Existence and Uniqueness of Stationary Probability Vector for Stochastic Matrices using Farkas' Lemma

Abstract

The purpose of this note is to provide a simple proof of existence of stationary probability vectors (fixed points) for stochastic matrices using Farkas' lemma. This result as well as the uniqueness of stationary probability vectors also holds for a certain subclass of quasi-stochastic matrices.

1. Introduction: In this note we provide a new and simple proof of the existence of stationary probability vectors (fixed points) for stochastic matrices as defined for instance in section 6 of Chapter 5 of the book by Margalit and Rabinoff (2017), using Farkas' lemma-a very simple proof of the latter, using purely combinatorial arguments, is available in topic 3 of Lahiri (2022). The usual approaches to the proof rely either on Brouwer's fixed point theorem (for instance the first theorem in section 24.6 of https://python.quantecon.org/finite_markov.html), or the contraction mapping theorem- the latter being the proof in Lalley (undated). The proof that there exists a unique stationary probability vector for regular stochastic matrices is an easy consequence of our main result. Both results hold for a subclass of quasi-stochastic matrices (i.e., non-negative matrices with row sums less than or equal to one) which satisfy the property that either there is a square submatrix in the upper left-hand corner with all row sums equal to one or there is a square submatrix in the lower right-hand corner with all row sums equal to one.

We thus provide an alternative and considerably easier route to the proofs of theorems in section 7 of chapter 4 in Kemeny, Snell and Thompson (1974) concerning convergence results for stochastic matrices.

2. The model and the main existence result: An n-**dimensional probability row vector** is a row vector u in $\mathbb{R}^n_+$ such that $\sum_{j=1}^{n} u_j = 1$.

An n×n matrix P (of real numbers) is said to be **a non-negative matrix** if all entries in P are non-negative, i.e., if p_{ij} denotes the $(i,j)^{th}$ entry of P then for all $i,j \in \{1,\ldots,n\}$, $p_{ij} \geq 0$.

An n×n non-negative matrix P is said to be a **stochastic matrix** if all row sums of P are equal to one, i.e. for all $i \in \{1,\ldots,n\}$: $\sum_{j=1}^{n} p_{ij} = 1$.

An n-dimensional probability row vector is said to be a **stationary probability vector** for an n×n matrix P, if uP = u.

Given a n n×n matrix P and $i,j \in \{1,\ldots,n\}$, let P_i denote the i^{th} row of P and $P^{(j)}$ denote the j^{th} column of j.

The simple proof of the following theorem uses Farkas' lemma.

Theorem 1: Let P be a n×n stochastic matrix. Then there exists a stationary probability vector for P.

Proof: Let A = P-I, where I is the n×n identity matrix, and towards a contradiction suppose there does not exist any probability row vector u such that uA = 0.

Then the system u[A|e(n)] = [0|1] has no non-negative solution, where

(a) [A|e(n)] is the n×(n+1) matrix whose first n columns are the columns of the matrix A and its $(n+1)^{st}$ column e(n) is the n-dimensional column vector all whose entries are 1; and

(b) [0|1] is the (n+1)-dimensional row vector whose first 'n' entries are 0 and the last entry is 1.

Thus, by Farkas' Lemma there exists an n- dimensional column vector z and a real number α such that $Az + \alpha e(n) \geq 0$ and $\alpha < 0$.

Thus, Pz-z = Az >> 0, i.e., $P_i z > z_i$ for all i = 1,…,n, where P_i is the i^{th} row of P.

Let $z_h = \max\{z_i | i = 1,\ldots,n\}$.

Thus $P_h z > z_h$ which implies that a convex combination of the set of real numbers $\{z_i | i = 1,\ldots,n\}$ is greater than its maximum which is not possible.

Thus, it must be the case that there exists a probability row vector u such that uP = u. Q.E.D.

An n×n non-negative matrix P is said to be a **quasi- stochastic matrix** if all row sums of P are "positive" and "less" than or equal to one, i.e. for all $i \in \{1,\ldots,n\}$: $0 < \sum_{j=1}^{n} p_{ij} \leq 1$.

Note: Theorem 1 is not valid if we replace "stochastic matrix" by "quasi-stochastic matrix" as is apparent if we choose P = $\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$. That however does not imply that it is never true for quasi-stochastic matrices. If we let P = $\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}$, then the probability vector (0,1) is a stationary probability vector $\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}$. In fact, quasi-stochastic matrices of the form $\begin{bmatrix} P_1 & Q \\ R & P_2 \end{bmatrix}$, where P_1 and P_2 are square sub-matrices where either all row sums of P_1 are equal to one or all row sums of P_2 are equal to one, will have at least one stationary probability vector. This follows easily by adapting the proof of Theorem 1 to this context. If all row sums of P_1 are equal to one then Q has to be the zero matrix and if all row sums of P_2 are equal to one then R has to be the zero matrix.

3. Some properties of products of Quasi-Stochastic and Stochastic Matrices: Given any quasi-stochastic matrix P and an n-dimensional probability row vector u, [uP = u] implies [$u_i = 0$ whenever $\sum_{j=1}^{n} p_{ij} < 1$]. The reason is as follows:

If e(n) is the n-dimensional column vector all whose entries are 1, then uP = u implies uPe(n) = ue(n) = 1 and hence $\sum_{i=1}^{n} u_i \sum_{j=1}^{n} p_{ij} = 1$. Thus, $\sum_{i=1}^{n} u_i \left(\sum_{j=1}^{n} p_{ij} - 1\right) = 0$, which given $\sum_{j=1}^{n} p_{ij} \leq 1$ for all $i \in \{1,\ldots,n\}$ implies $u_i = 0$ if $\sum_{j=1}^{n} p_{ij} < 1$.

Note that if P and Q are quasi-stochastic matrices of size n×n then PQ is a non-negative matrix whose i^{th} row is P_iQ and j^{th} entry in row i of PQ is $P_iQ^{(j)}$.

Hence sum of terms in the i^{th} row of PQ is $P_i \sum_{j=1}^{n} Q^{(j)}$.

Since Q is a quasi-stochastic matrix each coordinate of $\sum_{j=1}^{n} Q^{(j)}$ is positive and less than or equal to one and since P quasi-stochastic matrix $P_i \sum_{j=1}^{n} Q^{(j)}$ is positive and less than or equal to one.

Thus, PQ is a quasi-stochastic matrix.

In particular P^r (i.e. P multiplied by itself r times for some positive integer r) is a quasi-stochastic matrix for all $r \in \mathbb{N}$. Clearly, $P^1 = P$.

Note the difference between P^r and $P^{(j)}$, where the latter denotes the j^{th} column of P.

If x is a non-negative row vectors and Q is a quasi-stochastic matrix, then $\sum_{j=1}^{n} xQ^{(j)} = \sum_{j=1}^{n} \sum_{i=1}^{n} x_i q_{ij} = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i q_{ij} = \sum_{i=1}^{n} x_i \sum_{j=1}^{n} q_{ij} \leq \sum_{i=1}^{n} x_i$.

If P and Q are stochastic matrices of size n×n then PQ is a non-negative matrix whose i^{th} row is P_iQ and j^{th} entry in row i of PQ is $P_iQ^{(j)}$.

Hence sum of terms in the i^{th} row of PQ is $P_i \sum_{j=1}^{n} Q^{(j)}$.

Since Q is a stochastic matrix implies $\sum_{j=1}^{n} Q^{(j)} = e(n)$ and since P stochastic matrix $P_i \sum_{j=1}^{n} Q^{(j)} = P_ie(n) = 1$.

Thus, PQ is a stochastic matrix.

In particular P^r (i.e. P multiplied by itself r times for some positive integer r) is a stochastic matrix for all $r \in \mathbb{N}$. Clearly, $P^1 = P$.

If x is a row vectors and Q is a stochastic matrix, then $\sum_{j=1}^{n} xQ^{(j)} = \sum_{j=1}^{n} \sum_{i=1}^{n} x_i q_{ij} = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i q_{ij} = \sum_{i=1}^{n} x_i \sum_{j=1}^{n} q_{ij} = \sum_{i=1}^{n} x_i$.

Note that if Q is a stochastic matrix, the above holds for all (and not just non-negative) row vectors which has as many co-ordinates as the number of rows in Q.

Thus, the sum of the coordinates of xQ^r, where Q^r is Q multiplied by itself r times, is equal to the sum of the coordinates of x for all positive integers r.

Given $x,y \in \mathbb{R}^n$, the **Manhattan distance** between x and y denoted $||x-y||_{MD}$ is defined as $\sum_{i=1}^{n} |x_i - y_i|$.

If x and y are n-dimensional non-negative row vectors and P is a quasi-stochastic matrix, then $\|xP-yP\|_{MD} = \sum_{j=1}^{n} |(x-y)P^{(j)}| = \sum_{j=1}^{n} |\sum_{i=}^{n}(x_i - y_i)p_{ij}| \leq \sum_{j=1}^{n}\sum_{i=1}^{n} |x_i - y_i|p_{ij} = \sum_{i=1}^{n}\sum_{j=1}^{n} |x_i - y_i|p_{ij} = \sum_{i=1}^{n} |x_i - y_i| \sum_{j=1}^{n} p_{ij} \leq \sum_{i=1}^{n} |x_i - y_i| = \|x-y\|_{MD}$ since $0 < \sum_{j=1}^{n} p_{ij} \leq 1$ for all $i \in N$.

Thus, for all $r \in \mathbb{N}$ and n-dimensional non-negative row vectors x, y, $\|x-y\|_{MD} \geq \|xP^r-yP^r\|_{MD} \geq \|xP^{r+1}-yP^{r+1}\|_{MD}$.

4. Regular Quasi-Stochastic Matrices: Results in this section lead to the second theorem in section 24.6 of https://python.quantecon.org/finite_markov.html.

A quasi-stochastic matrix P is said to be **regular** if for some positive integer r, all entries of P^r are positive.

For $i,j \in \{1,\ldots,n\}$ and any positive integer r, let $p_{ij}^{[r]}$ denote the j^{th} entry in the i^{th} row of P^r, P_i^r denote the i^{th} row of P^r and $P^{r(,j)}$ denote the j^{th} column of P^r.

The proof of the following- intended to be a slight generalization of Lalley (undated)- closely follows page 10 of Lalley's notes on "Markov Chains: Basic Theory" available at: http://galton.uchicago.edu/~lalley/Courses/312/MarkovChains.pdf .

Lemma 1: If P is a regular quasi-stochastic matrix with all entries of P^K strictly positive for some positive integer K, then there exists $\varepsilon > 0$ with $0 < 1-n\varepsilon < 1$ such that for $x,y \in \{u| \text{ u is an n-dimensional row vector satisfying } \sum_{j=1}^{n} u_j = 1\}$, then $\|xP^K-yP^K\|_{MD} \leq (1-n\varepsilon)\|x-y\|_{MD}$. Thus, for all positive integers m, $\|xP^{mK}-yP^{mK}\|_{MD} \leq (1-n\varepsilon)^m\|x-y\|_{MD}$.

Proof: Suppose P is a regular quasi-stochastic matrix with all entries of P^K strictly positive for some positive integer K and let $x,y \in \{u| \text{ u is an n-dimensional row vector satisfying } \sum_{j=1}^{n} u_j = 1\}$.

Since all entries of P^K are positive, there exists $\varepsilon > 0$, such that $p_{ij}^{[K]} > \varepsilon$ for all $i,j \in \{1,\ldots,n\}$.

Since P^K is also a quasi-stochastic matrix, it must be the case that $0 < \sum_{j=1}^{n} p_{ij}^{[K]} \leq 1$.

Thus, $1 \geq \sum_{j=1}^{n} p_{ij}^{[K]} > n\varepsilon$ for all $j \in \{1,\ldots,n\}$.

Thus, $1 > 1-n\varepsilon > 0$.

For $i,j \in \{1,\ldots,n\}$, let $q_{ij} = \frac{p_{ij}^{[K]}-\varepsilon}{1-n\varepsilon}$. Since $p_{ij}^{[K]} > \varepsilon$ for all $i,j \in \{1,\ldots,n\}$ and $1-n\varepsilon > 0$, it must be the case that $q_{ij} > 0$ for all $i,j \in \{1,\ldots,n\}$. Further, for all $i \in \{1,\ldots,n\}$, $0 < \sum_{j=1}^{n} q_{ij} = \sum_{j=1}^{n} \frac{p_{ij}^{[K]}-\varepsilon}{1-n\varepsilon} \leq 1$.

$\sum_{j=1}^{n} |xP^{K(j)} - yP^{K(j)}| = \sum_{j=1}^{n} |\sum_{i=1}^{n}(x_i - y_i)p_{ij}^{[K]}| = \sum_{j=1}^{n} |\sum_{i=1}^{n}(x_i - y_i)[q_{ij}(1-n\varepsilon) + \varepsilon]| = \sum_{j=1}^{n} |(1-n\varepsilon)\sum_{i=1}^{n}(x_i - y_i)q_{ij} + \varepsilon\sum_{i=1}^{n}(x_i - y_i)| = \sum_{j=1}^{n} |(1-n\varepsilon)\sum_{i=1}^{n}(x_i - y_i)q_{ij}|$, since $\varepsilon\sum_{i=1}^{n}(x_i - y_i) = 0$.

Thus, $\sum_{j=1}^{n}|xP^{K(j)} - yP^{K(j)}| = (1-n\varepsilon)\sum_{j=1}^{n}|\sum_{i=1}^{n}(x_i - y_i)q_{ij}| \le (1\text{-}n\varepsilon)\sum_{j=1}^{n}\sum_{i=1}^{n}|(x_i - y_i)q_{ij}| = (1\text{-}n\varepsilon)\sum_{j=1}^{n}\sum_{i=1}^{n}|(x_i - y_i)|q_{ij}$, since $q_{ij} > 0$ for all $i,j \in \{1,\ldots,n\}$.

Hence $\sum_{j=1}^{n}|xP^{K(j)} - yP^{K(j)}| \le (1\text{-}n\varepsilon)\sum_{i=1}^{n}\sum_{j=1}^{n}|(x_i - y_i)|q_{ij} = (1\text{-}n\varepsilon)\sum_{i=1}^{n}|(x_i - y_i)|\sum_{j=1}^{n}q_{ij} \le (1\text{-}n\varepsilon)\sum_{i=1}^{n}|(x_i - y_i)|$, since $0 < \sum_{j=1}^{n}q_{ij} \le 1$ for all $i \in \{1,\ldots,n\}$.

Thus, $||xP^K\text{-}yP^K||_{MD} \le (1\text{-}n\varepsilon)||x\text{-}y||_{MD}$ where $0 < 1\text{-}n\varepsilon < 1$.

Thus, for all positive integers m, $||xP^{(m+1)K}\text{-}yP^{(m+1)K}||_{MD} \le (1\text{-}n\varepsilon)||xP^{mK}\text{-}yP^{mK}||_{MD}$ and hence $||xP^{mK}\text{-}yP^{mK}||_{MD} \le (1\text{-}n\varepsilon)^m||x\text{-}y||_{MD}$, where $0 < 1\text{-}n\varepsilon < 1$. Q.E.D.

The following result is an immediate consequence of Theorem 1 and Lemma 1.

Theorem 2: If P is a regular stochastic matrix, then $\{u|$ u is an n-dimensional row vector satisfying $\sum_{j=1}^{n}u_j = 1$ and $uP = u\}$ is a singleton with all coordinates strictly positive.

Proof: Suppose P is a regular stochastic matrix with all entries of P^K being strictly positive for some positive integer K.

By Theorem 1, $\{u|$ u is an n-dimensional row vector satisfying $\sum_{j=1}^{n}u_j = 1$ and $uP = u\}$ is non-empty.

Further $uP = u$ implies $uP^r = u$ for all positive integers r.

Towards a contradiction suppose, u and v are any two distinct (i.e., $u \neq v$) n-dimensional row vectors satisfying $uP = u$, $vP = v$, $\sum_{j=1}^{n}u_j = \sum_{j=1}^{n}v_j = 1$.

Thus $uP^K = u$ and $vP^K = v$.

By lemma 1, there exists $\varepsilon > 0$ with $0 < 1\text{-}n\varepsilon < 1$ such that $||uP^K\text{-}vP^K||_{MD} \le (1\text{-}n\varepsilon)||u\text{-}v||_{MD}$.

Since $1 > 1\text{-}n\varepsilon > 0$, $u \neq v$ implies $||uP^K\text{-}vP^K||_{MD} < ||u\text{-}v||_{MD}$, contradicting $uP^K = u$ and $vP^K = v$.

Thus, $\{u|$ u is an n-dimensional row vector satisfying $\sum_{j=1}^{n}u_j = 1$ and $uP = u\}$ must be a singleton and by Theorem 1 it follows that there is a unique non-zero n-dimensional row vector u satisfying $uP = u$ and this u is an n-dimensional probability row vector.

Since all entries of P^K are strictly positive, $u_j = uP^{K(j)} > 0$ for all $j \in \{1,\ldots,n\}$ and hence u is a strictly positive probability row vector. Q.E.D.

Note: It is easy to see that the above result holds for quasi-stochastic matrices of the form $\begin{bmatrix} P_1 & Q \\ R & P_2 \end{bmatrix}$, where P_1 and P_2 are square sub-matrices where <u>either</u> all row sums of P_1 are equal to one *and* P_1 is regular <u>or</u> all row sums of P_2 are equal to one *and* P_2 is regular.

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