SOME THEOREMS ON THE DEMEANOUR OF PROBABILISTIC UNCERTAINTY-LIKE FUNCTIONAL UNDER THE BOUNDS

Abstract

The resulting mean of the optimal solutions of minimization problems, whose objective functions are the uncertainty like functionals, are known as uncertainty mean. The uncertainty mean satisfies all the basic properties of the classical mean, weighted homogeneous mean as well as many others are special cases of uncertainty mean. The indeed paper deals with comparison property and asymptotic demeanour of the uncertainty mean.

Keywords: Uncertainty mean, Uncertainty like functional, Comparison theorem, Asymptotic demeanour, Weighted mean, Homogeneous mean etc.

1. Introduction:

In 2007, Ulrich Bodenhofer [2, 3] considered two T-equivalences $E_1 : X_1^2 \to [0,1]$, $E_2 : X_2^2 \to [0,1]$, a T-E_1-ordering $L_1 : X_1^2 \to [0,1]$, and a T-E_2-ordering $L_2 : X_2^2 \to [0,1]$. Moreover, let $\tilde{T}$ be a t–norm that dominates T. Then the fuzzy relation [10] $\text{Lex}_{\tilde{T},T}(L_1, L_2)$: $(X_1 \times X_2)^2 \to [0,1]$ defined as

$$Lex_{\tilde{T},T}(L_1, L_2)((x_1, x_2),(y_1, y_2)) =$$

$$\max\left(\tilde{T}(L_1(x_1, y_1), L_2(x_2, y_2)), \min\left(L_1(x_1, y_1), N_T(L_1(y_1, x_1))\right)\right) \tag{1.1}$$

is a fuzzy ordering [11] with respect to T and the T-equivalence $\text{Cart}_{\tilde{T}}(E_1, E_2) : (X_1 \times X_2)^2 \to [0,1]$ defined as the Cartesian product [10] of E_1 and E_2:

$$Cart_{\tilde{T}}(E_1, E_2)((x_1, x_2),(y_1, y_2)) = \tilde{T}(E_1(x_1, y_1), E_2(x_2, y_2))$$

Definition 1.1 If L_1 is a crisp ordering, then $Lex_{\tilde{T},T}(L_1,L_2)$ defined as above and coincides with the fuzzy relation L defined as follows.

Let us consider a crisp ordering $L_1 : X_1^2 \to \{0,1\}$, a $T-$equivalence $E_2 : X_2^2 \to [0,1]$, and a T-E_2- ordering $L_2 : X_2^2 \to [0,1]$. Then the fuzzy relation $L : (X_1 \times X_2)^2 \to [0,1]$ defined as

$$L((x_1,x_2),(y_1,y_2)) = \begin{cases} 1 & \textit{if } x_1 \neq y_1 \textit{ and } L_1(x_1,y_1)=1, \\ L_2(x_2,y_2) & \textit{if } x_1 = y_1, \\ 0 & \text{otherwise}, \end{cases}$$

is a fuzzy ordering with respect to T and the T- equivalence $E : (X_1 \times X_2)^2 \to [0,1]$ defined as

$$E((x_1,x_2),(y_1,y_2)) = \begin{cases} E_2(x_2,y_2) & \textit{if } x_1 = y_1, \\ 0 & \text{otherwise.} \end{cases}$$

Note that, if both components L_1 and L_2 are crisp orderings, then L as defined above is equivalent to the following constructions (1.2) and (1.3).

Definition 1.2 For any given two orderings $\leq_1$ and $\leq_2$ on non-empty domains X_1 and X_2, respectively, the lexicographic composition is an ordering $\leq'$ on the cartesian product $X_1 \times X_2$, where $(x_1,x_2) \leq' (y_1,y_2)$ if and only if

$$(x_1 \neq y_1 \wedge x_1 \leq_1 y_1) \vee (x_1 = y_1 \wedge x_2 \leq_2 y_2). \quad (1.2)$$

Rewriting $x_1 \neq y_1 \wedge x_1 \leq_1 y_1$ as $x_1 <_1 y_1$ (i.e. the strict ordering induced by $\leq_1$)and taking into account that $x_1 = y_1 \vee x_1 \neq y_1$ is a tautology and that $\leq_1$ is reflexive, we obtain that (1.2) is equivalent to

$$(x_1 \leq_1 y_1 \wedge x_2 \leq_2 y_2) \vee x_1 <_1 y_1. \quad (1.3)$$

Moreover, E as defined above is nothing else but the Cartesian product of the crisp equality with E_2.

Consequently, if both components L_1 and L_2 are crisp orderings, then $\text{Lex}_{\tilde{T},T}(L_1,L_2)$ is equivalent to the constructions (1.2) and (1.3). Construction (1.1) is based on one specific formulation of lexicographic composition, namely (1.3).

Definition 1.3 A function $D(P,Q)$ of P and Q will be considered as a measure of directed divergence [9] of the probability distribution P from the probability distribution Q if

(i) $D(P,Q) \geq 0$,

(ii) $D(P,Q) = 0$ iff $p_i = q_i$ for each i,

(iii) $D(P,Q)$ is a convex function of both $p_1, p_2, ..., p_n$ and $q_1, q_2, ..., q_n$.

Let the logarithmic mean [5, 8] of the positive number x and y be defined by

$$L(x,y) = \frac{x-y}{\log x - \log y}, \qquad x \neq y,$$

$$L(x,x) = x.$$

Note that L is symmetric and homogeneous in x and y and continuous at $x = y$. It is not widely known that L separates the arithmetic and geometric [12] means:

$$(xy)^{\frac{1}{2}} \leq L(x,y) \leq \frac{x-y}{2},$$

with strict inequalities if $x \neq y$. Division by y shows that the above result is equivalent to well-known inequalities in the single variable $w = x/y$, but the beauty of the above result comes from its symmetry in two variables. The symmetry is somewhat slighted by retaining the unnecessary condition $x \geq y$. The left hand inequality is stated by Carlson [7], who obtained the above result by specializing some rather general integral inequalities to the case of the representation

$$\frac{1}{L(x,y)} = \int_0^1 \frac{du}{ux+(1-u)y}.$$

If x and y are the two unequal positive numbers, then

$$(xy)^{\frac{1}{2}} < (xy)^{\frac{1}{4}} \frac{\sqrt{x}+\sqrt{y}}{2} < L(x,y) < \left(\frac{\sqrt{x}+\sqrt{y}}{2}\right)^2 < \frac{x+y}{2}.$$

Since the asymptotic property for the mean of order p $i.e.$

$$z_\delta(\alpha_1, \ldots \ldots, \alpha_n) = \left(\sum_{i=1}^n w_i {\alpha_i}^p\right)^{\frac{1}{p}}$$

is given by $z_p(\alpha_1 + \tau, \ldots \ldots, \alpha_n + \tau) - \tau \to \sum_{i=1}^n w_i \alpha_i, \quad \tau \to +\infty$. This result is proved for some special values of p. This results were extended to unweighted power means of all orders and homogeneous means.

Given a set of positive numbers $a_1, a_2, \ldots\ldots, a_n$, we define their mean value as any function $f(a_1, a_2, \ldots\ldots, a_n)$ which satisfies the following six conditions:

(i) $\min(a_1, a_2, \ldots\ldots, a_n) \leq f(a_1, a_2, \ldots\ldots, a_n) \leq \max(a_1, a_2, \ldots\ldots, a_n)$,

(ii) $f(a_1, a_2, \ldots\ldots, a_n)$ is a permutationally symmetric function of $a_1, a_2, \ldots\ldots, a_n$ i.e. it does not change when $a_1, a_2, \ldots\ldots, a_n$ are interchanged among themselves,

(iii) If $a_1 = a_2 = \ldots\ldots = a_n = a$, then $f(a_1, a_2, \ldots\ldots, a_n) = a$,

(iv) $\min(a_1, a_2, \ldots\ldots, a_n) < \max(a_1, a_2, \ldots\ldots, a_n)$

$$\Rightarrow \min(a_1, a_2, \ldots\ldots, a_n) < f(a_1, a_2, \ldots\ldots, a_n) < \max(a_1, a_2, \ldots\ldots, a_n),$$

(v) $f\left(a_1, a_2, \ldots\ldots, a_n\right)$ is homogeneous or scale invariant, i.e.

$$f\left(\lambda a_1, \lambda a_2, \ldots. \lambda a_n\right) = \lambda\, f\left(a_1, a_2, \ldots. a_n\right),$$

(vi) $f\left(a_1, a_2, \ldots\ldots, a_n\right)$ is a monotonic increasing function of each of its arguments.

Sometimes, we consider weighted means [4], when positive weights $w_1, w_2, \ldots\ldots, w_n$ are associated with $a_1, a_2, \ldots\ldots, a_n$. A weighted mean is a number which lies between $\min\left(a_1, a_2, \ldots\ldots, a_n\right)$ and $\max\left(a_1, a_2, \ldots\ldots., a_n\right)$, which does not change when pairs $\left(a_i, w_i\right)$ are permuted among themselves, which reduces to a when $a_1 = a_2 = \ldots\ldots. = a_n = a$ and which satisfies conditions (iv), (v) and (vi) above. Some important means are [7, 9]

	unweighted	weighted
Arithmetic mean [1,8], A	$\sum_{i=1}^{n} \frac{a_i}{n}$	$\frac{\sum_{i=1}^{n} w_i a_i}{\sum_{i=1}^{n} w_i}$
Geometric mean [5], G	$\sqrt[n]{a_1 a_2 \ldots\ldots.. a_n}$	$\left(a_1^{w_1} a_2^{w_2} \ldots\ldots. a_n^{w_n}\right)^{\frac{1}{\sum_{i=1}^{n} w_i}}$
Harmonic mean, H	$\left[\sum_{i=1}^{n} \frac{a_i^{-1}}{n}\right]^{-1}$	$\left[\frac{\sum_{i=1}^{n} w_i a_i^{-1}}{\sum_{i=1}^{n} w_i}\right]^{-1}$
Root mean square $\sqrt{\mu_2'}$	$\left[\sum_{i=1}^{n} \frac{a_i^2}{n}\right]^{\frac{1}{2}}$	$\left[\frac{\sum_{i=1}^{n} w_i a_i^2}{\sum_{i=1}^{n} w_i}\right]^{\frac{1}{2}}$
Mean of Order r $\sqrt[r]{\mu'_r}$	$\left[\sum_{i=1}^{n} \frac{a_i^r}{n}\right]^{\frac{1}{r}}$	$\left[\frac{\sum_{i=1}^{n} w_i a_i^r}{\sum_{i=1}^{n} w_i \frac{1}{r}}\right]$

Lehmer mean $\qquad \dfrac{\sum_{i=1}^{n} a_i^r}{\sum_{i=1}^{n} a_i^{r-1}} \qquad \dfrac{\sum_{i=1}^{n} w_i a_i^r}{\sum_{i=1}^{n} w_i a_i^{r-1}}$

Gini mean $\qquad \left[\dfrac{\sum_{i=1}^{n} a_i^r}{\sum_{i=1}^{n} a_i^s}\right]^{\frac{1}{r-s}} \qquad \left[\dfrac{\sum_{i=1}^{n} w_i a_i^r}{\sum_{i=1}^{n} w_i a_i^s}\right]^{\frac{1}{r-s}}$

The last mean is the most general mean [6, 11] since the other six means can be obtained as its special cases. Another general mean is given by

$$f^{-1}\left(\frac{\sum_{i=1}^{n} w_i f(a_i)}{\sum_{i=1}^{n} w_i}\right),$$

where $f(.)$ is a one-one function defined from $R^+ \to R^+$.All these means satisfy the six conditions given above.

2. Our Results:

Theorem 2.1 Let $\delta_1, \delta_2 \in \Delta$ and $\delta_\epsilon(t) = \epsilon\delta_1(t) + (1-\epsilon)\delta_2(t)$. Then $\forall\, \epsilon \in [0,1]$

$$\min\{\bar{z}_{\delta_1}(\alpha), \bar{z}_{\delta_2}(\alpha)\} \le \bar{z}_{\delta_i}(\alpha) \le \max\{\bar{z}_{\delta_i}(\alpha), \bar{z}_{\delta_2}(\alpha)\}.$$

Proof. $\forall\ \epsilon \in [0,1]$ and $\delta_\epsilon \in \Delta$. Now $\bar{z}_{\delta_\epsilon} \in \Delta$. Now $\bar{z}_{\delta_\epsilon}$ is obtained from

$$\sum_{i=1}^{n} w_i \left\{\epsilon\delta_1'\left(\frac{\bar{z}_{\delta_\epsilon}}{\alpha_i}\right) + (1-\epsilon)\delta_2'\left(\frac{\bar{z}_{\delta_\epsilon}}{\alpha_i}\right)\right\} \qquad (2.1.1)$$

Letting, $\bar{z}_{\delta_\epsilon} < \min(\bar{z}_{\delta_1}, \bar{z}_{\delta_2})$ then since δ_1', δ_2' are strictly increasing we have with (2.2.1)

$$\epsilon \sum_{i=1}^{n} w_i \delta_1'\left(\frac{\bar{z}_{\delta_1}}{\alpha_i}\right) + (1-\epsilon)\sum_{i=1}^{n} w_i \delta_2'\left(\frac{\bar{z}_{\delta_2}}{\alpha_i}\right) > 0. \qquad (2.1.2)$$

But, from the optimality conditions for $\bar{z}_{\delta_1}$ and $\bar{z}_{\delta_2}$, the left hand of (2.1.2) is equal to zero, thus the contradiction. Similarly for $\bar{z}_{\delta_\epsilon} > \max\{\bar{z}_{\delta_i}(\alpha), \bar{z}_{\delta_2}(\alpha)\}$. This completes the proof.

Example 2.1 If $\delta_1(s) = -\ln s + s - 1$ and $\delta_2(s) = (s-1)^2$. Also $\bar{z}_{\delta_1} = A(\alpha)$ and $\bar{z}_{\delta_2} = H(\alpha)$. Consider for $\epsilon = \frac{2}{3}$, $\delta_\lambda(s) = \epsilon\delta_1(s) + (1-\epsilon)\delta_2(s)$. Then $\bar{z}_{\delta_\lambda} = \sqrt{A(\alpha).H(\alpha)}$. Indeed as predicted by theorem 2.1, since $A(a) \ge H(a)$ and $A(a) \ge \sqrt{A(\alpha).H(\alpha)} \ge H(a)$.

Theorem 2.2 Let $\delta_1, \delta_2 \in \Delta$ and $\bar{z}_{\delta_1}, \bar{z}_{\delta_2}$ denotes their corresponding uncertainty means. If there exists a constant $K \neq 0$ such that

$$K\delta_1'(s) \le \delta_2'(s) \qquad \forall s \in \boldsymbol{R}_+ \backslash \{1\}$$

with optimality condition $\quad \sum_{i=1}^{n} w_i \delta'\left(\frac{\bar{z}_\delta}{\alpha_i}\right) = 0$

then, $\quad \bar{z}_{\delta_1}(\alpha) \ge \bar{z}_{\delta_2}(\alpha)$

Proof: By the optimality condition $\quad \sum_{i=1}^{n} w_i \delta'\left(\frac{\bar{z}_\delta}{\alpha_i}\right) = 0$

we get the following results

$$\sum_{i=1}^{n} w_i \delta_1'\left(\frac{\bar{z}_{\delta_1}}{\alpha_i}\right) = 0 \tag{2.2.1}$$

and

$$\sum_{i=1}^{n} w_i \delta_2'\left(\frac{\bar{z}_{\delta_2}}{\alpha_i}\right) = 0. \tag{2.2.2}$$

Again, if $\bar{z}_{\delta_1} < \bar{z}_{\delta_2}$. Since δ_2 is strictly convex δ_2' is strictly increasing. Now, using the given condition $K\delta_1'(s) \le \delta_2'(s), \forall s \in \boldsymbol{R}_+ \backslash \{1\}$ we get the following result

$$K\delta_1'\left(\frac{\bar{z}_{\delta_1}}{\alpha_i}\right) \le \delta_2'\left(\frac{\bar{z}_{\delta_1}}{\alpha_i}\right) < \delta_2'\left(\frac{\bar{z}_{\delta_2}}{\alpha_i}\right). \qquad 1 \le i \le n. \tag{2.2.3}$$

Now, multiplying by $w_i > 0$ and summing the inequality (2.2.3) from $1 \le i \le n$ we get the following result

$$K\sum_{i=1}^{n} w_i \delta_1'\left(\frac{\bar{z}_{\delta_1}}{\alpha_i}\right) < \sum_{i=1}^{n} w_i \delta_2'\left(\frac{\bar{z}_{\delta_2}}{\alpha_i}\right).$$

Hence from (2.2.1) and (2.2.2), it implies $0 < 0$ a contradiction. This completes the proof.

Example 2.2 We show that the classical inequalities $A(\alpha) \ge G(\alpha) \ge H(\alpha)$ are an easy consequence of the Theorem 2.2. If $\delta_1(s) = -\ln s + s - 1$ and $\delta_2(s) = s\ln s - s + 1$ also, $\bar{z}_{\delta_1} = A$ and $\bar{z}_{\delta_2} = G$. Given $K\delta_1'(s) \le \delta_2'(s)$ is satisfied with $K = 1$. Indeed, by the convexity of $s\ln s$ it follows that $\forall s > 0$, $\ln s \ge 1 - \frac{1}{s}$, that means $\delta_2'(s) \ge \delta_1'(s)$ and hence $A \ge G$. Again, if $\delta_1(s) = s\ln s - s + 1$ and $\delta_2(s) = (s-1)^2$ also, $\bar{z}_{\delta_1} = G$ and $\bar{z}_{\delta_2} = H$. From the concavity of $\ln s$ it follows that $\forall s > 0$, $s - 1 \ge \ln s$ with $K = 2$ the condition $K\delta_1'(s) \le \delta_2'(s)$ is satisfied that means $\delta_2'(s) \le 2\delta_1'(s)$ and hence $G \ge H$.

Theorem 2.3 Suppose $\delta \in \Delta$ and assume that δ is three times continuously differentiable in the neighborhood of $s = 1$. If $\alpha_1, \ldots\ldots, \alpha_n$ are fixed, $\tau \to +\infty$ and $z_\delta(\alpha)$ is homogeneous mean satisfying $z_\delta(1) = z_\delta(1, \ldots\ldots, 1) = 1$. Then, the asymptotic result

$$z_\delta(\alpha_1+\tau,\dots\dots,\alpha_n+\tau)=\tau+\sum_{i=1}^{n} w_i\alpha_i+\eta\left(\frac{1}{\tau}\right).$$

Proof: Suppose z_δ is a weighted mean and given that $z_\delta(\alpha)$ is homogeneous mean. So

$$\frac{\partial z_\delta}{\partial a_j}(1,\dots\dots,1)=w_j, \qquad j=1,\dots\dots,n.$$

$$\sum_{i=1}^{n} w_i\delta'\left(\frac{z_\delta(\alpha)}{a_i}\right)=0.$$

On differentiating the identity in terms of α with respect to a_j we get

$$\frac{\partial z_\delta(\alpha)}{\partial a_j}\sum_{i=1}^{n}\frac{w_i}{a_i}\delta''\left(\frac{z_\delta(\alpha)}{a_i}\right)=\frac{w_j}{\alpha_j^2}\delta''\left(\frac{z_\delta(\alpha)}{a_j}\right)z_\delta(\alpha), \quad j=1,\dots\dots,n. \tag{2.3.1}$$

Taking $a_i=1$, $1\le i\le n$ and using $z_\delta(1)=1$, $\delta''(1)=1$ and $\sum_{i=1}^{n} w_i=1$. Hence from (2.3.1) we get $\left(\frac{\partial z_\delta}{\partial a_j}\right)(1,\dots\dots,1)=w_j$, $j=1,\dots\dots,n$. On the other hand, the differentiability assumption of δ implies that $z_\delta(.)$ Is twice continuously differentiable in the neighborhood of $(1,\dots\dots,1)$. Hence, the asymptotic result follows.

Example 2.3 Suppose the mean of order p is $\delta_p(s)=\left(\frac{1}{p-1}\right)(s^{1-p}-p)+s$, $p\neq1$, $p>0$. Then the optimality condition equation $\sum_{i=1}^{n} w_i\delta'\left(\frac{z}{a_i}\right)=0$ yields $-z^{-p}\sum w_i{\alpha_i}^p+1=0$, hence $\bar{z}_p(\alpha)=\left(\sum_{i=1}^{n} w_i{\alpha_i}^p\right)^{\frac{1}{p}}$. To extend $\bar{z}_p(\alpha)$ for negative order, one may choose $\delta_q(s)=\frac{(s^q-sq)}{(q-1)}+1$, $q>0$, $q\neq1$, which yields $\bar{z}_q(q)=\left(\left(\sum_{i=1}^{n} w_i\alpha_i^{1-q}\right)\right)^{\frac{1}{1-q}}$. Note that $\tilde{\delta}_q(s)=s\delta_q\left(\frac{1}{s}\right)$, hence strictly convex for all $s>0$ and $\tilde{\delta}_q(s)=\delta_p(s)$ for $p=q=\frac{1}{2}$ yielding the root mean square R, while $q=2$ gives the harmonic mean H. A simple application of L' Hospital's rule shows that the arithmetic and geometric mean are re-obtained respectively by choosing for δ, the limiting cases $\lim_{p\to1}\delta_p(s)=-\ln s+s-1$ and $\lim_{q\to1}\tilde{\delta}_q(s)=s\ln s-s+1$.

Conclusion: However, by the help of homogeneous and weighted means, we defines the asymptotic demeanour and also the comparison theorem for the uncertainty means are developed under the optimality conditions but, both are useful to derive classical inequalities.

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