Original Research Article

~~ON~~ MIXTURE MODEL ON DEVELOPMENT OF BIVARIATEPRODUCT DISTRIBUTION ~~DEVELOPMENT~~ AND ITS PROPERTIES

ABSTRACT

In the study, some bivariate distributions were developed from mixture model offspring,using the Independent (Product) distribution approach. These developments are categorized under the IID and II_nD: where the Bivariate Exponential distribution, Bivariate Lindleydistribution and Bivariate Juchez distribution are constructed asIIDs; and Bivariate Exponential-Lindleydistribution, Bivariate Exponential-Juchezdistribution and Bivariate Lindley-Juchez distribution as (II_nDs). The properties of these distributions which involve: the shape of the bivariate PDFs, moments, moment generating function, mean, covariance, coefficient of variance and coefficient of correlation, maximum likelihood estimator, reliability analysis, renewal property, inverse cumulative distribution and probability patterns, are studied randomly across the distributions. Finally, under renewal properties, functions are derived which can model two-dimensional queuing and renewal processes, for events where the arrival and service times are dependent.

Keywords: mixture model, bivariate derivations, product distribution, renewal property, 3D PDF shapes.

1. INTRODUCTION

Distribution development~~is one such that~~ constantly evolves as events unfold. Thesedevelopments are theoretically categorized into three kinds namely: univariate, bivariate and multivariate representations. The exhaustive nature of univariate development is a huge credit to researchers who have developed very many distributions within the past three decades.Specifically, mixture model which recently received an avalanche usehas played a huge part in the recent univariate innovations. Examples are Exponential and Lindley distribution as developed in the works of Epstein (1954) and Lindley (1958). Recently,Juchez distribution was developed byEchebiri and Mbegbu (2022); other univariate distribution types are Frechet, Weibull, Pareto, Sujatha, Odoma, Shanker, Pranav, Aradhana, Amarendra, Devya and Shambhu distributions. However, univariate distributions by default do not capture sufficiently all real life phenomena, that is, cases where it is inevitable to model bivariate outcomes; hence thise necessitated development of bivariate distributions.

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Some of the well-known, classic bivariate distributions are bivariate normal, bivariate-t, bivariate log-normal, bivariate gamma, bivariate extreme value, bivariate Birnbaum-Saunders distributions, bivariate skew normal distribution, bivariate geometric skew normal distribution etc., with the usual purpose of modeling the two marginal and finding the association betweenthem. Now, the development, study and applications of bivariate distributions are one of the vitalareas of research in statistics field. Hutchinson & Lai (1990) and recently: Joe (1997), Arnold (1999), Kotz&Nadarajah (2000), Kotz&Nadarajah (2004), Nelsen (2006) among many, have published several papers and books on bivariate theory.Extensive reviews have been done by Lai (2004), Lai (2006)~~)~~ and Sarabia& Gomez (2008) in the bivariate construction methods; where the scope covers both for discrete and continuous bivariate distributions. Product distribution, Marginal transformation, Copula Method, Method of Mixing and Compounding, Trivariate Reduction Method, Frailty Approach and Conditional Specification Method make up the different handy bivariate construction methods as seen in their reviews.

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Product distribution represents a multivariate statistical distribution whose j^{th} marginal distributions are independent component probability density functions.; The component distributions may be continuous or discrete, univariate or multivariate, and is given as:

$$g(x_1, \ldots, x_n) = \prod_{i=1}^{n} f_i(x_i) = f_1(x_1) f_2(x_2) \ldots f_n(x_n) \qquad (1)$$

In as much as the study of product distributions backdated to 1940s, the first thorough treatment of the topic was a paper by Springer and Thompson (1966). This is followed by an improvement in both theoretical and algorithmic study, and the extensive use of Monte Carlo theory and other numerical methods.Mixing and Compounding Method as another way for bivariate construction, is majorly used to particularly combine two different bivariate distributions. If F_1 and $F_2$are two bivariate distribution functions, then the new type derivation is given by

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$$F(x_1, x_2) = \theta F_1(x_1, x_2) + (1-\theta) F_2(x_1, x_2), \quad 0 \le \theta \le 1 \qquad (2)$$

This was employed byFrechet(1960)Mardia (1970) in the derivation of bivariate families of distributions.Copula distribution is also another method for developing multivariate distributions. Unlike the product counterpart mentioned above, it has different developmental approaches that describe the dependence between variables. The various copula predefined kernels are used for the parameterization, andalso they allow for the investigation of the different degrees of these dependence.Some examples are Archimedean kernel, Clayton kernels,Gaussian kernel, Plackettkernel, Independent or Product kernel and Frank kernel.Gumbel (1960 and 1961)used Farlie-Gumbel-Morgenstern (FGM) copula for the development of Gumbel bivariate exponential distribution and bivariate logistic distributions and it is given as

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$$F(x_1 x_2) = x_1 x_2 (1 + \theta[1 - x_1][1 - x_2]) \qquad (3)$$

Another approach to copula method is Sklar's theorem which adopts product kernelthat combines the marginal cumulative distribution functions (CDF) unto n-dimensional copula C such that for all real $x_1, x_2, x_3, \ldots, x_n$:

$$F(x_1, x_2, x_3, \ldots, x_n) = C[F(x_1), F(x_2), \ldots, F(x_n)] \qquad (4)$$

While there is strong commendation over the different bivariate distribution development and methodologies, it is worthy of note that mixture-model-derived distributions which in recent years have made significant contributions in univariate data modeling, have not been exhaustively explored "bivariately"; hence the development of this paper. This paper aims at development of bivariate distributions using independent or product distribution method; where the baseline distributions are strictly mixture model derived distributions. Some relevant properties will be derived as well.

For the arrangement structure of the paper, the antecedent abstract and introduction is preceded by the mixture model bivariate development: the baseline distribution review, methodical derivations using the Product Distribution for the Independence approach. Finally the derivations of some relevant properties follow suit, as occasion serves.

2. MIXTURE MODEL BIVARIATE DEVELOPMENT

2.1 Baseline Distributions

Lindley and Juchez distributions are derived from the composition of exponential and gamma distributions with suitable mixing probabilities; where the gamma distribution is characterized by a constant scale parameter θ: and shape parameter $\alpha = 2$ for Lndley distribution; and two different shape parameters: $\alpha = 2$ and 4 for Juchez distribution. The PDFs of Lindley and Juchez distributions are derived from the mixture model, Lindsay (1995).

$$f(x) = \sum_{i=1}^{k} d_i g_i \; ; \; \text{where} \sum_{i=1}^{k} d_i = 1, \; d_i > 0 \qquad (5)$$

with their corresponding CDFs, and given as:

$$lind(x) = \frac{\theta^2}{\theta+1}(1 + x)e^{-\theta x}, \; x > 0, \; \theta > 0 \qquad (6)$$

$$Lind(x) = 1 - \left(\frac{\theta + 1 + \theta x}{\theta + 1}\right)e^{-\theta x}$$

$$juc(x) = \frac{\theta^4}{\theta^3+\theta^2+6}(1 + x + x^3)e^{-\theta x}, \; x > 0, \; \theta > 0 \qquad (7)$$

$$Juc(x) = 1 - \left(1 + \frac{\theta x\,[\theta^2 + \theta^2 x^2 + 3\theta x + 6]}{\theta^3 + \theta^2 + 6}\right)e^{-\theta x}$$

In addition, we include the Exponential distribution, which is not a mixture model offspring; however, it is a vital baseline distribution in most of those mixture developments.

$$exp(x) = \theta e^{-\theta x}, \; x > 0, \theta > 0 \qquad (8)$$

$$Exp(x) = 1 - e^{-\theta x}$$

2.2 Product Distribution (Independent Approach)

All the bivariate constructions here are categorized under these two different phenomena:

- Independent and Identical distributions (IID) and
- Independent and Non - Identical distributions (II_nD)

Independent and Identical Distributions (IID)
These simply represent bivariate combinations of the same distribution. So, under this category, Bivariate Exponential Distribution, Bivariate Lindley Distribution and Bivariate Juchez Distribution are derived together with their individual multivariate extensions.

Let $X_1, X_2, X_3, \dots, X_n$ be multivariate random variables, with marginal distributions $f_1(x_1), f_2(x_2), \dots, f_n(x_n)$, then the product distribution is given by:

$$g(x_1, x_2, \dots, x_n) = \prod_i^n f_i(x_i) = f_1(x_1) f_2(x_2) \dots f_n(x_n) \qquad (9)$$

Now, if $X_1 \sim Exp(x_1; \theta_1)$ and $X_2 \sim Exp(x_2; \theta_2)$, then the bivariate pdf derivation is

$$g(x_1 x_2) = \left[\theta_1 e^{-(\theta_1 x_1)}\right] \times \left[\theta_2 e^{-(\theta_2 x_2)}\right] = \theta_1 \theta_2 e^{-(\theta_1 x_1 + \theta_2 x_2)} \qquad (10)$$

and the multivariate extension is given as :

$$g(x_1, x_2, x_3 \dots x_n) = \left(\prod_{i=1}^{n} \theta_i\right)\left[e^{-[\sum_{i=1}^{n} \theta_i x_i]}\right] \qquad (11)$$

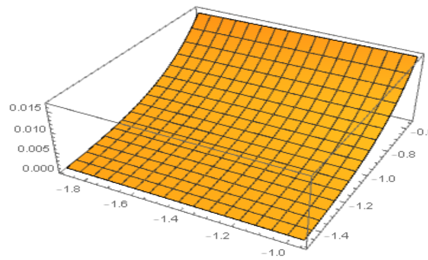


Figure 1. The shape of the PDF of the Bivariate Exponential Distribution

If $X_1 \sim Lind(x_1;\theta_1)\quad and \quad X_2 \sim Lind(x_2;\theta_2)$, then the bivariate Lindley derivation is given as:

$$g(x_1 x_2) = \left[\frac{\theta_1^2\theta_2^2}{(\theta_1+1)(\theta_2+1)}\right][(1+x_1)(1+x_2)]e^{-(\theta_1 x_1+\theta_2 x_2)} \quad (12)$$

and the multivariate extension is given as :

$$g(x_1,x_2,x_3\ldots x_n) = \frac{\prod_{i=1}^{n}\theta_i^2}{\prod_{i=1}^{n}\gamma_i}\{\prod_{i=1}^{n}\alpha_i\}\left[e^{-[\sum_{i=1}^{n}\theta_i x_i]}\right] \quad (13)$$

where

$$\begin{Bmatrix} \gamma_1 = (\theta_1+1) \\ \gamma_2 = (\theta_2+1) \\ \gamma_3 = (\theta_3+1) \\ \quad \cdots \quad \cdots \\ \vdots \\ \quad \cdots \quad \cdots \\ \gamma_n = (\theta_n+1) \end{Bmatrix} \; and \; \begin{Bmatrix} \alpha_1 = 1+x_1 \\ \alpha_2 = 1+x_2 \\ \alpha_3 = 1+x_3 \\ \quad \cdots \quad \cdots \\ \vdots \\ \quad \cdots \quad \cdots \\ \alpha_n = 1+x_n \end{Bmatrix}$$

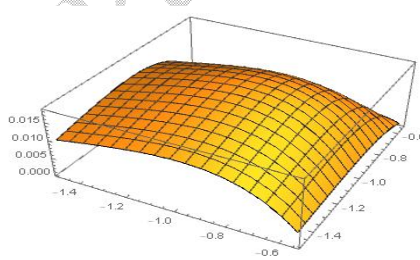


Figure 2. The shape of the PDF of the Bivariate Lindley Distribution

More so, if $X_1 \sim Juc(x_1;\theta_1)\quad and \quad X_2 \sim Juc(x_2;\theta_2)$, then the bivariate Lindley derivation is given as:

$$g(x_1 x_2) = \left[\frac{\theta_1^4\theta_2^4}{(\theta_1^3+\theta_1^2+6)(\theta_2^3+\theta_2^2+6)}\right][(1+x_1^2+x_1^3)(1+x_2^2+x_2^3)]e^{-(\theta_1 x_1+\theta_2 x_2)} \quad (14)$$

The multivariate extension is given as:

$$g(x_1,x_2,x_3\ldots x_n) = \frac{\prod_{i=1}^{n}\theta_i^4}{\prod_{i=1}^{n}a_i}\{\prod_{i=1}^{n}\varphi_i\}\left[e^{-[\sum_{i=1}^{n}\theta_i x_i]}\right] \quad (15)$$

where

$$\begin{Bmatrix} a_1 = \theta_1^3 + \theta_1^2 + 6 \\ a_2 = \theta_2^3 + \theta_2^2 + 6 \\ a_3 = \theta_3^3 + \theta_3^2 + 6 \\ \quad \dots \quad \dots \quad \dots \\ \vdots \\ \quad \dots \quad \dots \quad \dots \\ a_n = \theta_n^3 + \theta_n^2 + 6 \end{Bmatrix} \quad and \quad \begin{Bmatrix} \varphi_1 = 1 + x_1^2 + x_1^3 \\ \varphi_2 = 1 + x_2^2 + x_2^3 \\ \varphi_3 = 1 + x_3^2 + x_3^3 \\ \quad \dots \quad \dots \quad \dots \\ \vdots \\ \quad \dots \quad \dots \quad \dots \\ \varphi_n = 1 + x_n^2 + x_n^3 \end{Bmatrix}$$

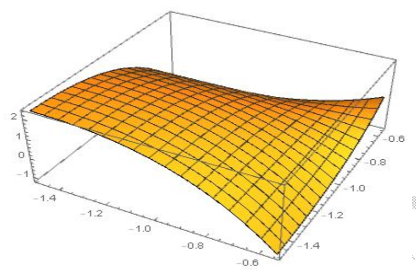


Figure 3. The shape of the PDF of the Bivariate Juchez Distribution

The bivariate cumulative distribution function (CDF) for the IID is derived thus

$G(x_1x_2) = \int_{-\infty}^{x_2}\int_{-\infty}^{x_1} f(s,t)\,dt\,ds$(16)

The CDF of the bivariate exponential distribution is given as

$G(x_1x_2) = \int_0^{x_2}\int_0^{x_1} \theta_1\theta_2\, e^{-(\theta_1 s+\theta_2 t)}\,dt\,ds$(17)

#mathematica software

Input: $Integrate[(a*b)*(Exp[-((a*x)+(b*y))]),x,y]$

Output: $e^{-(ax+by)}\big]_{\substack{a=\theta_1\\ b=\theta_2\\ x=x_1\\ y=\theta_2}} \rightarrow \quad G(x_1x_2) = e^{-(\theta_1x_1+\theta_2x_2)}$(18)

The CDF of the bivariate Lindley distribution is given as

$G(x_1x_2) = \int_{-\infty}^{x_2}\int_{-\infty}^{x_1}\left[\frac{\theta_1^2\theta_2^2}{(\theta_1+1)(\theta_2+1)}\right][(1+s)(1+t)]e^{-(\theta_1 s+\theta_2 t)}\,dt\,ds$(19)

*Input: Integrate[((a^2*b^2)/((a+1)*(b+1)))*((1+x)*(1+y))*(Exp[-((a*x)+(b*y))]),x,y]*

Output: $\frac{e^{-(ax+by)}(1+a+ax)(1+b+by)}{(1+a)(1+b)}\Big]_{\substack{a=\theta_1\\ b=\theta_2\\ x=x_1\\ y=\theta_2}}$

$\rightarrow \quad G(x_1x_2) = \left[1 + \frac{\theta_1x_1+\theta_2x_2+\theta_1\theta_2x_1+\theta_1\theta_2x_2+\theta_1\theta_2x_1x_2}{1+\theta_1+\theta_2+\theta_1\theta_2}\right]e^{-(\theta_1x_1+\theta_2x_2)}$(20)

The CDF of the bivariate Juchez distribution is given as

$$F(x_1x_2) = \int_{-\infty}^{x_2}\int_{-\infty}^{x_1}\left[\frac{\theta_1^4\theta_2^4}{(\theta_1^3+\theta_1^2+6)(\theta_2^3+\theta_2^2+6)}\right][(1+s^2+s^3)(1+t^2+t^3)]e^{-(\theta_1 s+\theta_2 t)}\,dt\,ds \quad (21)$$

Input: $\{Integrate[(((a\text{^}4*b\text{^}4)/((a\text{^}3+a\text{^}2+6)*(b\text{^}3+b\text{^}2+6)))*(1+x+y+(x*y)+x\text{^}3+y\text{^}3+(x\text{^}3*y)+(x*y\text{^}3)+(x\text{^}3*y\text{^}3))*(Exp[-((a*x)+(b*y))])),x,y]\}$

Output: $$\left[\frac{e^{-ax-by}(6+6ax+a^2(1+3x^2)+a^3(1+x+x^3))(6+6by+b^2(1+3y^2)+b^3(1+y+y^3))}{(6+a^2+a^3)(6+b^2+b^3)}\right]\Bigg|_{\substack{a=\theta_1\\ b=\theta_2\\ x=x_1\\ y=x_2}}$$

$$\rightarrow G(x_1x_2) = \left[\begin{array}{c} 1+\frac{\theta_1^3x_1+3\theta_1^2x_1^2+\theta_1^3x_1^3}{6+\theta_1^2+\theta_1^3}+\frac{\theta_2^3x_2+3\theta_2^2x_2^2+\theta_2^3x_2^3}{6+\theta_2^2+\theta_2^3} \\ +\frac{(\theta_1^3x_1+3\theta_1^2x_1^2+\theta_1^3x_1^3)(\theta_2^3x_2+3\theta_2^2x_2^2+\theta_2^3x_2^3)}{(6+\theta_1^2+\theta_1^3)(6+\theta_2^2+\theta_2^3)}\end{array}\right]e^{-(\theta_1x_1+\theta_2x_2)} \quad (22)$$

To verify the validity of the IIDs, it suffices to state that:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x_1x_2)dx_1dx_2 = 1 \quad (23)$$

$$X_1X_2 \sim Exp(\theta)$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x_1x_2)dx_1dx_2 = \int_0^{\infty}\int_0^{\infty}\theta_1\theta_2e^{-(\theta_1x_1+\theta_2x_2)}dx_1dx_2 \quad (24)$$

Input: $Integrate[(a*b)*(Exp[-((a*x)+(b*y))]),\{x,0,Infinity\},\{y,0,Infinity\}]$
Output: $ConditionalExpression[1,Re[a,b]>0]$; *where* $a=\theta_1, b=\theta_2,\ x=x_1,\ y=x_2$

$$\rightarrow \qquad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\theta_1\theta_2e^{-(\theta_1x_1+\theta_2x_2)}dx_1dx_2 = 1 \quad (25)$$

$$X_1X_2 \sim Lind(\theta)$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x_1x_2)dx_1dx_2 = \int_0^{\infty}\int_0^{\infty}\left[\frac{\theta_1^2\theta_2^2}{(\theta_1+1)(\theta_2+1)}\right][(1+x_1)(1+x_2)]e^{-(\theta_1x_1+\theta_2x_2)}dx_1dx_2 \quad (26)$$

Input: $Integrate[((a\text{^}2*b\text{^}2)/((a+1)*(b+1)))*((1+x)*(1+y))*(Exp[-((a*x)+(b*y))]),\{x,0,Infinity\},\{y,0,Infinity\}]$
Output: $ConditionalExpression[1,Re[a,b]>0]$; *where* $a=\theta_1, b=\theta_2,\ x=x_1,\ y=x_2$

$$\rightarrow \quad \int_0^{\infty}\int_0^{\infty}\left[\frac{\theta_1^2\theta_2^2}{(\theta_1+1)(\theta_2+1)}\right][(1+x_1)(1+x_2)]e^{-(\theta_1x_1+\theta_2x_2)}dx_1dx_2 = 1 \quad (27)$$

$$X_1X_2 \sim Juc(\theta)$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x_1x_2)dx_1dx_2 = \int_0^{\infty}\int_0^{\infty}\left[\frac{\theta_1^4\theta_2^4}{(\theta_1^3+\theta_1^2+6)(\theta_2^3+\theta_2^2+6)}\right][(1+x_1^2+x_1^3)(1+x_2^2+x_2^3)] * e^{-(\theta_1x_1+\theta_2x_2)}dx_1dx_2 \quad (28)$$

XInput: $Integrate[((a\text{^}4*b\text{^}4)/((a\text{^}3+a\text{^}2+6)*(b\text{^}3+b\text{^}2+6)))*(1+x+y+(x*y)+x\text{^}3+y^3+(x\text{^}3*y)+(x*y\text{^}3)+(x\text{^}3*y\text{^}3))*(Exp[-((a*x)+(b*y))]),\{x,0,Infinity\},\{y,0,Infinity\}]$
0utput: ConditionalExpression[1, Re[a,b] > 0]; *where* $a=\theta_1, b=\theta_2,\ x=x_1,\ y=x_2$

$$\rightarrow \qquad \int_0^{\infty}\int_0^{\infty}\left[\frac{\theta_1^4\theta_2^4}{(\theta_1^3+\theta_1^2+6)(\theta_2^3+\theta_2^2+6)}\right][(1+x_1^2+x_1^3)(1+x_2^2+x_2^3)]e^{-(\theta_1x_1+\theta_2x_2)} = 1 \quad (29)$$

Independent and Non-Identical distributions (II_nD)
Here, we project bivariate combinations of different distributions following same product model inequation (9); say, Bivariate Exponential-Lindley Distribution, Bivariate Exponential-Juchez

Distribution, Bivariate Lindley-Juchez Distribution and Multivariate Exponential-Lindley-Juchez Distribution.

Let$X_1 \sim Exp(x_1;\theta_1)$ $\ and \ $ $X_2 \sim Lind(x_2;\theta_2)$, then the bivariate Exponential-Lindley derivation is given thus

$g(x_1x_2) = \left[\frac{\theta_1\theta_2^2}{(\theta_2+1)}\right](1 + x_2)\ e^{-(\theta_1x_1+\theta_2x_2)}$(30)

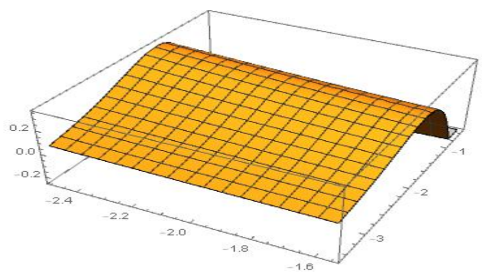


Figure 4. The shape of the PDF of the Bivariate Exponential-Lindley Distribution

Let$X_1 \sim Exp(x_1;\theta_1)$ $\ and \ $ $X_2 \sim Juc(x_2;\theta_2)$, then the bivariate Exponential-Juchezderivation is given as

$g(x_1x_2) = \left[\frac{\theta_1\theta_2^4}{(\theta_2^3+\theta_2^2+6)}\right](1 + x_2^2 + x_2^3)\ e^{-(\theta_1x_1+\theta_2x_2)}$(31)

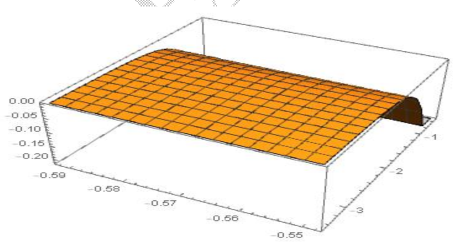


Figure 5. The shape of the PDF of the Bivariate Exponential-Juchez Distribution

Let $X_1 \sim Lind(x_1;\theta_1)$ $\ and \ $ $X_2 \sim Juc(x_2;\theta_2)$, then the bivariate Lindley-Juchezderivation is

$g(x_1x_2) = \left[\frac{\theta_1^2\theta_2^4}{(1+\theta_1)(\theta_2^3+\theta_2^2+6)}\right](1 + x_1)(1 + x_2^2 + x_2^3)\ e^{-(\theta_1x_1+\theta_2x_2)}$(32)

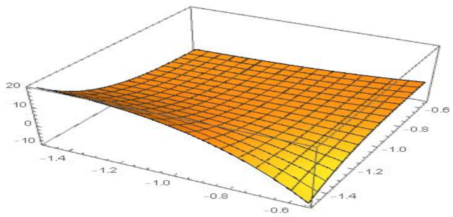


Figure 6. The shape of the PDF of the Bivariate Lindley-Juchez Distribution

Let$X_1 \sim Exp(x_1;\theta_1)$, $X_2 \sim Lind(x_2;\theta_2)$ and $X_3 \sim Juc(x_3;\theta_3)$, then the multivariateExponential-Lindley-Juchez derivation is

$$f(x_1x_2x_3) = \left[\frac{\theta_1\theta_2^2\theta_3^4}{(1+\theta_2)(\theta_3^3+\theta_3^2+6)}\right](1+x_2)(1+x_3^2+x_3^3)\, e^{-(\theta_1+\theta_2x_2+\theta_3x_3)} (33)$$

From equation (16) the bivariate cumulative distribution function (CDF) for the II_nD is derived thus:

The CDF of the bivariate Exponential-Lindley distribution is given as

$$G(x_1x_2) = \int_{-\infty}^{x_2}\int_{-\infty}^{x_1}\left[\frac{\theta_1\theta_2^2}{(\theta_2+1)}\right](1+t)\, e^{-(\theta_1 s+\theta_2 t)}\, dt\, ds (34)$$

Input:$Integrate[((a*b^\wedge 2)/(b+1))*(1+y)*(Exp[-((a*x)+(b*y))]),x,y]$

Output$\left.\frac{e^{-ax-by}(1+b+by)}{1+b}\right|_{\substack{a=\theta_1\\ b=\theta_2\\ x=x_1\\ y=x_2}}$

$$\rightarrow \qquad F(x_1x_2) = \left[1+\frac{\theta_2x_2}{1+\theta_2}\right]e^{-(\theta_1x_1+\theta_2x_2)} (35)$$

The CDF of the bivariate Exponential-Juchez distribution is given as

$$F(x_1x_2) = \int_0^{x_2}\int_0^{x_1}\left[\frac{\theta_1\theta_2^4}{(\theta_2^3+\theta_2^2+6)}\right](1+t^2+t^3)\, e^{-(\theta_1 s+\theta_2 t)}\, ds\, dt \qquad (36)$$

Input: $Integrate[((a*b^\wedge 4)/(b^\wedge 3+b^\wedge 2+6))*(1+y+y^\wedge 3)*(Exp[-((a*x)+(b*y))]),x,y]$

Output: $\left.\frac{e^{-ax-by}(6+6ax+a^2(1+3x^2)+a^3(1+x+x^3))}{6+a^2+a^3}\right|_{\substack{a=\theta_1\\ b=\theta_2\\ x=x_1\\ y=x_2}}$

$$\rightarrow \qquad F(x_1x_2) = \left[1+\frac{6\theta_2x_2+\theta_2^3x_2+3\theta_2^2x_2^2+\theta_2^3x_2^3}{6+\theta_2^2+\theta_2^3}\right]e^{-(\theta_1x_1+\theta_2x_2)} \qquad (37)$$

The CDF of the bivariate Lindley-Juchez distribution is given as

$$F(x_1x_2) = \int_0^{x_2}\int_0^{x_1}\left[\frac{\theta_1^2\theta_2^4}{(1+\theta_1)(\theta_2^3+\theta_2^2+6)}\right](1+s)(1+t^2+t^3)\; e^{-(\theta_1 s+\theta_2 t)}\, ds\, dt \qquad (38)$$

*Input: Integrate[((a^4*b^2)/((a^3+a^2+6)*(b+1)))*((1+x+x^3)*(1+y))*(Exp[-((a*x)+(b*y))]),x,y]*

Output: $\frac{e^{-ax-by}(6+6ax+a^2(1+3x^2)+a^3(1+x+x^3))(1+b+by)}{(6+a^2+a^3)(1+b)}\Bigg|\begin{bmatrix} a=\theta_2 \\ b=\theta_1 \\ x=x_2 \\ y=x_1 \end{bmatrix}$

$$\rightarrow\ F(x_1x_2) = \left[1 + \frac{\begin{matrix}\theta_2^3+\theta_2^2+6+12\theta_2x_2+2\theta_2^3x_2+6\theta_2^2x_2^2+2\theta_2^3x_2^3+6\theta_2\theta_1x_2+3\theta_2^2\theta_1x_2^2 \\ \theta_2^3\theta_1x_2+\theta_2^3\theta_1x_2^3+6\theta_1x_1+6\theta_2\theta_1x_2x_1+\theta_2^2\theta_1x_1+3\theta_2^2\theta_1x_2^2x_1 \\ +\theta_2^3\theta_1x_1+\theta_2^3\theta_1x_2x_1+\theta_2^3\theta_1x_2^3x_1\end{matrix}}{6+\theta_2^2+\theta_2^3+6\theta_1+\theta_2^2\theta_1+\theta_2^3\theta_1}\right] e^{-(\theta_1x_1+\theta_2x_2)} \qquad (39)$$

The CDF of the bivariate Exponential-Lindley-Juchez distribution is given as

$$F(x_1x_2x_3) = \int_0^{x_3}\int_0^{x_2}\int_0^{x_1}\left[\frac{\theta_1\theta_2^2\theta_3^4}{(1+\theta_2)(\theta_3^3+\theta_3^2+6)}\right](1+s)(1+t+t^3)\ e^{-(\theta_1r+\theta_2s+\theta_3t)}\ drdsdt \qquad (40)$$

Input: $Integrate[((a*b^2*c^4)/((c^3+c^2+6)*(b+1)))*((1+z+z\hat{}3)*(1+y))*(Exp[-((a*x)\ +(b*y)+(c*z))]),x,y,z]$

Output: $-\frac{e^{-ax-by-cz}(1+b+by)(6+6cz+c^2(1+3z^2)+c^3(1+z+z^3))}{(1+b)(6+c^2+c^3)}\Bigg|\begin{bmatrix} a=\theta_1 \\ b=\theta_2 \\ c=\theta_3 \\ x=x_1 \\ y=x_2 \\ z=x_3 \end{bmatrix}$

$$\rightarrow\quad F(x_1x_2x_3) = -\left[1 + \frac{\begin{matrix}6\theta_3x_3+3x_3^2+\theta_3^3x_3+\theta_3^3x_3^3+6\theta_2\theta_3x_3+3\theta_2x_3^3+ \\ \theta_2\theta_3^3x_3+\theta_2\theta_3^3x_3^3+6\theta_2x_2+6\theta_2\theta_3x_2x_3+b\theta_3^2x_2+ \\ 3\theta_2x_2x_3^3+\theta_2\theta_3^3x_2+\theta_2\theta_3^3x_2x_3+\theta_2\theta_3^3x_2x_3^3\end{matrix}}{6+6\theta_2+\theta_3^2+\theta_2\theta_3^2+\theta_3^3+\theta_2\theta_3^3}\right] e^{-\theta_1x_1-\theta_2x_2-\theta_3x_3} \qquad (41)$$

To verify the validity of the II_nDs, we recall equation (23):

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x_1x_2)dx_1dx_2 = 1$$

$$X_1X_2 \sim Exp - Lind(\theta)$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x_1x_2)dx_1dx_2 = \int_0^{\infty}\int_0^{\infty}\left[\frac{\theta_1\theta_2^2}{(\theta_2+1)}\right](1+x_2)\ e^{-(\theta_1x_1+\theta_2x_2)}\ dx_1dx_2 \qquad (42)$$

Input: $Integrate[((a*b^2)/(b+1))*(1+y)*(Exp[-((a*x)+(b*y))]),\{x,0,In\ inity\},\{y,0,In\ inity\}]$
Output: $ConditionalExpression[1, Re[a,b]>0]$; *where* $a=\theta_1, b=\theta_2,\ x=x_1,\ y=x_2$

$$\rightarrow\quad \int_0^{\infty}\int_0^{\infty}\left[\frac{\theta_1\theta_2^2}{(\theta_2+1)}\right](1+x_2)\ e^{-(\theta_1x_1+\theta_2x_2)}\ dx_1dx_2 = 1 \qquad (43)$$

$$X_1X_2 \sim Exp - Juc(\theta)$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x_1x_2)dx_1dx_2 = \int_0^{\infty}\left[\frac{\theta_1\theta_2^4}{(\theta_2^3+\theta_2^2+6)}\right](1+x_2^2+x_2^3)\ e^{-(\theta_1x_1+\theta_2x_2)}\ dx_1dx_2 \qquad (44)$$

Input: $Integrate[((a^4*b)/(a^3+a^2+6))*(1+x+x\hat{}3)*(Exp[-((a*x)+(b*y))]),\{x,0,Infinity\},\{y,0,Infinity\}]$
Output: $ConditionalExpression[1, Re[a,b]>0]$; *where* $a=\theta_1, b=\theta_2,\ x=x_1,\ y=x_2$

$$\rightarrow \quad \int_0^\infty \int_0^\infty \left[\frac{\theta_1\theta_2^4}{(\theta_2^3+\theta_2^2+6)}\right](1+x_2^2+x_2^3)\ e^{-(\theta_1x_1+\theta_2x_2)}\,dx_1dx_2 = 1 \quad (45)$$

$$X_1X_2 \sim Lind - Juc(\theta)$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x_1x_2)dx_1dx_2 = \int_0^\infty \left[\frac{\theta_1^2\theta_2^4}{(1+\theta_1)(\theta_2^3+\theta_2^2+6)}\right](1+x_1)(1+x_2^2+x_2^3)e^{-(\theta_1x_1+\theta_2x_2)}\,dx_1dx_2 \quad (46)$$

Input: Integrate[$((a^4 * b^2)/((a^3 + a^2 + 6) * (b + 1))) * ((1 + x + x$^3) * (1 + y)) * (Exp[$-((a * x) + (b * y))$]), {x, 0, In inity}, {y, 0, In inity}]
Output: ConditionalExpression[1, $Re[a, b] > 0$]; *where* $a = \theta_2, b = \theta_1, x = x_2, y = x_1$

$$\rightarrow \quad \int_0^\infty \int_0^\infty \left[\frac{\theta_1^2\theta_2^4}{(1+\theta_1)(\theta_2^3+\theta_2^2+6)}\right](1+x_1)(1+x_2^2+x_2^3)\ e^{-(\theta_1x_1+\theta_2x_2)}\,dx_1dx_2 = 1 \quad (47)$$

$$X_1X_2X_3 \sim Exp - Lind - Juc(\theta)$$

$$\int_{-\infty}^{\infty}\int_{\infty}^{\infty}\int_{-\infty}^{\infty} f(x_1x_2x_3)dx_1dx_2dx_3 = \int_0^\infty\int_0^\infty\int_0^\infty \left[\frac{\theta_1\theta_2^2\theta_3^4}{(1+\theta_2)(\theta_3^3+\theta_3^2+6)}\right](1+x_2)(1+x_3^2+x_3^3) * e^{-(\theta_1+\theta_2x_2+\theta_3x_3)}dx_1dx_2dx_3 \quad (48)$$

Input: Integrate[$((a * b$^2 $* c$^4)/((c^3 + c^2 + 6) * (b + 1))) * ((1 + z + z^3) * (1 + y)) * (Exp[$-((a * x) + (b * y) + (c * z))$]), {$x$, 0, In inity}, {$y$, 0, In inity}, {$z$, 0, In inity}]
Output: ConditionalExpression[1, $Re[a, b] > 0$]; *where* $a = \theta_1, b = \theta_2, c = \theta_3, x = x_1, y = x_2, z = x_3$

$$\rightarrow \quad \int_0^\infty\int_0^\infty\int_0^\infty \left[\frac{\theta_1\theta_2^2\theta_3^4}{(1+\theta_2)(\theta_3^3+\theta_3^2+6)}\right](1+x_2)(1+x_3^2+x_3^3)\,e^{-(\theta_1x_1+\theta_2x_2+\theta_3x_3)} = 1 \quad (49)$$

3. BIVARIATE PROPERTIES FOR THE PRODUCT DISTRIBUTIONS

3.1 Conditional Bivariate Development

If $X_1$and X_2have a joint probability density function $f(x_1x_2)$, then the conditional pdf of X, given that$X_2 = x_2$, is defined for all values of $x_2$such that $f(x_2) > 0$0, by

$$h(x_1|x_2) = \frac{f(x_1x_2)}{f(x_2)} = f(x_1) \quad (50)$$
$$h(x_2|x_1) = \frac{f(x_1x_2)}{f(x_1)} = f(x_2) \quad (51)$$

By implication, the conditional properties for the independent identical and non-identical distributions are given thus:

$$h_{a_1:a_2}(x_1|x_2) = \frac{f_{a_1:a_2}(x_1x_2)}{f_{a_1:a_2}(x_2)} = f_{a_1:a_2}(x_1) \quad (52)$$
$$h_{a_1:a_2}(x_2|x_1) = \frac{f_{a_1:a_2}(x_1x_2)}{f_{a_1:a_2}(x_1)} = f_{a_1:a_2}(x_2) \quad (53)$$

$$h_{a:b}(x_1|x_2) = \frac{f_{a:b}(x_1x_2)}{f_{a:b}(x_2)} = f_{a:b}(x_1) \quad (54)$$
$$h_{a:b}(x_2|x_1) = \frac{f_{a:b}(x_1x_2)}{f_{a:b}(x_1)} = f_{a:b}(x_2) \quad (55)$$

where$a_1: a_2$and $a : b$implies independent identical and non-identical distributions respectively.

3.2 Moments

The r^{th} moment of the Bivariate and Multivariate distributions is obtained thus:

$E(x^r x^s) = \int_0^\infty \int_0^\infty x_1^r x_2^s f(x_1 x_2) dx_1 dx_2$ (56)

$E(x^r x^s x^t) = \int_0^\infty \int_0^\infty x_1^r x_2^s x_3^t f(x_1 x_2 x_3) dx_1 dx_2 dx_3$ (57)

$E(x^r x^s x^t \ldots x^z) = \int_0^\infty \int_0^\infty \int_0^\infty \ldots \int_0^\infty x_1^r x_2^s x_3^t \ldots x_n^z f(x_1 x_2 x_3 \ldots x_n) dx_1 dx_2 dx_3 \ldots x_n$(58)

$$X_1 X_2 \sim Exp(\theta)$$

$E(x_1^r x_2^s) = \int_0^\infty \int_0^\infty x_1^r x_2^s \theta_1 \theta_2 e^{-(\theta_1 x_1 + \theta_2 x_2)} dx_1 dx_2$ (59)

Output: $[\theta_1{}^{-r} \theta_2{}^{-s}$Gamma$[1 + r]$Gamma$[1 + s]$, Re$[r] > -1$&&Re$[\theta_1 \theta_2] > 0]$

$\rightarrow \quad E(x_1^{r=1} x_2^{s=1}) = \frac{1}{\theta_1 \theta_2} = \mu_{x_1 x_2}$ (60)

$$X_1 X_2 \sim Lind(\theta)$$

$E(x_1^r x_2^s) = \int_0^\infty \int_0^\infty x_1^r x_2^s \frac{\theta_1^2 \theta_2^2}{(1+\theta_1)(1+\theta_2)} (1 + x_1)(1 + x_2) e^{-(\theta_1 x_1 + \theta_2 x_2)} dx_1 dx_2$(61)

Output: $[\frac{\theta_1{}^{-r} \theta_2{}^{-s} (1+\theta_1+r)(1+\theta_2+s) \text{Gamma}[1+r] \text{Gamma}[1+s]}{(1+\theta_1)(1+\theta_2)}$, Re$[r] > -1$&&Re$[\theta_1, \theta_2] > 0]$

$E(x_1^{r=1} x_2^{s=1}) = \frac{(2+\theta_1)(2+\theta_2)}{\theta_1 \theta_2 (1+\theta_1)(1+\theta_2)} = \mu_{x_1 x_2}$ (62)

$$X_1 X_2 \sim Juc(\theta)$$

$E(x_1^r x_2^s) = \int_0^\infty \int_0^\infty x_1^r x_2^s \frac{\theta_1^4 \theta_2^4}{(\theta_1^3+\theta_1^2+6)(\theta_2^3+\theta_2^2+6)} (1 + x_1 + x_1^3)(1 + x_2 + x_2^3) e^{-(\theta_1 x_1 + \theta_2 x_2)} dx_1 dx_2$(63)

Output:

$$[\frac{\theta_1{}^{-r} \theta_2{}^{-s} \left(\theta_1^3 + \theta_1^2(1 + r) + (1 + r)(2 + r)(3 + r)\right)\left(\theta_2^3 + \theta_2^2(1 + s) + (1 + s)(2 + s)(3 + s)\right) \text{Gamma}[1 + r]\text{Gamma}[1 + s]}{\left(\theta_1^3 + \theta_1^2 + 6\right)\left(\theta_2^3 + \theta_2^2 + 6\right)},$$

Re$[r] > -1$&&Re$[\theta_1, \theta_2] > 0]$

$E(x_1^{r=1} x_2^{s=1}) = \frac{(\theta_1^3+2\theta_1^2+24)(\theta_2^3+2\theta_2^2+24)}{\theta_1 \theta_2 (\theta_1^3+\theta_1^2+6)(\theta_2^3+\theta_2^2+6)} = \mu_{x_1 x_2}$ (64)

$$X_1 X_2 \sim Exp - Lind(\theta)$$

$E(x_1^r x_2^s) = \int_0^\infty \int_0^\infty x_1^r x_2^s \frac{\theta_1 \theta_2^2}{(1+\theta_2)} (1 + x_2) e^{-(\theta_1 x_1 + \theta_2 x_2)} dx_1 dx_2$ (65)

Output: $[\frac{\theta_1{}^{-r} \theta_2{}^{-s} (1+\theta_2+s) Gamma[1+r] Gamma[1+s]}{1+\theta_2}, Re[r] > -1\&\&Re[\theta_1, \theta_2] > 0]$

$E(x_1^{r=1} x_2^{s=1}) = \frac{2+\theta_2}{\theta_1 \theta_2 (1+\theta_2)}$ (66)

$$X_1 X_2 \sim Exp - Juc(\theta)$$

$E(x_1^r x_2^s) = \int_0^\infty \int_0^\infty x_1^r x_2^s \frac{\theta_1 \theta_2^4}{(\theta_2^3+\theta_2^2+6)} (1 + x_2 + x_2^3) e^{-(\theta_1 x_1 + \theta_2 x_2)} dx_1 dx_2$ (67)

Output: $[\frac{\theta_1{}^{-r}\theta_2{}^{-s}\left(\theta_2^3+\theta_2^2(1+r)+(1+r)(2+r)(3+r)\right)\text{Gamma}[1+r]\text{Gamma}[1+s]}{6+\theta_2^2+\theta_2^3}, \text{Re}[r] > -1\&\&\text{Re}[\theta_1,\theta_2] > 0]$

$$E(x_1^{r=1}x_2^{s=1}) = \frac{(\theta_2^3+2\theta_2^2+24)}{\theta_1\theta_2(\theta_2^3+\theta_2^2+6)} \quad (68)$$

$$X_1X_2 \sim Lind - Juc(\theta)$$

$$E(x_1^r x_2^s) = \int_0^\infty\int_0^\infty x_1^r x_2^s \frac{\theta_1^2\theta_2^4}{(1+\theta_1)(\theta_2^3+\theta_2^2+6)}(1+x_1)(1+x_2+x_2^3)e^{-(\theta_1x_1+\theta_2x_2)}dx_1dx_2 \quad (69)$$

Output: $[\frac{\theta_1{}^{-r}\theta_2{}^{-s}(1+a+r)(b^3+b^2(1+s)+(1+s)(2+s)(3+s))\text{Gamma}[1+r]\text{Gamma}[1+s]}{(1+a)(6+b^2+b^3)}, \text{Re}[r] > -1\&\&\text{Re}[a] > 0]$

$$E(x_1^{r=1}x_2^{s=1}) = \frac{(2+\theta_1)(\theta_2^3+2\theta_2^2+24)}{\theta_1\theta_2(1+\theta_1)(\theta_2^3+\theta_2^2+6)} \quad (70)$$

$$X_1X_2X_3 \sim Exp - Lind - Juc(\theta)$$

$$E(x_1^r x_2^s x_3^t) = \int_0^\infty\int_0^\infty x_1^r x_2^s x_3^t \frac{\theta_1\theta_2^2\theta_3^4}{(1+\theta_2)(\theta_3^3+\theta_3^2+6)}(1+x_1)(1+x_2+x_2^3)e^{-(\theta_1x_1+\theta_2x_2+\theta_3x_3)} dx_1dx_2dx_3 \quad (71)$$

Output: $[\frac{\theta_1{}^{-r}\theta_2{}^{-s}\theta_3{}^{-t}(1+\theta_2+s)\left(\theta_3^3+\theta_3^2(1+t)+(1+t)(2+t)(3+t)\right)Gamma[1+r]Gamma[1+s]Gamma[1+t]}{(1+\theta_2)(6+\theta_3^2+\theta_3^3)},$
$Re[r] > -1\&\&Re[\theta_1\theta_2\theta_3] > 0]$

$$E(x_1^{r=1}x_2^{s=1}x_3^{t=1}) = \frac{(2+\theta_2)(\theta_3^3+2\theta_3^2+24)}{\theta_1\theta_2\theta_3(1+\theta_2)(\theta_3^3+\theta_3^2+6)} = \mu_{x_1\,x_2\,x_3} \quad (72)$$

3.3 Mean, Covariance and Coefficient of Variation

Having obtained the r^{th} moment of the bivariate and multivariate distributions, we hence recall that evaluation: $E(x^r x^s)]_{r,s=1}$ and $E(x^r x^s x^t \dots x^z)]_{r,s,t,\ \dots\ ,z=1}$ are the means of the various derived distributions. The covariance of a bivariate distribution is given as: $Cov(X_1X_2) = E(X_1X_2) - E(X_1)E(X_2)$.However,coefficient of variation (CV) is dependent on the covariance; since by induction from the univariate CV model given as $CV = \frac{\sigma}{\mu}$,

$$CV_{bivariate} = \frac{\sqrt{Cov(x_1x_2)}}{\mu_{x_1x_2}} \quad (73)$$

Consequently, the developed models are independent, and following calculations:$Cov(X_1,X_2) = 0$. This implies that the coefficient of variation is zero as well, for all the distributions in this study.More so, the correlation coefficient $\rho = \frac{Cov(x_1x_2)}{\sigma_{x_1}\sigma_{x_2}}$ by the same implication is zero also.

3.4 Moment Generating Function

The moment generating function of Bivariate or Multivariate Distribution is derived thus:

$$M_{x_1x_2}(t_1,t_2) = E(e^{t_1x_1+t_2x_2}) = \int_0^\infty\int_0^\infty e^{t_1x_1+t_2x_2} f(x_1x_2)\, dx_1dx_2 \quad (74)$$

$$M_{x_1x_2x_3}(t_1,t_2,t_3) = E\left(e^{t_1x_1+t_2x_2+t_3x_3}\right) = \int_0^\infty\int_0^\infty e^{t_1x_1+t_2x_2} f(x_1x_2)\ dx_1dx_2 \quad (75)$$

$X_1X_2 \sim Exp(\theta)$

$$M_{x_1x_2}(t_1,t_2) = \frac{\theta_1\theta_2}{(\theta_1-t_1)(\theta_2-t_1)} \quad (76)$$

$X_1X_2 \sim Lind(\theta)$

$$M_{x_1x_2}(t_1,t_2) = \frac{\theta_1^2\theta_2^2(1+\theta_1-t_1)(1+\theta_2-t_2)}{(1+\theta_1)(1+\theta_2)(\theta_1-t_1)^2(\theta_2-t_2)^2} \quad (77)$$

$X_1X_2 \sim Juc(\theta)$

$$M_{x_1x_2}(t_1,t_2) = \frac{\theta_1^4\theta_2^4(6+(\theta_1-t_1)^2+(\theta_1-t_1)^3)(6+(\theta_2-t_2)^2(1+\theta_2-t_2))}{(\theta_1^3+\theta_1^2+6)(\theta_2^3+\theta_2^2+6)(\theta_1-t_1)^4(\theta_2-t_2)^4} \quad (78)$$

$X_1X_2 \sim Exp - Lind(\theta)$

$$M_{x_1x_2}(t_1,t_2) = \frac{\theta_1\theta_2^2(1+\theta_2-t_2)}{(1+\theta_2)(\theta_1-t_1)(\theta_2-t_2)^2} \quad (79)$$

$X_1X_2 \sim Exp - Juc(\theta)$

$$M_{x_1x_2}(t_1,t_2) = \frac{\theta_1\theta_2^4(6+(\theta_2-t_2)^2+(\theta_2-t_2)^3)}{(\theta_2^3+\theta_2^2+6)(\theta_1-t_1)(\theta_2-t_2)^4} \quad (80)$$

$X_1X_2 \sim Juc - Lind(\theta)$

$$M_{x_1x_2}(t_1,t_2) = \frac{\theta_1^4\theta_2^2(6+(\theta_1-t_1)^2+(\theta_1-t_1)^3)(1+\theta_2-t_2)}{(\theta_1^3+\theta_1^2+6)(1+\theta_2)(\theta_1-t_1)^4(\theta_2-t_2)^2} \quad (81)$$

$X_1X_2X_3 \sim Exp - Lind - Juc(\theta)$

$$M_{x_1x_2x_3}(t_1,t_2,t_3) = \frac{\theta_1\theta_2^2\theta_3^4(1+\theta_2-t_2)(6+(\theta_3-t_2)^2(1+\theta_3-t_3))}{(1+\theta_2)(\theta_3^3+\theta_3^2+6)(\theta_1-t_1)(\theta_2-t)^2(\theta_3-t_3)^4} \quad (82)$$

3.5 Maximum Likelihood Estimator

For brevity purposes we estimate the parameters of Exponential-Lindley-Juchez distribution only. However, further derivations could be made for the parameters of every other independent distribution. Let(X_{1i}, X_{2i}, X_{3i}) $i = 1,2,3,\ldots,n$, be random vectors from Exponential-Lindley-Juchez multivariate distribution, the maximum likelihood estimator (MLE) is obtained thus:

$$\mathrm{L}f(x_1x_2x_3,\theta) = \left(\frac{\theta_1\theta_2^2\theta_3^4}{(1+\theta_2)(\theta_3^3+\theta_3^2+6)}\right)^n \prod_{i=1}^n (1+x_{i2})(1+x_{i3}^2+x_{i3}^3)$$

$e^{-(\theta_1\sum_{i=1}^n x_{i_1} + \theta_2\sum_{i=1}^n x_{i_2} + \theta_3\sum_{i=1}^n x_{i_3})}$ (83)$ln\mathrm{L}f(x_1x_2x_3,\theta_1\theta_2\theta_3) = nln\theta_1 + 2nln\theta_2 + 4nln\theta_3 -$ $nln(1+\theta_2) - nln(\theta_3^3+\theta_3^2+6)$

$$+\sum_{i=1}^n \ln(1+x_{i3}^2+x_{i3}^3+x_{i2}+x_{i2}x_{i3}^2+x_{i2}x_{i3}^3)$$

$-(\theta_1\sum_{i=1}^n x_{i_1} + \theta_2\sum_{i=1}^n x_{i_2} + \theta_3\sum_{i=1}^n x_{i_3})$ (84)

In estimation of the parameters, the estimator is maximized at

$\frac{\partial\ ln\mathrm{L}f(x_1x_2x_3,\theta_1\theta_2\theta_3)}{\partial\theta_1} = 0$, $\frac{\partial\ ln\mathrm{L}f(x_1x_2x_3,\theta_1\theta_2\theta_3)}{\partial\theta_2} = 0$ *and* $\frac{\partial\ ln\mathrm{L}f(x_1x_2x_3,\theta_1\theta_2\theta_3)}{\partial\theta_3} = 0$, then

$$\frac{\partial\ ln\mathrm{L}f(x_1x_2x_3,\theta_1\theta_2\theta_3)}{\partial\theta_1} = \frac{n}{\theta_1} - \sum_{i=1}^n x_{i_1} = 0 \quad (85)$$

$$\rightarrow \qquad \hat{\theta}_1 = \frac{1}{\bar{x}_1}$$

$$\frac{\partial\ ln\mathrm{L}f(x_1x_2x_3,\theta_1\theta_2\theta_3)}{\partial\theta_2} = \frac{2n}{\theta_2} - \frac{n}{1+\theta_2} - \sum_{i=1}^n x_{i_2} = 0 \quad (86)$$

$$\frac{\partial\, ln\mathrm{L}f(x_1x_2x_3,\theta_1\theta_2\theta_3)}{\partial\theta_3} = \frac{4n}{\theta_3} - \frac{n(3\theta_3^2+2\theta_3)}{\theta_3^3+\theta_3^2+6} - \sum_{i=1}^{n} x_{i_3} = 0 \qquad (87)$$

The likelihood equations in (85), (86) and (87) can easily be solved iteratively using Fisher's scoring method due to the closed form equations obtained. We have thus:

$$\frac{\partial^2\, ln\mathrm{L}f(x_1x_2x_3,\theta_1\theta_2\theta_3)}{\partial\theta_1\partial\theta_2} = \frac{\partial^2\, ln\mathrm{L}f(x_1x_2x_3,\theta_1\theta_2\theta_3)}{\partial\theta_1\partial\theta_3} = \frac{\partial^2\, ln\mathrm{L}f(x_1x_2x_3,\theta_1\theta_2\theta_3)}{\partial\theta_2\partial\theta_3} = 0 \qquad (88)$$

$$\frac{\partial^2\, ln\mathrm{L}f(x_1x_2x_3,\theta_1\theta_2\theta_3)}{\partial\theta_1^2} = -\frac{n}{\theta_1^2} \qquad (89)$$

$$\frac{\partial^2\, ln\mathrm{L}f(x_1x_2x_3,\theta_1\theta_2\theta_3)}{\partial\theta_2^2} = -\frac{2n}{\theta_2^2} + \frac{n}{(1+\theta_2)^2} \qquad (90)$$

$$\frac{\partial^2\, ln\mathrm{L}f(x_1x_2x_3,\theta_1\theta_2\theta_3)}{\partial\theta_3^2} = -\frac{4n}{\theta_3^2} + \frac{(2\theta_3+3\theta_3^2)^2}{(6+\theta_3^2+\theta_3^3)^2} - \frac{2+6\theta_3}{6+\theta_3^2+\theta_3^2} \qquad (91)$$

Resolving the following matrix equations, the solutions of MLE $(\hat{\theta}_1, \hat{\theta}_2\ and\ \hat{\theta}_3)$ for $f(x_1x_2x_3, \theta_1\theta_2\theta_3)$ are obtained:

$$\begin{bmatrix} \frac{\partial^2\, ln\mathrm{L}f(x_1x_2x_3,\theta_1\theta_2\theta_3)}{\partial\theta_1^2} & \frac{\partial^2\, ln\mathrm{L}f(x_1x_2x_3,\theta_1\theta_2\theta_3)}{\partial\theta_1\partial\theta_2} & \frac{\partial^2\, ln\mathrm{L}f(x_1x_2x_3,\theta_1\theta_2\theta_3)}{\partial\theta_1\partial\theta_3} \\ \frac{\partial^2\, ln\mathrm{L}f(x_1x_2x_3,\theta_1\theta_2\theta_3)}{\partial\theta_2\partial\theta_1} & \frac{\partial^2\, ln\mathrm{L}f(x_1x_2x_3,\theta_1\theta_2\theta_3)}{\partial\theta_2^2} & \frac{\partial^2\, ln\mathrm{L}f(x_1x_2x_3,\theta_1\theta_2\theta_3)}{\partial\theta_2\partial\theta_3} \\ \frac{\partial^2\, ln\mathrm{L}f(x_1x_2x_3,\theta_1\theta_2\theta_3)}{\partial\theta_3\partial\theta_1} & \frac{\partial^2\, ln\mathrm{L}f(x_1x_2x_3,\theta_1\theta_2\theta_3)}{\partial\theta_3\partial\theta_1} & \frac{\partial^2\, ln\mathrm{L}f(x_1x_2x_3,\theta_1\theta_2\theta_3)}{\partial\theta_3^2} \end{bmatrix} \begin{bmatrix} \hat{\theta}_1 = \theta_{01} \\ \hat{\theta}_2 = \theta_{02} \\ \hat{\theta}_3 = \theta_{03} \end{bmatrix} = \begin{bmatrix} \frac{\partial\, ln\mathrm{L}f(x_1x_2x_3,\theta_1\theta_2\theta_3)}{\partial\theta_1} \\ \frac{\partial\, ln\mathrm{L}f(x_1x_2x_3,\theta_1\theta_2\theta_3)}{\partial\theta_2} \\ \frac{\partial\, ln\mathrm{L}f(x_1x_2x_3,\theta_1\theta_2\theta_3)}{\partial\theta_3} \end{bmatrix} \qquad (92)$$

whereθ_{01}, $\theta_{02}\ and\ \theta_{03}$ are initial values of$\hat{\theta}_1\hat{\theta}_2\ and\ \hat{\theta}_3$.

3.6 Bivariate Reliability Analysis

The Reliability function models the probability that a component will survive after a given time; whereas, hazard function is the likelihood that a system will terminate after a given period of time or cycle. The reliability$R(x,\theta)$ and hazard rate models $H(x,\theta)$ of bivariate distributions are given as:

$$R(x_1x_2,\theta) = P(X_1 \geq x_1, X_2 \geq x_2) = \int_{x_1}^{\infty}\int_{x_2}^{\infty} f(t_1,t_2)\, dt_1 dt_2$$

$$R(x_1,x_2,x_3,\theta) = P(X_1 \geq x_1, X_2 \geq x_2, X_3 \geq x_3) = \int_{x_1}^{\infty}\int_{x_2}^{\infty}\int_{x_3}^{\infty} f(t_1,t_2,t_3)\, dt_1 dt_2 dt_3 \qquad (93)$$

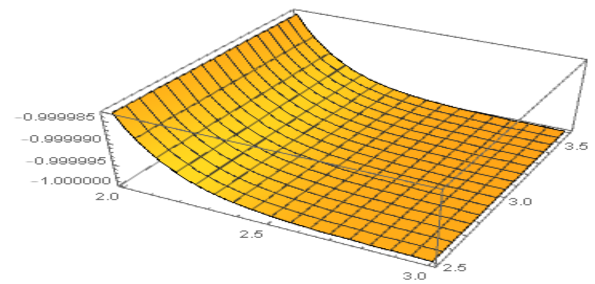
$$\rightarrow \qquad R(x_1,x_2,\theta) = 1 - G(X_1) - G(X_2) + G(X_1,X_2) \qquad (94)$$

$$R(x_1,x_2,x_3,\theta) = 1 - G(X_1) - G(X_2) - G(X_3) + G(X_1,X_2,X_3)$$

We consider (random selection of) two bivariate: say, Exponential-Exponential from IID and Exponential-Lindley; and the Exponential-Lindley-Juchezmultivariate distributions from $\mathrm{II_nD}$.

$$R_{Exp-Exp}(x_1,x_2,\theta) = 1 - \left[1 - e^{-\theta_1 x_1}\right] - \left[1 - e^{-\theta_2 x_2}\right] + \left[e^{-(\theta_1 x_1 + \theta_2 x_2)}\right] \qquad (95)$$

$$R_{Exp-Lind}(x_1, x_2, \theta) = 1 - \left[1 - e^{-\theta_1 x_1}\right] - \left[1 - \left(\frac{\theta_2 + 1 + \theta_2 x_2}{\theta_2 + 1}\right) e^{-\theta_2 x_2}\right] + \left[\left(1 + \frac{\theta_2 x_2}{1+\theta_2}\right) e^{-(\theta_1 x_1 + \theta_2 x_2)}\right] \quad (96)$$



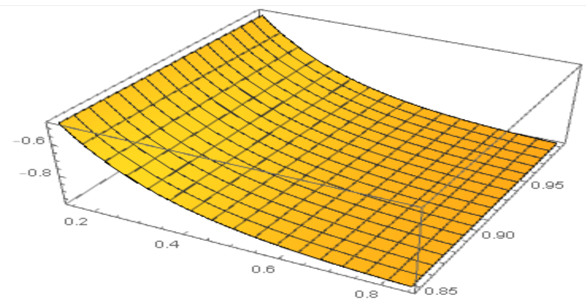


Figure 7 (a)and (b). The shape of the reliability function of the Bivariate Exponential-Exponential distribution and the Bivariate Exponential-Lindley Distribution.

$$R_{Exp-Lind-Juc}(x_1, x_2, x_3, \theta) = 1 - \left[1 - e^{-\theta_1 x_1}\right] - \left[1 - \left(\frac{\theta_2 + 1 + \theta_2 x_2}{\theta_2 + 1}\right) e^{-\theta_2 x_2}\right] - \left[1 - \left(1 + \frac{\theta x\left[\theta^2 + \theta^2 x^2 + 3\theta x + 6\right]}{\theta^3 + \theta^2 + 6}\right) e^{-\theta x}\right] \quad (97)$$

$$+ \left[- \left(1 + \frac{\begin{matrix} 6\theta_3 x_3 + 3x_3^2 + \theta_3^3 x_3 + \theta_3^3 x_3^3 + 6\theta_2\theta_3 x_3 + 3\theta_2 x_3^3 + \\ \theta_2\theta_3^3 x_3 + \theta_2\theta_3^3 x_3^3 + 6\theta_2 x_2 + 6\theta_2\theta_3 x_2 x_3 + b\theta_3^2 x_2 + \\ 3\theta_2 x_2 x_3^3 + \theta_2\theta_3^3 x_2 + \theta_2\theta_3^3 x_2 x_3 + \theta_2\theta_3^3 x_2 x_3^3 \end{matrix}}{6 + 6\theta_2 + \theta_3^2 + \theta_2\theta_3^2 + \theta_3^3 + \theta_2\theta_3^3}\right) e^{-\theta_1 x_1 - \theta_2 x_2 - \theta_3 x_3}\right] \quad (98)$$

$$\rightarrow \quad H(x_1 x_2, \theta) = \frac{f(x_1 x_2, \theta)}{R(x_1 x_2, \theta)} \quad and \quad H(x_1, x_2, x_2, \theta) = \frac{f(x_1, x_2, x_2, \theta)}{R(x_1, x_2, x_2, \theta)} \quad (99)$$

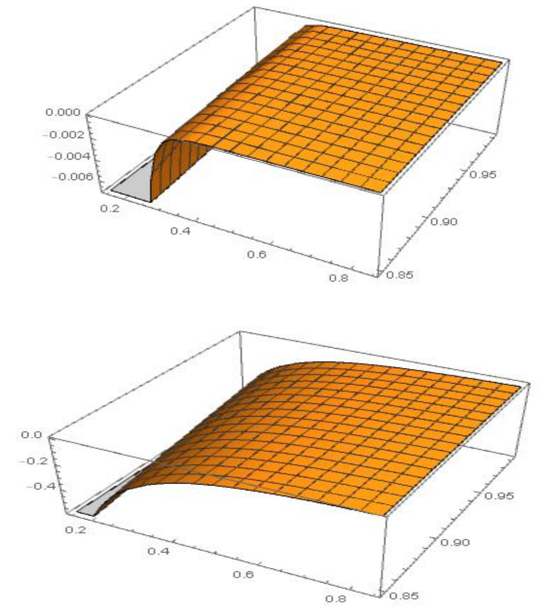


Figure 8 (a)and (b). The shape of the Hazard Function of the Bivariate Exponential-Exponential distribution and the Bivariate Exponential-Lindley Distribution.

3.7 Renewal Property

In the modeling of two-dimensional queuing and renewal processes, in scenarios, where the arrival and service times are dependent; bivariate distributions can be employed. Laplace transformation as a function makes this mathematically possible.

Let X_1X_2be a random vector with joint PDF $f(x_1x_2)$ then the function is derived thus:

$$\varphi_{X_1X_2}(t_1,t_2) = E[\exp(-t_1x_1 - t_2x_2)] \qquad (100)$$

$$= \int_0^\infty \int_0^\infty \exp(-t_1x_1 - t_2x_2)f(x_1x_2)dx_1dx_2 \qquad (101)$$

Concisely, the bivariate distributions randomly chosen for this representation are Bivariate Exponential-Juchez, Bivariate Lindley-Lindley and Bivariate Juchez-Lindley distributions:

$$\varphi_{Exp-Juc}(t_1,t_2) = \frac{\theta_1\theta_2^4(6+(\theta_2+t_2)^2+(\theta_2+t_2)^3)}{(\theta_1-t_1)(6+\theta_2^2+\theta_2^3)(\theta_2+t_2)^4}, \quad [\theta_2+t_2] > 0 \qquad (102)$$

$$\varphi_{Lind-Lind}(t_1,t_2) = \frac{(1+\theta_1+t_1)(1+\theta_2-t_2)}{(\theta_1+t_1)^2(\theta_2-t_2)^2}, [\theta_1+t_1] > 0 \qquad (103)$$

$$\varphi_{Juc-Lind}(t_1,t_2) = \frac{\theta_1^4\theta_2^2(6+(\theta_1+t_1)^2+(\theta_1+t_1)^3)(1+\theta_2-t)}{(6+\theta_1^2+\theta_1^3)(1+\theta_2)(\theta_1+t_1)^4(\theta_2-t_2)^2}, [\theta_1+t_1] > 0 \qquad (104)$$

Which are the Laplace transform of our bivariate distributions.

3.8 Bivariate Probability Patterns

In this section, random choice is also made in the selection of the bivariate distributions, with the intent to investigate the trend or the shape of the distributions; and at same time test whether there is consistency in respect of the probability axiom: $0 \leq P(x) \leq 1$.

Table 1. Statistical Table for theBivariate Juchez and Bivariate Lindley-JuchezDistribution

	Bivariate Juchez($\theta_1 = 0.5$,$\theta_2 = 0.5$)				**Bivariate Lindley-Juchez**			
$X_1 \| \| X_2$	1	**2**	3	4	1	2	3	4
1	0.000318	0.007077	0.001209	0.001633	0.003606	0.003281	0.002654	0.002011
2	0.000707	0.001574	0.002690	0.003632	0.008021	0.007297	0.005902	0.004474
3	0.001209	0.002690	0.004599	0.006208	0.013710	0.012474	0.010087	0.007648
4	0.001633	0.003632	0.006208	0.008381	0.018509	0.016840	0.013618	0.010325
5	0.001881	0.004182	0.007149	0.009651	0.021314	0.019391	0.015682	0.011890
6	0.001942	0.004318	0.007381	0.009965	0.022007	0.020022	0.016192	0.012276
7	0.001854	0.004123	0.007047	0.009513	0.021009	0.019114	0.015457	0.011719
8	0.001669	0.003711	0.006344	0.008565	0.018914	0.017208	0.013916	0.010551
9	0.001436	0.003193	0.005458	0.007368	0.016272	0.014805	0.011973	0.009077
10	0.001191	0.002650	0.004529	0.006114	0.013502	0.012284	0.009934	0.007532
11	0.000960	0.002135	0.003649	0.004926	0.010878	0.009897	0.008004	0.006069
12	0.000755	0.001678	0.002869	0.003873	0.008554	0.007782	0.006294	0.004772
13	0.000581	0.001293	0.002210	0.002984	0.006589	0.005994	0.004847	0.003675
14	0.000440	0.000978	0.001673	0.002258	0.004987	0.004537	0.003669	0.002782
15	0.000328	0.000729	0.001246	0.001683	0.003717	0.003382	0.002735	0.002074
16	0.000241	0.000537	0.000917	0.001238	0.002735	0.002488	0.002012	0.001525
17	0.000175	0.000390	0.000667	0.000901	0.001989	0.001809	0.001463	0.001109
18	0.000126	0.000281	0.000480	0.000648	0.001431	0.001302	0.001053	0.000798
19	0.000090	0.000200	0.000342	0.000462	0.001021	0.000929	0.000751	0.000569
20	0.000063	0.000142	0.000242	0.000326	0.000722	0.000657	0.000531	0.000403

The goal in Table 1 is to observe probability patterns; and indeed the output validated for the probability axioms. The choice for the derived bivariate distribution used here is purely random; and carefully examining across the variables we notice first, an upward movement as seen at$X_1: 1 - 6$; followed by a monotonic downward movement from $X_1: 7 - 20$. This trend is similar for eachX_2. Consequently, keeping both parameters constant, we could deduce that the two bivariate distributions are unimodal and positively skewed to the right.

3.9 SIMULATION

In bivariate analysis, the quantile function specifies the values of the random vector and is also known as the Inverse cumulative distribution.The quantile function for a univariate distribution can be obtained from this expression $x = F^{-1}(u)$, which is derived from $F(x) = u$; where$F(x)$ is the distribution function; and 0 < u < 1. However, with respect to bivariate study the inverse cumulative distribution is given as

$$G(X_1, X_2) = (u_1, u_2) \quad \rightarrow \quad (X_1, X_2) = G^{-1}(u_1, u_2)$$

CONCLUSION

The paper aimed at developing different bivariate distributions, using mixture model offspring as the baseline distributions. Product distribution approach was used for the model developmental constructionsof bivariate Independent Identical Distributions (IIDs) and Independent Non-Identical Distributions (II_nD). Under IIDs, A New Bivariate Exponential, Bivariate Lindley, Bivariate Juchez Distributions are constructed; whereas, Bivariate Exponential-Lindley, Bivariate Exponential-Juchez and Bivariate Lindley-Juchez Distributions are derived under (II_nDs). The validity test for these bivariate derivations, the verification of the statistical properties and all the mathematical bottlenecks were verified, carried out in both R and Mathematica software. Some properties considered are: the shape of the bivariate PDFs, moments, moment generating function, mean, covariance, coefficient of variance and coefficient of correlation, maximum likelihood estimator, reliability analysis, renewal property, inverse cumulative distribution and probability patterns.

The probability patterns confirm with the shape of the PDFs that the various bivariate distributions, as derived in this study, is unimodal and non-normal. The covariance was calculated to be zero; and by implication coefficient of both covariance and correlation is zero. Finally the models derived under renewal properties could be used to model two-dimensional queuing and renewal processes, for situations where the arrival and service times are dependent.

FUTURE WORK

Having adopted product distribution in the development of bivariate models, using the category of distributions from mixture distribution, other developmental approaches like Marginal transformation, Copula Method, Method of Mixing and Compounding, Trivariate Reduction Method, Frailty Approach and Conditional Specification Method, could be explored as well; and the performance comparison among other counterpart bivariate models could be done to test for better fit across models with respect to various bivariate data emanating from different fields of life.

REFERENCE

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Epstein, B. and Sobel, M. (1954). Some theorems relevant to life testing from an exponential distribution. *The Annals of Mathematics Statistics, 373-381*

Lindley D.V., 1958. Fiducial distributions and Bayes' Theorem. *Journal of the Royal Statistical Society. Series B.*; 20(1):102–107.

Echebiri U.V., Mgbebu J.I. (2022). Juchez Probability Distribution: Properties and Application. *Asian Journal of Probability and Statistics*, 20(2): 56-71.

Hutchinson, T. P., & Lai, C. D. (1990). Continuous bivariate distributions emphasizing applications (Tech. Rep.).

Arnold, B. C., Castillo, E., &Sarabia, J. M. (1999b). Conditional specification of statistical models. Springer-Verlag.

Kotz, S., &Nadarajah, S. (2000). Extreme value distributions Vol. (31). Imperial College Press, London.

Kotz, S., &Nadarajah, S. (2004). Multivariate t-distributions and their applications. Cambridge University Press.

Nelsen, R. B. (2006). An introduction to copulas, 2nd edition. New York: Springer-Science Business Media.

Lai, C.D., (2004). Constructions of continuous bivariate distributions. *Journal of the Indian Society for Probability & Statistics*, 8, 21-43.

Lai, C.D., (2006). Constructions of discrete bivariate distributions. Advances in Distribution Theory, Order Statistics, and Inference, 1 (1), 29-58.

Sarabia, J. M., & Gomez, D. E. (2008). Construction of multivariate distributions: A review of some recent results. *SORT,* 32 (1), 3-36.

Springer. M.D., and William E. (1966). Bayesian confidence limits for the product of N binomial parameters. *Biometrika, 53*(3-4), 611-613.

Fréchet, M. (1960). On arrays whose margins and bounds are given. Review of the International Statistical Institute, 10-32.

Mardia, K.V. (1970). Families of bivariate distributions (Vol. 27). Griffin London.

Gumbel, E.J. (1960). Bivariate exponential distributions. *Journal of the American Statistical Association*, 55 (292), 698-707.

Gumbel, E.J. (1961). Bivariate logistic distributions. *Journal of the American Statistical Association, 56* (294), 335-349.

Lindsay B.G., 1995. Mixture models: theory, geometry and applications, NSF-CBMS Regional Conference Series in Probability and Statistics, Hayward, CA, USA: Institute of Mathematical Statistics, ISBN 0-940600-32-3, JSTOR 4153184.