Original Research Article

Modified Ratio Estimators for Population Means with Two Auxiliary Parameters using Calibration Weights

ABSTRACT

Many researchers as seen in literature have used different auxiliary parameters such as coefficient of variation, coefficient of kurtosis, coefficient of skewness, quartiles, deciles etc., to improve the precision of estimators under various sampling schemes. This paper suggested a class of ratio estimator with two known auxiliary variable parameters for estimation of population means under simple random sample without replacement (SRSWOR) using the calibration weighting method. The calibrated weight was obtained using new calibration constraint, which includes known standard deviation of the auxiliary variable. The biases and mean square errors of the proposed estimators were derived and compared with the biases and mean square errors of the existing modified ratio estimators in Upadhyaya & Singh (1999), Singh (2003), Lu & Yan (2014), and Yan & Tian (2010). Furthermore, we derived the condition for which the proposed estimators perform better than the existing estimators. The results from using real data sets showed that the suggested estimators perform better than the existing ratio estimators.

Key words: Calibration; Estimator; Stratified sampling; Ratio; Mean square error; Bias;

1. Introduction

The improvement on precision of estimates of population parameters in sampling theory is a continuous issue. Kanwai, Asiribo, & Isah, (2016) established that by increasing the sampling size, the precision of estimate can be improved, but the cost of the sampling survey increases by doing so, therefore an appropriate estimation procedure that makes use of auxiliary parameter which is closely related to the study variable can be used to increase the precision of the estimates. In survey sampling, the availability of more auxiliary information can be used to further increase the precision of an estimate by adjusting the design weights based on all the auxiliary information (Rao, Khan & Khan 2012). Calibration is one of the methods in survey sampling that can be used to achieve this purpose. Calibration weighting was originally developed as a method for reducing sampling errors while retaining randomization consistency (Kott, 2006). This procedure adjusts the sampling weights by multipliers known as calibration factors that make the estimates agree with known totals. In literature, many researchers including Yan & Tian (2010), Kadilar (2006), Sarndal (2007), Upadhyaya and Singh (1999), Singh (2003), Lu and Yan (2014), Koyuncu & Kadilar (2013), Subramani (2013), Kim & Rao (2012), and Deville & Sarndal (1992) etc., have contributed to the improvement of estimators precision using auxiliary parameters. In this paper, we obtained a calibrated weight using new calibration constraint, which includes known standard deviation of the auxiliary variable. A class of ratio estimator with two known auxiliary variable parameters for estimation of population means under simple random sample without replacement (SRSWOR) was suggested using the calibration weight obtained. The biases and mean square errors (MSE) of the proposed estimators were derived and used to check the efficiency compared to some existing modified ratio estimators.

2. Notation definition

Let population Ω = {1, 2,..., N}, and let a probability sample s be drawn with sampling design denoted by p, and the probabilities of inclusion f_i = Pr(i ∈ s). For the ith population unit, let y_i be the value of the variable of interest and x_i be the value of auxiliary variable associated with this unit. Let $\bar{Y}$ and $\bar{X}$ be the population means of *y* and *x* respectively.

N = Population size, n = Sample size, $f = n/N$ = Sampling fraction, Y − Study variable,

X − Auxiliary variable, $\bar{X}$ and $\bar{Y}$ − Population means, x and y - Sample totals,

$\bar{x}$ and $\bar{y}$ − Sample means, S_x and S_y − Population standard deviations,

C_x and C_y − Coefficient of variations, ρ − Coefficient of correlation,

$\beta_1 = \dfrac{N\sum_{i=1}^{N}\left(x_i - \bar{X}\right)^3}{(N-1)(N-2)S_x^3}$ = Coefficient of skewness of the auxiliary variable

$$\beta_2 = \frac{N(N+1)\sum_{1}^{N}(x_I - \bar{X})^4}{(N-1)(N-2)(N-3)S_X^4} - \frac{3(N-1)^2}{(N-2)(N-3)}$$ = Coefficient of kurtosis of the auxiliary variable

$B(.)$ − Bias of the estimator, $MSE(.)$ − Mean squared error of the estimator

3. Proposed ratio estimators

In this section, we proposed a modified generalized class of ratio estimator of population mean in simple random sampling using two parameters of the auxiliary variable and also obtained the bias and the mean square errors.

The calibration weights W_i is chosen by minimizing the average distance L

$$L = \sum_{i=1}^{n} (w_i - d_i)^2 / (d_i q_i) \tag{1}$$

while satisfying a calibration constraint.

$$\sum_{i=1}^{n} W_i \bar{X} = S_x \tag{2}$$

which gives the calibration weight in simple random sampling as

$$W_i = d_i + \frac{\bar{X} d_i q_i}{\sum_{i=1}^{n} \bar{X}^2 d_i q_i}\left(S_x - \sum_{i=1}^{n} d_i \bar{X}\right) \tag{3}$$

where W_i is the design weight such that $0 < W_i < 1$, S_x is the population standard deviation, the design weight $d_i = 1/\pi_1$, where the q_i's are known positive weights unrelated to d_i. The inclusion probability denoted by $\pi_1 = n/N$ so that $d_i = N/n$

According to Deville & Sarndal (1992), the calibrated estimator of the population mean $\bar{Y}$ was given as:

$$\hat{\bar{Y}}_{DS} = \sum_{i=1}^{n} W_i y_i \tag{4}$$

Substituting (3) into (4), and setting $q_i = \bar{X}^{-1}$ gives the proposed class of ratio estimators $\left(\hat{\hat{\bar{Y}}}_k\right)$ for estimating the population mean under SRSWOR as

$$\hat{\hat{\bar{Y}}}_k = \frac{\sum_{i=1}^{n} d_i y_i}{\sum_{i=1}^{n} d_i \bar{X}} S_x \left[\frac{A\bar{X}+B}{A\bar{x}+B} \right] \tag{5}$$

where A and B can either be real values or known population parameters of the auxiliary variable such as coefficient of skewness $(\beta_1 x)$, coefficient of kurtosis $(\beta_2 x)$, coefficient of variation (C_x), correlation coefficient (ρ_{xy}), median (M_x), standard deviation (S_x), quartiles (Q_x) e.t.c.

To obtain the bias and the MSE of $\left(\hat{\hat{\bar{Y}}}_k\right)$, up to the first order of approximation, we define

$$\bar{x} = \bar{X}\left(1+\Delta_x\right), \bar{y} = \bar{Y}\left(1+\Delta_y\right),$$

such that

$$E(\Delta_x) = E(\Delta_y) = 0$$

$$E(\Delta_x^2) = C_x^2, E(\Delta_y^2) = C_y^2, E(\Delta_x \Delta_y) = \rho_{xy} C_x C_y$$

expressing (5) in terms of Δ_x and Δ_y we have

$$\hat{\hat{\bar{Y}}}_k = \frac{\bar{Y}\left(1+\Delta_y\right)}{\bar{X}} S_x \left[\frac{A\bar{X}+B}{A\bar{X}\left(1+\Delta_x\right)+B} \right]$$

$$\hat{\hat{\bar{Y}}}_k = \left(\bar{Y}S_x + \bar{Y}\Delta_y S_x\right)\left(\bar{X}\right)^{-1} \left[\frac{A\bar{X}+B}{\left(A\bar{X}+B\right)\left(1+\frac{A\bar{X}\Delta_x}{A\bar{X}+B}\right)} \right] \tag{6}$$

Let $V_k = \frac{A\bar{X}}{A\bar{X}+B}$

Substitute V_k into (6), we have

$$\hat{\hat{\bar{Y}}}_k = \left(\bar{Y}S_x + \bar{Y}S_x\Delta_y\right)\left(1+V_k\Delta_x\right)^{-1}\left(\bar{X}\right)^{-1} \tag{7}$$

If we assume, $|\Delta_x| < 1$ and $|\Delta_y| < 1$, the expression $\left(1+V_k\Delta_x\right)^{-1}$ can be expanded to a convergent infinite series using binomial expansion. Expanding the term Δ's up to power 2, hence,

$$\left[1+V_k\Delta_x\right]^{-1} = \left(1-V_k\Delta_x + V_k^2\Delta_x^2\right)$$

$$\hat{\bar{Y}}_k = \left(\bar{Y}S_x + \bar{Y}S_x\Delta_y\right)\left(1-V_k\Delta_x + V_k^2\Delta_x^2\right)\left(\bar{X}\right)^{-1} \tag{8}$$

$$= RS_x + R\Delta_y S_x - RV_k S_x\Delta_x - RV_k S_x\Delta_x\Delta_y + RV_k^2 S_x\Delta_x^2 \tag{9}$$

Subtracting $\bar{Y}$ from both sides of (9) and taking the expectation, the bias of the estimator $\left(\hat{\bar{Y}}_k\right)$ to the first degree of approximation is

$$B\left(\hat{\bar{Y}}_k\right) = E\left(\hat{\bar{Y}}_k - \bar{Y}\right)$$

$$= E\left(RS_x + R\Delta_y S_x - RV_k S_x\Delta_x - RV_k S_x\Delta_x\Delta_y + RV_k^2 S_x\Delta_x^2 - \bar{Y}\right)$$

$$B\left(\hat{\bar{Y}}_k\right) = \frac{1-f}{n}\left(RS_x - RV_k S_x C_x C_y\rho_{xy} + RV_k^2 S_x C_X^2 - \bar{Y}\right)$$

$$= \frac{1-f}{n}\bar{Y}\left(C_x - V_k C_x^2 C_y\rho_{xy} + V_k^2 C_x^3 - 1\right) \tag{10}$$

From (9) the mean square error of the estimator $\left(\hat{\bar{Y}}_k\right)$ to the first degree of approximation is

$$MSE\left(\hat{\bar{Y}}_k\right) = E\left(\hat{\bar{Y}}_k - \bar{Y}\right)^2$$

$$= E\left(RS_x + R\Delta_y S_x - RV_k S_x\Delta_x - RV_k S_x\Delta_x\Delta_y + RV_k^2 S_x\Delta_x^2 - \bar{Y}\right)^2 \tag{11}$$

$$MSE\left(\hat{\bar{Y}}_k\right) = \frac{1-f}{n}\begin{pmatrix} R^2C_x + 3R^2V_k^2S_x^2C_x^2 - 4R^2V_kS_x^2C_xC_y\rho_{xy} - 2R\bar{Y}S_x + R^2S_x^2C_x^2 \\ -2R\bar{Y}V_k^2S_xC_x^2 + 2R\bar{Y}V_kS_xC_xC_y\rho_{xy} + \bar{Y}^2 \end{pmatrix}$$

$$= \frac{1-f}{n}\bar{Y}^2\left(1 - 2C_x + C_x^2 + C_x^2C_y^2 - 2V_k^2C_x^3 + 3V_k^2C_x^4 + 2V_kC_x^2C_y\rho_{xy} - 4V_kC_x^3C_y\rho_{xy}\right)$$

$$= \frac{1-f}{n}\bar{Y}^2\left[1 + C_x^2\left(1+C_y^2\right) - 2C_x\left(1 + V_k^2C_x^2 - V_kC_xC_y\rho_{xy}\right) + V_kC_x^3\left(3V_kC_x - 4C_y\rho_{xy}\right)\right]$$

To the first degree of approximation the biases and mean square errors (MSEs) of the proposed set of estimators are given as

$$B\left(\hat{\bar{Y}}_k\right) = \frac{1-f}{n}\bar{Y}\left(C_x - V_kC_x^2C_y\rho_{xy} + V_k^2C_x^3 - 1\right)$$

$$MSE\left(\hat{\bar{Y}}_k\right) = \frac{1-f}{n}\bar{Y}^2\left[1 + C_x^2\left(1+C_y^2\right) - 2C_x\left(1 + V_k^2C_x^2 - V_kC_xC_y\rho_{xy}\right) + V_kC_x^3\left(3V_kC_x - 4C_y\rho_{xy}\right)\right] \tag{12}$$

where $V_k = \dfrac{A\bar{X}}{A\bar{X}+B}$

3.1 Analytical study

The existing ratio estimators considered in this work and the proposed estimators are given in Table 1 with their respective auxiliary variables, Table 2 consist of the bias of the proposed and existing ratio estimators with their constants, while Table 3 consist of the mean square errors of the proposed and existing ratio estimators with their constants

The MSEs of the proposed estimators is compared with MSEs of some existing estimators as listed in Table 1. The proposed estimator $\hat{\hat{Y}}_k$ in (7) will be better than the existing estimators in the Table 1 and if and only if $MSE\left(\hat{\hat{Y}}_k\right) < MSE\left(\hat{\hat{Y}}_j\right)$, that is if

$$f_j\hat{\hat{Y}}^2\left(1+C_x^2+C_x^2C_y^2+3V_1^2C_x^4-2V_1^2C_x^3-4V_1C_x^3C_y\rho-2C_x-2V_1C_x^2C_y\rho\right)\leq f_1\hat{\hat{Y}}^2\left(C_y^2+\theta_{14}^2C_x^2-2\theta_{14}C_xC_y\rho\right)$$

$$\Rightarrow\left(1-C_x\right)^2-C_y^2\left(1-C_x^2\right)+2V_kC_xC_y\rho_{xy}\left(1+C_x-2C_x^2\right)-V_k^2C_x^2\left(1+2C_x-3C_x^2\right)\leq 0$$

The percent relative efficiency (PRE) of the proposed estimators $\left(\hat{\hat{Y}}_k\right)$ with respect to the existing estimators $\left(\hat{\hat{Y}}_j\right)$ by Upadhyaya and Singh (1999), Singh (2003), Lu and Yan (2014), Yan and Tian (2010) are computed as

$$\%RE\left[\hat{\hat{Y}}_k\right]=\frac{MSE\left[\hat{\hat{Y}}_j\right]}{MSE\left[\hat{\hat{Y}}_k\right]}\times 100$$

3.3 Empirical study

Two different populations were considered in this work to assess the performances of the proposed and existing ratio estimators.

The data used for population 1 was obtained from (Murthy, 1967, p. 228). The population parameters and constants computed from the data are given as:

$N=80,\ n=15,\ \bar{Y}=51.8264,\ \bar{X}=2.8513,\ \rho_{xy}=0.9150,\ S_x=2.7042,\ S_y=18.3569,\ C_x=0.9484,$
$C_y=0.3542,\ \beta_{1(x)}=0.6978,\ \beta_{2(x)}=1.3005$

The data for population 2 is a Household Kerosene (HHK) distribution statistics for Enugu State taken from Nigeria Bureau of Statistics website https://bit.ly/2JBk24f. The data represent the price of one gallon (4.5ltrs) of the product (Y variable) and the number of trucks loaded out to the state (X

variable). Data for four years are considered for this work (Jan., 2016 to Dec., 2019). The population constants computed from the data are given as:

$N = 48, n = 10, \bar{Y} = 1,041.8980, \bar{X} = 49.4375, \rho_{xy} = -0.6124, S_x = 34.7593, S_y = 198.8129, C_x = 0.7031,$
$C_y = 0.1908, \beta_{1(x)} = 1.6432, \beta_{2(x)} = 2.9879$

Based on the two data sets considered, the computation of the biases and the mean square errors of the estimators in Tables 1 were obtained. The results of the computation are presented in the Table 4.

4 Discussion of Results

From Table 3, the proposed ratio estimators have a smaller mean square errors and a higher percent relative efficiency when compared to the existing ratio estimators by Upadhyaya and Singh (1999), Singh (2003), Lu and Yan (2014), Yan and Tian (2010) in the two populations. Also in population 1, the biases of the proposed estimators are smaller to that of the existing estimators. In population 2, the biases of the proposed estimators are smaller to that of the existing estimators, except for estimator $\hat{\bar{Y}}_8$ where the bias of proposed estimators is negative.

5 Conclusion

In this paper, some class of ratio-type estimators $\hat{\bar{Y}}_k$ for estimating the population mean using two parameters of the auxiliary variable are proposed and evaluated. From the results obtained, the mean square errors of the proposed ratio-type estimators $\hat{\bar{Y}}_k$ are less than the mean square errors of the existing ratio-type estimators considered in this paper. This shows that all the proposed ratio estimators have a significant improvement on the existing ratio estimators. The results proved that the proposed estimators are better when we have two known parameters of the auxiliary variable.

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Table 1: Existing ratio estimators and the proposed ratio estimators

Existing Estimators (Y_j)	**Proposed Estimators** $\left(\hat{\bar{Y}}_k\right)$	A	B

$\hat{\hat{Y}}_1 = \bar{y}\left[\frac{\beta_{2(x)}\bar{X}+C_x}{\beta_{2(x)}\bar{x}+C_x}\right]$ Upadhyaya and Singh (1999)	$\hat{\hat{Y}}_1 = \frac{\sum_{i=1}^{n} d_i y_i}{\sum_{i=1}^{n} d_i \bar{X}} S_x\left[\frac{\beta_2\bar{X}+C_x}{\beta_2\bar{x}+C_x}\right]$	$\beta_{2(x)}$	C_x
$\hat{\hat{Y}}_2 = \bar{y}\left[\frac{C_x\bar{X}+\beta_{2(x)}}{C_x\bar{x}+\beta_{2(x)}}\right]$ Upadhyaya and Singh (1999)	$\hat{\hat{Y}}_2 = \frac{\sum_{i=1}^{n} d_i y_i}{\sum_{i=1}^{n} d_i \bar{X}} S_x\left[\frac{C_x\bar{X}+\beta_2}{C_x\bar{x}+\beta_2}\right]$	C_x	$\beta_{2(x)}$
$\hat{\hat{Y}}_3 = \bar{y}\left[\frac{\beta_{1(x)}\bar{X}+C_x}{\beta_{1(x)}\bar{x}+C_x}\right]$ Yan and Tian (2010)	$\hat{\hat{Y}}_3 = \frac{\sum_{i=1}^{n} d_i y_i}{\sum_{i=1}^{n} d_i \bar{X}} S_x\left[\frac{\beta_1\bar{X}+C_x}{\beta_1\bar{x}+C_x}\right]$	$\beta_{1(x)}$	C_x
$\hat{\hat{Y}}_4 = \bar{y}\left[\frac{C_x\bar{X}+\beta_{1(x)}}{C_x\bar{x}+\beta_{1(x)}}\right]$ Yan and Tian (2010)	$\hat{\hat{Y}}_4 = \frac{\sum_{i=1}^{n} d_i y_i}{\sum_{i=1}^{n} d_i \bar{X}} S_x\left[\frac{C_x\bar{X}+\beta_1}{C_x\bar{x}+\beta_1}\right]$	C_x	$\beta_{1(x)}$
$\hat{\hat{Y}}_5 = \bar{y}\left[\frac{\rho_{xy}\bar{X}+C_x}{\rho_{xy}\bar{x}+C_x}\right]$ Lu and Yan (2014)	$\hat{\hat{Y}}_5 = \frac{\sum_{i=1}^{n} d_i y_i}{\sum_{i=1}^{n} d_i \bar{X}} S_x\left[\frac{\rho\bar{X}+C_x}{\rho\bar{x}+C_x}\right]$	ρ	C_x
$\hat{\hat{Y}}_6 = \bar{y}\left[\frac{C_x\bar{X}+\rho}{C_x\bar{x}+\rho}\right]$ Lu and Yan (2014)	$\hat{\hat{Y}}_6 = \frac{\sum_{i=1}^{n} d_i y_i}{\sum_{i=1}^{n} d_i \bar{X}} S_x\left[\frac{C_x\bar{X}+\rho}{C_x\bar{x}+\rho}\right]$	C_x	ρ
$\hat{\hat{Y}}_7 = \bar{y}\left[\frac{S_x\bar{X}+C_x}{S_x\bar{x}+C_x}\right]$ Singh (2003)	$\hat{\hat{Y}}_7 = \frac{\sum_{i=1}^{n} d_i y_i}{\sum_{i=1}^{n} d_i \bar{X}} S_x\left[\frac{S_x\bar{X}+C_x}{S_x\bar{x}+C_x}\right]$	S_x	C_x
$\hat{\hat{Y}}_8 = \bar{y}\left[\frac{C_x\bar{X}+S_x}{C_x\bar{x}+S_x}\right]$ Singh (2003)	$\hat{\hat{Y}}_8 = \frac{\sum_{i=1}^{n} d_i y_i}{\sum_{i=1}^{n} d_i \bar{X}} S_x\left[\frac{C_x\bar{X}+S_x}{C_x\bar{x}+S_x}\right]$	C_x	S_x
$\hat{\hat{Y}}_9 = \bar{y}\left[\frac{\beta_{1(x)}\bar{X}+\beta_{2(x)}}{\beta_{1(x)}\bar{x}+\beta_{2(x)}}\right]$ Yan and Tian (2010)	$\hat{\hat{Y}}_9 = \frac{\sum_{i=1}^{n} d_i y_i}{\sum_{i=1}^{n} d_i \bar{X}} S_x\left[\frac{\beta_1\bar{X}+\beta_2}{\beta_1\bar{x}+\beta_2}\right]$	$\beta_{1(x)}$	$\beta_{2(x)}$
$\hat{\hat{Y}}_{10} = \bar{y}\left[\frac{\beta_{2(x)}\bar{X}+\beta_{1(x)}}{\beta_{2(x)}\bar{x}+\beta_{1(x)}}\right]$ Yan and Tian (2010)	$\hat{\hat{Y}}_{10} = \frac{\sum_{i=1}^{n} d_i y_i}{\sum_{i=1}^{n} d_i \bar{X}} S_x\left[\frac{\beta_2\bar{X}+\beta_1}{\beta_2\bar{x}+\beta_1}\right]$	$\beta_{2(x)}$	$\beta_{1(x)}$
$\hat{\hat{Y}}_{11} = \bar{y}\left[\frac{\rho\bar{X}+\beta_{2(x)}}{\rho\bar{x}+\beta_{2(x)}}\right]$ Lu and Yan (2014)	$\hat{\hat{Y}}_{11} = \frac{\sum_{i=1}^{n} d_i y_i}{\sum_{i=1}^{n} d_i \bar{X}} S_x\left[\frac{\rho\bar{X}+\beta_2}{\rho\bar{x}+\beta_2}\right]$	ρ	$\beta_{2(x)}$
$\hat{\hat{Y}}_{12} = \bar{y}\left[\frac{\beta_{2(x)}\bar{X}+\rho}{\beta_{2(x)}\bar{x}+\rho}\right]$ Lu and Yan (2014)	$\hat{\hat{Y}}_{12} = \frac{\sum_{i=1}^{n} d_i y_i}{\sum_{i=1}^{n} d_i \bar{X}} S_x\left[\frac{\beta_2\bar{X}+\rho}{\beta_2\bar{x}+\rho}\right]$	$\beta_{2(x)}$	ρ

$\hat{\bar{Y}}_{13} = \bar{y}\left[\frac{S_x\bar{X}+\beta_{2(x)}}{S_x\bar{x}+\beta_{2(x)}}\right]$ Singh (2003)	$\hat{\bar{Y}}_{13} = \frac{\sum_{i=1}^{n} d_i y_i}{\sum_{i=1}^{n} d_i \bar{X}} S_x \left[\frac{S_x\bar{X}+\beta_2}{S_x\bar{x}+\beta_2}\right]$	S_x	$\beta_{2(x)}$
$\hat{\bar{Y}}_{14} = \bar{y}\left[\frac{\beta_{2(x)}\bar{X}+S_x}{\beta_{2(x)}\bar{x}+S_x}\right]$ Singh (2003)	$\hat{\bar{Y}}_{14} = \frac{\sum_{i=1}^{n} d_i y_i}{\sum_{i=1}^{n} d_i \bar{X}} S_x \left[\frac{\beta_2\bar{X}+S_x}{\beta_2\bar{x}+S_x}\right]$	$\beta_{2(x)}$	S_x
$\hat{\bar{Y}}_{15} = \bar{y}\left[\frac{S_x\bar{X}+\beta_{1(x)}}{S_x\bar{x}+\beta_{1(x)}}\right]$ Singh (2003)	$\hat{\bar{Y}}_{15} = \frac{\sum_{i=1}^{n} d_i y_i}{\sum_{i=1}^{n} d_i \bar{X}} S_x \left[\frac{S_x\bar{X}+\beta_1}{S_x\bar{x}+\beta_1}\right]$	S_x	$\beta_{1(x)}$
$\hat{\bar{Y}}_{16} = \bar{y}\left[\frac{\beta_{1(x)}\bar{X}+S_x}{\beta_{1(x)}\bar{x}+S_x}\right]$ Singh (2003)	$\hat{\bar{Y}}_{16} = \frac{\sum_{i=1}^{n} d_i y_i}{\sum_{i=1}^{n} d_i \bar{X}} S_x \left[\frac{\beta_1\bar{X}+S_x}{\beta_1\bar{x}+S_x}\right]$	$\beta_{1(x)}$	S_x

Table 2: The constant and bias of the Existing and Proposed ratio estimators $\left(f_j = \frac{1-f}{n}\right)$

Constants(θ_i)	**Existing Bias B(.)**	**Constants** V_i	**Proposed Bias B(.)**

$\theta_1 = \frac{\beta_{2(x)}\bar{X}}{\beta_{2(x)}\bar{X}+C_x}$	$f_1\bar{Y}\left(\theta_1^2 C_x^2 - \theta_1 C_x C_y \rho\right)$	$V_1 = \frac{\beta_2 \bar{X}}{\beta_2 \bar{X}+C_x}$	$f_j\bar{Y}\left(C_x - V_1 C_x^2 C_y \rho + V_1^2 C_x^3 - 1\right)$
$\theta_2 = \frac{C_x\bar{X}}{C_x\bar{X}+\beta_{2(x)}}$	$f_1\bar{Y}\left(\theta_2^2 C_x^2 - \theta_2 C_x C_y \rho\right)$	$V_2 = \frac{C_x \bar{X}}{C_x \bar{X}+\beta_2}$	$f_j\bar{Y}\left(C_x - V_2 C_x^2 C_y \rho + V_2^2 C_x^3 - 1\right)$
$\theta_3 = \frac{\beta_{1(x)}\bar{X}}{\beta_{1(x)}\bar{X}+C_x}$	$f_1\bar{Y}\left(\theta_3^2 C_x^2 - \theta_3 C_x C_y \rho\right)$	$V_3 = \frac{\beta_1 \bar{X}}{\beta_1 \bar{X}+C_x}$	$f_j\bar{Y}\left(C_x - V_3 C_x^2 C_y \rho + V_3^2 C_x^3 - 1\right)$
$\theta_4 = \frac{C_x\bar{X}}{C_x\bar{X}+\beta_{1(x)}}$	$f_1\bar{Y}\left(\theta_4^2 C_x^2 - \theta_4 C_x C_y \rho\right)$	$V_4 = \frac{C_x \bar{X}}{C_x \bar{X}+\beta_1}$	$f_j\bar{Y}\left(C_x - V_4 C_x^2 C_y \rho + V_4^2 C_x^3 - 1\right)$
$\theta_5 = \frac{\rho\bar{X}}{\rho\bar{X}+C_x}$	$f_1\bar{Y}\left(\theta_5^2 C_x^2 - \theta_5 C_x C_y \rho\right)$	$V_5 = \frac{\rho \bar{X}}{\rho \bar{X}+C_x}$	$f_j\bar{Y}\left(C_x - V_5 C_x^2 C_y \rho + V_5^2 C_x^3 - 1\right)$
$\theta_6 = \frac{C_x\bar{X}}{C_x\bar{X}+\rho}$	$f_1\bar{Y}\left(\theta_6^2 C_x^2 - \theta_6 C_x C_y \rho\right)$	$V_6 = \frac{C_x \bar{X}}{C_x \bar{X}+\rho}$	$f_j\bar{Y}\left(C_x - V_6 C_x^2 C_y \rho + V_6^2 C_x^3 - 1\right)$
$\theta_7 = \frac{S_x\bar{X}}{S_x\bar{X}+C_x}$	$f_1\bar{Y}\left(\theta_7^2 C_x^2 - \theta_7 C_x C_y \rho\right)$	$V_7 = \frac{S_x \bar{X}}{S_x \bar{X}+C_x}$	$f_j\bar{Y}\left(C_x - V_7 C_x^2 C_y \rho + V_7^2 C_x^3 - 1\right)$
$\theta_8 = \frac{C_x\bar{X}}{C_x\bar{X}+S_x}$	$f_1\bar{Y}\left(\theta_8^2 C_x^2 - \theta_8 C_x C_y \rho\right)$	$V_8 = \frac{C_x \bar{X}}{C_x \bar{X}+S_x}$	$f_j\bar{Y}\left(C_x - V_8 C_x^2 C_y \rho + V_8^2 C_x^3 - 1\right)$
$\theta_9 = \frac{\beta_{1(x)}\bar{X}}{\beta_{1(x)}\bar{X}+\beta_{2(x)}}$	$f_1\bar{Y}\left(\theta_9^2 C_x^2 - \theta_9 C_x C_y \rho\right)$	$V_9 = \frac{\beta_1 \bar{X}}{\beta_1 \bar{X}+\beta_2}$	$f_j\bar{Y}\left(C_x - V_9 C_x^2 C_y \rho + V_9^2 C_x^3 - 1\right)$
$\theta_{10} = \frac{\beta_{2(x)}\bar{X}}{\beta_{2(x)}\bar{X}+\beta_{1(x)}}$	$f_1\bar{Y}\left(\theta_{10}^2 C_x^2 - \theta_{10} C_x C_y \rho\right)$	$V_{10} = \frac{\beta_2 \bar{X}}{\beta_2 \bar{X}+\beta_1}$	$f_j\bar{Y}\left(C_x - V_{10} C_x^2 C_y \rho + V_{10}^2 C_x^3 - 1\right)$
$\theta_{11} = \frac{\rho\bar{X}}{\rho\bar{X}+\beta_{2(x)}}$	$f_1\bar{Y}\left(\theta_{11}^2 C_x^2 - \theta_{11} C_x C_y \rho\right)$	$V_{11} = \frac{\rho \bar{X}}{\rho \bar{X}+\beta_2}$	$f_j\bar{Y}\left(C_x - V_{11} C_x^2 C_y \rho + V_{11}^2 C_x^3 - 1\right)$
$\theta_{12} = \frac{\beta_{2(x)}\bar{X}}{\beta_{2(x)}\bar{X}+\rho}$	$f_1\bar{Y}\left(\theta_{12}^2 C_x^2 - \theta_{12} C_x C_y \rho\right)$	$V_{12} = \frac{\beta_2 \bar{X}}{\beta_2 \bar{X}+\rho}$	$f_j\bar{Y}\left(C_x - V_{12} C_x^2 C_y \rho + V_{12}^2 C_x^3 - 1\right)$
$\theta_{13} = \frac{S_x\bar{X}}{S_x\bar{X}+\beta_{2(x)}}$	$f_1\bar{Y}\left(\theta_{13}^2 C_x^2 - \theta_{13} C_x C_y \rho\right)$	$V_{13} = \frac{S_x \bar{X}}{S_x \bar{X}+\beta_2}$	$f_j\bar{Y}\left(C_x - V_{13} C_x^2 C_y \rho + V_{13}^2 C_x^3 - 1\right)$
$\theta_{14} = \frac{\beta_{2(x)}\bar{X}}{\beta_{2(x)}\bar{X}+S_x}$	$f_1\bar{Y}\left(\theta_{14}^2 C_x^2 - \theta_{14} C_x C_y \rho\right)$	$V_{14} = \frac{\beta_2 \bar{X}}{\beta_2 \bar{X}+S_x}$	$f_j\bar{Y}\left(C_x - V_{14} C_x^2 C_y \rho + V_{14}^2 C_x^3 - 1\right)$
$\theta_{15} = \frac{S_x\bar{X}}{S_x\bar{X}+\beta_{1(x)}}$	$f_1\bar{Y}\left(\theta_{15}^2 C_x^2 - \theta_{15} C_x C_y \rho\right)$	$V_{15} = \frac{S_x \bar{X}}{S_x \bar{X}+\beta_1}$	$f_j\bar{Y}\left(C_x - V_{15} C_x^2 C_y \rho + V_{15}^2 C_x^3 - 1\right)$
$\theta_{16} = \frac{\beta_{1(x)}\bar{X}}{\beta_{1(x)}\bar{X}+S_x}$	$f_1\bar{Y}\left(\theta_{16}^2 C_x^2 - \theta_{16} C_x C_y \rho\right)$	$V_{16} = \frac{\beta_1 \bar{X}}{\beta_1 \bar{X}+S_x}$	$f_j\bar{Y}\left(C_x - V_{16} C_x^2 C_y \rho + V_{16}^2 C_x^3 - 1\right)$

Table 3: The constant and mean square errors of the Existing and Proposed ratio estimators $\left(f_j = \frac{1-f}{n}\right)$

Constants(θ_i)	Existing Mean Square Error MSE(.)	Constants V_i	Proposed Mean Square Error MSE(.)

$\theta_1=\dfrac{\beta_{2(x)}\bar{X}}{\beta_{2(x)}\bar{X}+C_x}$	$f_1\hat{\bar{Y}}^2\left(C_y^2+\theta_1^2C_x^2-2\theta_1C_xC_y\rho\right)$	$V_1=\dfrac{\beta_2\bar{X}}{\beta_2\bar{X}+C_x}$	$f_j\hat{\bar{Y}}^2\begin{pmatrix}1+C_x^2+C_x^2C_y^2+3V_1^2C_x^4-2V_1^2C_x^3\\-4V_1C_x^3C_y\rho-2C_x-2V_1C_x^2C_y\rho\end{pmatrix}$
$\theta_2=\dfrac{C_x\bar{X}}{C_x\bar{X}+\beta_{2(x)}}$	$f_1\hat{\bar{Y}}^2\left(C_y^2+\theta_2^2C_x^2-2\theta_2C_xC_y\rho\right)$	$V_2=\dfrac{C_x\bar{X}}{C_x\bar{X}+\beta_2}$	$f_j\hat{\bar{Y}}^2\begin{pmatrix}1+C_x^2+C_x^2C_y^2+3V_2^2C_x^4-2V_2^2C_x^3\\-4V_2C_x^3C_y\rho-2C_x-2V_2C_x^2C_y\rho\end{pmatrix}$
$\theta_3=\dfrac{\beta_{1(x)}\bar{X}}{\beta_{1(x)}\bar{X}+C_x}$	$f_1\hat{\bar{Y}}^2\left(C_y^2+\theta_3^2C_x^2-2\theta_3C_xC_y\rho\right)$	$V_3=\dfrac{\beta_1\bar{X}}{\beta_1\bar{X}+C_x}$	$f_j\hat{\bar{Y}}^2\begin{pmatrix}1+C_x^2+C_x^2C_y^2+3V_3^2C_x^4-2V_3^2C_x^3\\-4V_3C_x^3C_y\rho-2C_x-2V_3C_x^2C_y\rho\end{pmatrix}$
$\theta_4=\dfrac{C_x\bar{X}}{C_x\bar{X}+\beta_{1(x)}}$	$f_1\hat{\bar{Y}}^2\left(C_y^2+\theta_4^2C_x^2-2\theta_4C_xC_y\rho\right)$	$V_4=\dfrac{C_x\bar{X}}{C_x\bar{X}+\beta_1}$	$f_j\hat{\bar{Y}}^2\begin{pmatrix}1+C_x^2+C_x^2C_y^2+3V_4^2C_x^4-2V_4^2C_x^3\\-4V_4C_x^3C_y\rho-2C_x-2V_4C_x^2C_y\rho\end{pmatrix}$
$\theta_5=\dfrac{\rho\bar{X}}{\rho\bar{X}+C_x}$	$f_1\hat{\bar{Y}}^2\left(C_y^2+\theta_5^2C_x^2-2\theta_5C_xC_y\rho\right)$	$V_5=\dfrac{\rho\bar{X}}{\rho\bar{X}+C_x}$	$f_j\hat{\bar{Y}}^2\begin{pmatrix}1+C_x^2+C_x^2C_y^2+3V_5^2C_x^4-2V_5^2C_x^3\\-4V_5C_x^3C_y\rho-2C_x-2V_5C_x^2C_y\rho\end{pmatrix}$
$\theta_6=\dfrac{C_x\bar{X}}{C_x\bar{X}+\rho}$	$f_1\hat{\bar{Y}}^2\left(C_y^2+\theta_6^2C_x^2-2\theta_6C_xC_y\rho\right)$	$V_6=\dfrac{C_x\bar{X}}{C_x\bar{X}+\rho}$	$f_j\hat{\bar{Y}}^2\begin{pmatrix}1+C_x^2+C_x^2C_y^2+3V_6^2C_x^4-2V_6^2C_x^3\\-4V_6C_x^3C_y\rho-2C_x-2V_6C_x^2C_y\rho\end{pmatrix}$
$\theta_7=\dfrac{S_x\bar{X}}{S_x\bar{X}+C_x}$	$f_1\hat{\bar{Y}}^2\left(C_y^2+\theta_7^2C_x^2-2\theta_7C_xC_y\rho\right)$	$V_7=\dfrac{S_x\bar{X}}{S_x\bar{X}+C_x}$	$f_j\hat{\bar{Y}}^2\begin{pmatrix}1+C_x^2+C_x^2C_y^2+3V_7^2C_x^4-2V_7^2C_x^3\\-4V_7C_x^3C_y\rho-2C_x-2V_7C_x^2C_y\rho\end{pmatrix}$
$\theta_8=\dfrac{C_x\bar{X}}{C_x\bar{X}+S_x}$	$f_1\hat{\bar{Y}}^2\left(C_y^2+\theta_8^2C_x^2-2\theta_8C_xC_y\rho\right)$	$V_8=\dfrac{C_x\bar{X}}{C_x\bar{X}+S_x}$	$f_j\hat{\bar{Y}}^2\begin{pmatrix}1+C_x^2+C_x^2C_y^2+3V_8^2C_x^4-2V_8^2C_x^3\\-4V_8C_x^3C_y\rho-2C_x-2V_8C_x^2C_y\rho\end{pmatrix}$
$\theta_9=\dfrac{\beta_{1(x)}\bar{X}}{\beta_{1(x)}\bar{X}+\beta_{2(x)}}$	$f_1\hat{\bar{Y}}^2\left(C_y^2+\theta_9^2C_x^2-2\theta_9C_xC_y\rho\right)$	$V_9=\dfrac{\beta_1\bar{X}}{\beta_1\bar{X}+\beta_2}$	$f_j\hat{\bar{Y}}^2\begin{pmatrix}1+C_x^2+C_x^2C_y^2+3V_9^2C_x^4-2V_9^2C_x^3\\-4V_9C_x^3C_y\rho-2C_x-2V_9C_x^2C_y\rho\end{pmatrix}$
$\theta_{10}=\dfrac{\beta_{2(x)}\bar{X}}{\beta_{2(x)}\bar{X}+\beta_{1(x)}}$	$f_1\hat{\bar{Y}}^2\left(C_y^2+\theta_{10}^2C_x^2-2\theta_{10}C_xC_y\rho\right)$	$V_{10}=\dfrac{\beta_2\bar{X}}{\beta_2\bar{X}+\beta_1}$	$f_j\hat{\bar{Y}}^2\begin{pmatrix}1+C_x^2+C_x^2C_y^2+3V_{10}^2C_x^4-2V_{10}^2C_x^3\\-4V_{10}C_x^3C_y\rho-2C_x-2V_{10}C_x^2C_y\rho\end{pmatrix}$
$\theta_{11}=\dfrac{\rho\bar{X}}{\rho\bar{X}+\beta_{2(x)}}$	$f_1\hat{\bar{Y}}^2\left(C_y^2+\theta_{11}^2C_x^2-2\theta_{11}C_xC_y\rho\right)$	$V_{11}=\dfrac{\rho\bar{X}}{\rho\bar{X}+\beta_2}$	$f_j\hat{\bar{Y}}^2\begin{pmatrix}1+C_x^2+C_x^2C_y^2+3V_{11}^2C_x^4-2V_{11}^2C_x^3\\-4V_{11}C_x^3C_y\rho-2C_x-2V_{11}C_x^2C_y\rho\end{pmatrix}$
$\theta_{12}=\dfrac{\beta_{2(x)}\bar{X}}{\beta_{2(x)}\bar{X}+\rho}$	$f_1\hat{\bar{Y}}^2\left(C_y^2+\theta_{12}^2C_x^2-2\theta_{12}C_xC_y\rho\right)$	$V_{12}=\dfrac{\beta_2\bar{X}}{\beta_2\bar{X}+\rho}$	$f_j\hat{\bar{Y}}^2\begin{pmatrix}1+C_x^2+C_x^2C_y^2+3V_{12}^2C_x^4-2V_{12}^2C_x^3\\-4V_{12}C_x^3C_y\rho-2C_x-2V_{12}C_x^2C_y\rho\end{pmatrix}$
$\theta_{13}=\dfrac{S_x\bar{X}}{S_x\bar{X}+\beta_{2(x)}}$	$f_1\hat{\bar{Y}}^2\left(C_y^2+\theta_{13}^2C_x^2-2\theta_{13}C_xC_y\rho\right)$	$V_{13}=\dfrac{S_x\bar{X}}{S_x\bar{X}+\beta_2}$	$f_j\hat{\bar{Y}}^2\begin{pmatrix}1+C_x^2+C_x^2C_y^2+3V_{13}^2C_x^4-2V_{13}^2C_x^3\\-4V_{13}C_x^3C_y\rho-2C_x-2V_{13}C_x^2C_y\rho\end{pmatrix}$
$\theta_{14}=\dfrac{\beta_{2(x)}\bar{X}}{\beta_{2(x)}\bar{X}+S_x}$	$f_1\hat{\bar{Y}}^2\left(C_y^2+\theta_{14}^2C_x^2-2\theta_{14}C_xC_y\rho\right)$	$V_{14}=\dfrac{\beta_2\bar{X}}{\beta_2\bar{X}+S_x}$	$f_j\hat{\bar{Y}}^2\begin{pmatrix}1+C_x^2+C_x^2C_y^2+3V_{14}^2C_x^4-2V_{14}^2C_x^3\\-4V_{14}C_x^3C_y\rho-2C_x-2V_{14}C_x^2C_y\rho\end{pmatrix}$

$\theta_{15} = \dfrac{S_x\bar{X}}{S_x\bar{X}+\beta_{1(x)}}$	$f_1\hat{\bar{Y}}^2\left(C_y^2+\theta_{15}^2C_x^2-2\theta_{15}C_xC_y\rho\right)$	$V_{15} = \dfrac{S_x\bar{X}}{S_x\bar{X}+\beta_1}$	$f_j\hat{\bar{Y}}^2\begin{pmatrix}1+C_x^2+C_x^2C_y^2+3V_{15}^2C_x^4-2V_{15}^2C_x^3\\ -4V_{15}C_x^3C_y\rho-2C_x-2V_{15}C_x^2C_y\rho\end{pmatrix}$
$\theta_{16} = \dfrac{\beta_{1(x)}\bar{X}}{\beta_{1(x)}\bar{X}+S_x}$	$f_1\hat{\bar{Y}}^2\left(C_y^2+\theta_{16}^2C_x^2-2\theta_{16}C_xC_y\rho\right)$	$V_{16} = \dfrac{\beta_1\bar{X}}{\beta_1\bar{X}+S_x}$	$f_j\hat{\bar{Y}}^2\begin{pmatrix}1+C_x^2+C_x^2C_y^2+3V_{16}^2C_x^4-2V_{16}^2C_x^3\\ -4V_{16}C_x^3C_y\rho-2C_x-2V_{16}C_x^2C_y\rho\end{pmatrix}$

Table 4: The results of the biases and the mean square errors from the two populations

Estimators	POPULATION 1			POPULATION 2		
	Existing	Proposed	PRE	Existing	Proposed	PRE

	Bias	MSE	Bias	MSE		Bias	MSE	Bias	MSE	
$\hat{\hat{Y}}_{1}$	0.914	30.016	0.722	22.749	131.945	47.135	59264.90	8.651	16369.88	362.036
$\hat{\hat{Y}}_{2}$	0.569	17.528	0.394	13.269	132.097	40.816	52156.15	4.208	15604.22	334.244
$\hat{\hat{Y}}_{3}$	0.574	17.699	0.399	13.397	132.112	46.797	58886.22	8.414	16329.45	360.614
$\hat{\hat{Y}}_{4}$	0.910	29.844	0.718	22.617	131.954	43.648	55347.10	6.200	15949.71	347.010
$\hat{\hat{Y}}_{5}$	0.725	23.046	0.543	17.437	132.167	49.675	62113.26	10.437	16672.82	372.542
$\hat{\hat{Y}}_{6}$	0.765	24.484	0.581	18.529	132.139	49.149	61524.10	10.068	16610.33	370.397
$\hat{\hat{Y}}_{7}$	1.234	42.377	1.025	32.245	131.422	47.516	59692.84	8.919	16415.53	363.636
$\hat{\hat{Y}}_{8}$	0.200	6.249	0.045	4.995	125.105	13.582	20810.02	-14.940	11954.80	174.073
$\hat{\hat{Y}}_{9}$	0.402	12.023	0.236	9.164	131.198	44.470	56271.57	6.777	16049.24	350.618
$\hat{\hat{Y}}_{10}$	1.062	35.673	0.863	27.085	131.708	46.585	58648.34	8.265	16304.03	359.717
$\hat{\hat{Y}}_{11}$	0.549	16.846	0.376	12.757	132.053	57.713	71092.61	16.088	17615.55	403.579
$\hat{\hat{Y}}_{12}$	0.932	30.705	0.739	23.276	131.917	47.921	60146.61	9.204	16463.87	365.325
$\hat{\hat{Y}}_{13}$	1.110	37.538	0.908	28.518	131.629	47.399	59561.49	8.837	16401.52	363.146
$\hat{\hat{Y}}_{14}$	0.345	10.294	0.183	7.894	130.403	32.206	42399.55	1.845	14526.26	291.882
$\hat{\hat{Y}}_{15}$	1.332	46.281	1.118	35.258	131.264	47.468	59638.74	8.885	16409.76	363.435
$\hat{\hat{Y}}_{16}$	0.088	3.854	0.061	3.408	113.087	24.745	33855.13	7.091	13547.83	249.893