3})$. On the other hand, since S is a γ_{ri}-set of L_n then S must have at least $2(\frac{n+6}{3})$ number of vertices in L_n. Hence, $|S| \geq 2(\frac{n+6}{3})$. Therefore, $|S| = 2(\frac{n+6}{3})$.

Case 2: $n \equiv 1(\text{mod } 3)$.

On similar manner in case 1, let $K \subseteq V(L_n)$ with $K = \{(a,1),(a,4),(a,7),...,(a,n-3),(a,n)\} \cup \{(b,1),(b,4),(b,7),...,(b,n-3),(b,n)\}$. It can easily be seen that R is a rings dominating set and $K = V(P_2 \times P)$, where $P \subseteq V(P_n)$ and $P = \{1,4,7,...,n-3,n\}$. With this, $|K| = 2(\frac{n+2}{3})$. Since S is a γ_{ri}-set, $|K| \geq |S|$. Hence $|S| \leq |K| = 2(\frac{n+2}{3})$. Meanwhile, since S is a γ_{ri}-set of L_n, then S must have at least $2(\frac{n+2}{3})$ number of vertices in L_n. Hence, $|S| \geq 2(\frac{n+2}{3})$. Thus, $|S| = 2(\frac{n+2}{3})$.

Case 3: $n \equiv 2(\text{mod } 3)$.

Comparably, let $K \subseteq V(L_n)$ with $K = \{(a,1),(a,4),(a,7),(a,8),(a,11),(a,14),...,(a,n-3),(a,n)\} \cup \{(b,1),(b,4),(b,7),(b,8),(b,11),(b,14),...,(b,n-3),(b,n)\}$. Obviously, K is a rings dominating set and $R = V(P_2 \times P)$, where $P \subseteq V(P_n)$ and $P = \{1,4,7,8,11,14,...,n-3,n\}$. On this account, $|T| = 2(\frac{n+4}{3})$. Since S is a γ_{ri}-set, $|K| \geq |S|$. Hence $|S| \leq |K| = 2(\frac{n+4}{3})$. On the other hand, since S is a γ_{ri}-set of L_n then S must have at least $2(\frac{n+4}{3})$ number of vertices in L_n. Hence, $|S| \geq 2(\frac{n+4}{3})$. Therefore, $|S| = 2(\frac{n+4}{3})$. □

3.3 Maximum Cardinality of Rings Dominating Set in the Ladder Graph

Here, the maximum cardinality of rings dominating set is referred to as the upper rings domination number, denoted by γ'_{ri}.

Theorem 3.4. *Let $D \subset L_n$. The upper rings domination number in the ladder graph L_n is $\gamma'_{ri} = 2n - 4$.*

Proof. Let L_n be a ladder graph with order $2n$. Assume that the maximum cardinality of rings dominating set in the ladder graph is $2n - 3$. From the proof of Theorem 3.1, for every vertex $(u, w_i) \in D$, (v, w_i) must also be in D. Hence the cardinality of rings dominating set in the ladder graph is an even number. Note that $2n-3$ is an odd number for every natural number n. Thus $2n-3$ is not the maximum cardinality of rings dominating set in the ladder graph L_n. To this end, $2n-2$ is not the maximum cardinality since it will lead to an induced subgraph $L_n[V(L_n)-D]$ of order 2 that could either be a path or an empty graph, both of order 2. □

Example 3.5. *Consider the ladder graph L_7 in Figure 5 below. The possible rings dominating set of different cardinality for L_7 was provided. Based on the diagram, the maximum cardinality of the rings dominating set in the ladder graph L_7 is $2n-4=2(7)-4=10$.*

Remark 3.1. Suppose $D \subseteq L_n$. The rings dominating set of maximum cardinality is L_n itself. Hence, $\gamma'_{ri} = |L_n| = 2n$.

Note that if $\gamma'_{ri}(D) = 2n$, then $D = V(L_n)$. Consequently, $V(L_n) - D = \emptyset$. Hence, the condition of rings dominating set is not violated since there exists no vertex $v \in V(L_n) - D$ such that $deg(v) < 2$. In fact, no vertex in $V - D$ that should be considered and examined.

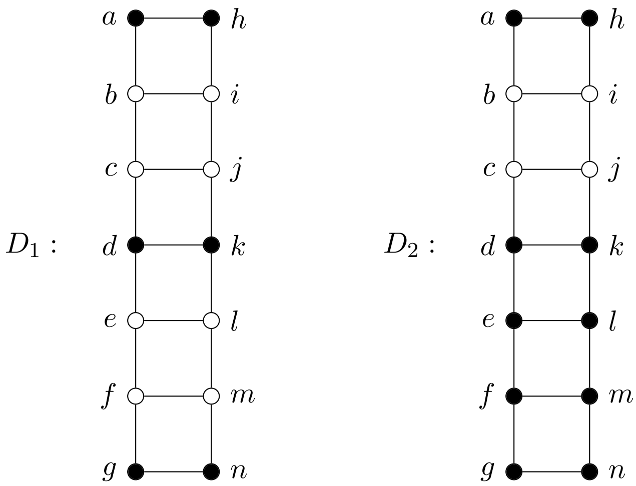

Figure 5: All possible rings dominating set of L_7 of different cardinality

Proposition 3.1. *Let D be a rings dominating set in the ladder graph L_n. Then the order of the induced subgraph $L_n[V(L_n)-D]$ is divisible by 4. Further, the order and size of $L_n[V(L_n)-D]$ are equal.*

Proof. This is immediate from Theorem 3.1. □

Example 3.6. *Again, refer to Figure 5. It can easily be seen that the order $L_7[V(L_7)-D_1]$ and $L_7[V(L_7)-D_2]$ are $8=4(2)$ and $4=4(1)$ respectively, which coincides to their sizes.*

4 Conclusion

In this article, rings dominating sets in the ladder graph generated on the cartesian product of path graph of order n and of order 2 are studied. Further, the maximum cardinality of rings dominating set is also determined and new approach for the rings domination number is presented. Lastly, we intend to examine the rings dominating set and rings domination number for few unstudied graph families in the future.

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Competing Interests

The authors declare that they have no competing interests.

References

[1] Abed, S. S., AL-Harere, M.N. (2022). *Rings domination in graphs.* International Journal of Nonlinear Analysis and Applications.

[2] Acosta, H. R., de Oro City, C., Eballe, P. R. G., Cabahug Jr, I. S. (2019). *Downhill domination in the tensor product of graphs.* International Journal of Mathematical Analysis, 13(12), 555-564.

[3] Cabahug Jr, I. S., Canoy Jr, S. R. (2016). *Independent dr-Power Dominating Sets in Graphs.* Applied Mathematical Sciences, 10(8), 377-387.

[4] Cabahug Jr, I. S., Canoy Jr, S. R. (2018). *Connected dr-Power Dominating Sets in Graphs.* Advances and Applications in Discrete Mathematics, 19(3), 171-182.

[5] Canoy Jr., S. R. (2014). *Restrained Domination in Graphs Under Some Binary Operations.* Applied Mathematical Sciences, 8(16), 6025 - 6031

[6] Chartrand, G., Lesniak, L., Zhang, P., (2015). *Graphs and Digraphs* (Discrete Mathematics and Its Applications) (6th ed.), Chapman and Hall/CRC.

[7] Consistente, L. F., Cabahug Jr, I. S., (2021). *Hinge Total Domination on Some Graph Families.*

[8] Damalerio, R. J. M., Eballe, R. G. (2021). *Triangular index of some graph products.* Applied Mathematical Sciences, 15(12), 587-594.

[9] Eballe, R., Cabahug, I. (2021). *Closeness centrality of some graph families.* International Journal of Contemporary Mathematical Sciences, 16(4), 127-134.

[10] Gayathri, A., Muneera, A., Rao, T.N., Rao, T.S. *Study of Various Dominations in Graph Theory and Its Applications.* International Journal of Scientific and Technology Research, 9, 3426 - 3429.

[11] Guichard, D., (2017). *An introduction to combinatorics and graph theory.* Hitman College - Creative Commons

[12] Mangubat, D. P., Cabahug Jr, I. S., (2021). *On the Restrained Cost Effective Sets of Some Special Classes of Graphs.*

[13] Militante, M., Eballe, R. (2021). *Weakly Connected 2-domination in some special graphs.* Applied Mathematical Sciences, 15(12), 579-586.

[14] Militante, M., Eballe, R., Leonida, R. (2021). *Weakly Connected 2-Domination in the Join of Graphs.* Applied Mathematical Sciences, 15(12), 202.

[15] Moussa M.I., Badr E.M. *Ladder and subdivision of ladder graphs with pendant edges are odd graceful.*

[16] Tan, K. S. R., Cabahug Jr, I. S., (2021). *Safe Sets in Some Graph Families.*

[17] Venkateswari, R. *Applications of Domination Graphs in Real Life.* A Journal of Composition Theory.