Original Research Article

MODELLING MOTHER-TO-CHILD HIV TRANSMISSION RATE IN NIGERIA USING A LINDLEY-LOMAX DISTRIBUTION

Abstract:
In this research a new extension of the Lomax distribution known as Lindley-Lomax distribution has been proposed by adding a shape parameter to the Lomax distribution using the Lindley-G family of distributions. The proposed article considered and extensively studied some properties of the new distribution such as moments, moment generating function, the characteristics function, survival function, hazard function and the distribution of order statistics. A graphical study of the proposed distribution and the other related functions was also done in the article. Estimation of the parameters of the proposed distribution was also done using the method of maximum likelihood estimation. The performance of the Lindley-Lomax distribution has also been tested by an application to the rate of mother-to-child HIV transmission.

Keywords: Lindley distribution, Lomax distribution, Lindley-Lomax distribution, statistical properties, parameter estimation, HIV data, application.

1 Background

Lomax [1] proposed the Lomax distribution for modeling business failure data and it has been widely used in numerous contexts. In [2] it was mentioned that the distribution can be used for reliability modeling and life testing. It has been shown by different authors that the distribution is used for modeling different kinds of data, it was used for modeling income and wealth data [3], [4] also applied the Lomax distribution in modeling business failure data, more so [5] applied the distribution in solving firm size and queuing problems. The applications of the Lomax distribution were also discovered in biological sciences and in the modeling of the sizes of computer files on servers, [6]. Other authors including [7] noted that the Lomax distribution can be used as an alternative model to the exponential distribution when the available data is heavily skewed.

The probability density function (pdf) of a Lomax distribution with parameters $\alpha \ and \beta$ is given as;

$$f(x)=\frac{\alpha}{\beta}\left[1+\left(\frac{x}{\beta}\right)\right]^{-(\alpha+1)} \tag{1}$$

and the corresponding cumulative distribution function (*cdf*) is given as;

$$F(x)=1-\left[1+\left(\frac{x}{\beta}\right)\right]^{-\alpha} \tag{2}$$

Where $x > 0, \alpha > 0, \beta > 0$ and α *and* β are the shape and scale parameters respectively.

A number of authors have carried out some research that involved different extensions of the Lomax distribution and these among others are; the Marshall–Olkin extended-Lomax [8] and [9], exponential–Lomax [10], McDonald-Lomax [11], Exponentiated Lomax [12]. In [13] a three-parameter Gamma–Lomax distribution was presented based on a versatile and flexible gamma generator proposed by [14] using Stacy's generalized gamma distribution and record value theory. In [15] the four parameters Weibull Lomax distribution was introduced, [16] introduced Poisson-Lomax distribution and also the Power Lomax distribution was introduced by [17]. The Extended Poisson-Lomax distribution was introduced by [18] and [19] proposed the transmuted exponentiated Lomax distribution.

This article intends to define and study a new probability distribution known as a Lindley-Lomax distribution (LinLomD) using the proposed Lindley-G family of distributions by [20]. The rest of the sections in this article are arranged as follows: In section two, the definition of the Lindley-Lomax distribution is presented. The properties of the proposed distribution are presented in section 3. Section 4 presents the maximum likelihood estimators of the proposed distribution. An application of the proposed distribution to the rate of mother-to-child transmission of HIV is given in section 5 and the conclusion is finally made in section 6.

2. The Lindley-Lomax Distribution *(LinLomD)*

According to [20], the cumulative distribution function (cdf) and the probability density function (pdf) of the Lindley-G family of distributions are defined as:

$$F(x) = \int_{-\infty}^{\frac{G(x)}{1-G(x)}} \frac{\theta^2}{\theta+1}(1+t)e^{-\theta t}dt = 1 - \frac{\theta + \left(1-G(x)\right)}{(1+\theta)\left(1-G(x)\right)} \exp\left\{-\theta\left[\frac{G(x)}{1-G(x)}\right]\right\} \tag{3}$$

and

$$f(x) = \frac{\theta^2 g(x)}{(1+\theta)(1-G(x))^3} \exp\left\{-\theta\left[\frac{G(x)}{1-G(x)}\right]\right\} \tag{4}$$

respectively, where $g(x)$ and $G(x)$ are the *pdf* and the *cdf* of any continuous distribution to be extended and $\theta > 0$ is the shape parameter of the Lindley-G.

Considering equation (1) and (2) and making use of it in equation (3) and (4) and simplifying, we derive the *cdf* and *pdf* of the Lindley-Lomax distribution as follows:

$$F(x) = 1 - \left[1 - \frac{\theta}{(\theta+1)}\left[\log\left(\left(1+\left(\frac{x}{\beta}\right)\right)^{-\alpha}\right)\right]\right]\left(1+\left(\frac{x}{\beta}\right)\right)^{-\alpha\theta} \tag{5}$$

And

$$f(x)=\frac{\alpha\theta^{2}}{\beta(\theta+1)}\left[1+\left(\frac{x}{\beta}\right)\right]^{-(\alpha+1)}\left[1-\log\left(1+\left(\frac{x}{\beta}\right)\right)^{-\alpha}\right]\left(1+\left(\frac{x}{\beta}\right)\right)^{-\alpha(\theta-1)} \quad (6)$$

respectively.

For $x>0$, $\alpha,\beta,\theta>0$ where $\alpha>0$ and $\theta>0$ are the shape parameters while $\beta>0$ is a scale parameter. Hence equation (5) and (6) are the cdf and pdf of the Lindley-Lomax distribution.

For any chosen values of the parameters α, β and θ, the graphs of the pdf and the cdf of the LinLomD as shown in figure 1 and 2 below:

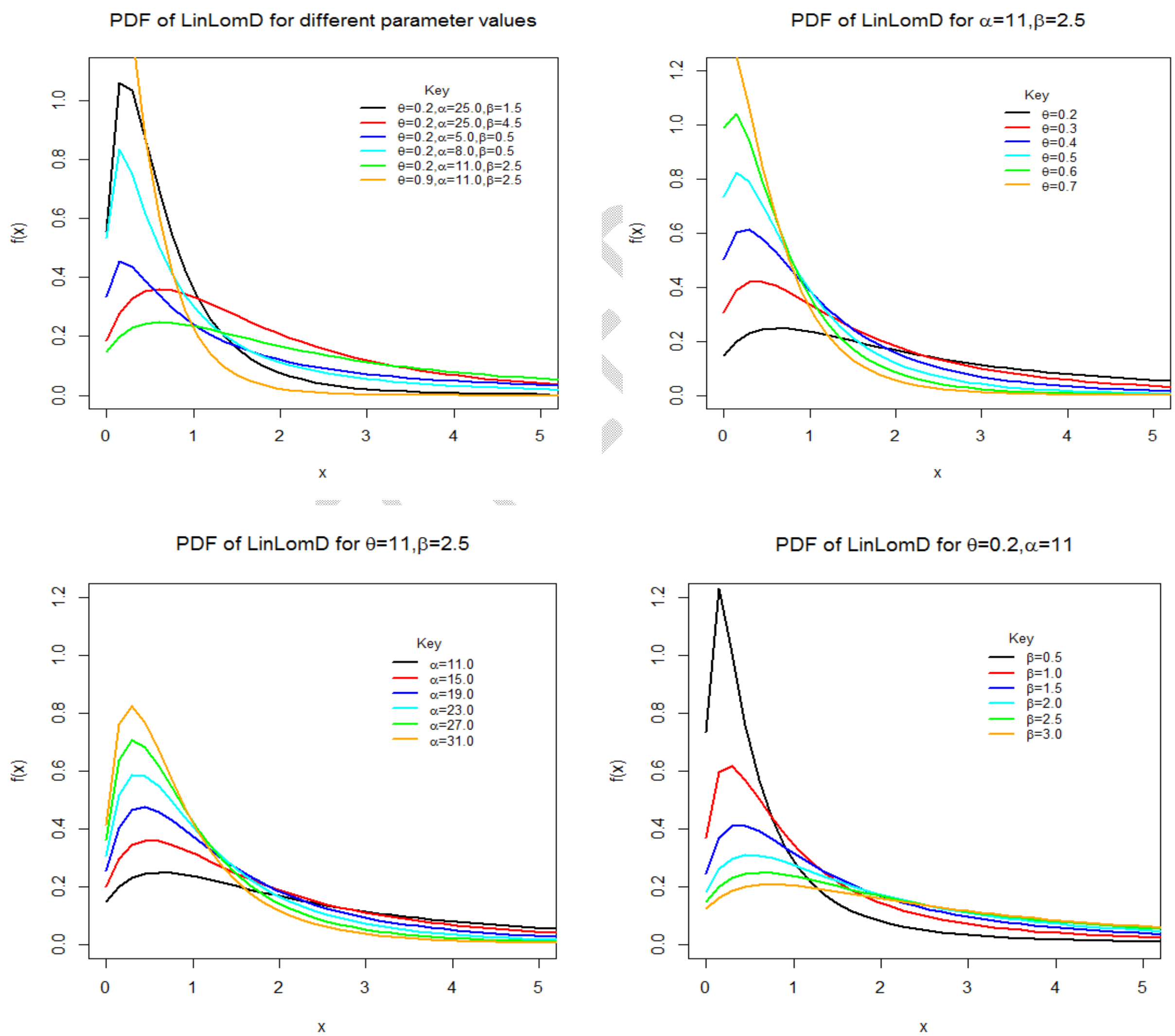

Figure 1: *PDF* of the LinLomD.

Figure **1** indicates that the LinLomD distribution has various shapes such as left-skewed or right-skewed shapes depending on the parameter values. This means that distribution can be very useful for datasets with different shapes.

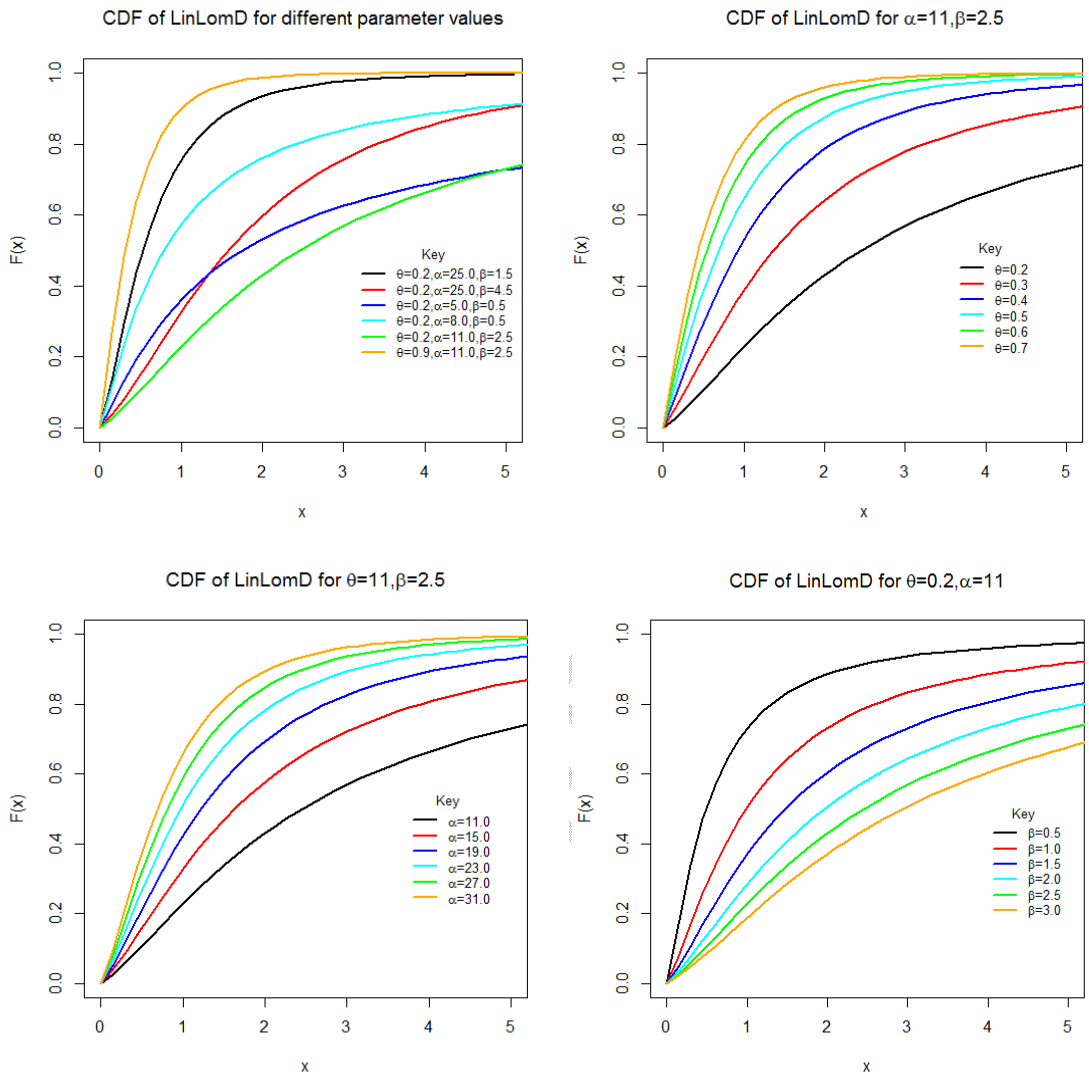

Fig. 2: *CDF* of the LinLomD.

From the above *cdf* plot, the *cdf* increases when X increases, and approaches 1 when X becomes large, as expected.

3. Properties

This section of the article presents and discuss useful properties of the proposed LinLomD distribution.

3.1 Moments

Let X denote a continuous random variable, the n^{th} moment of X is given by;

$$\mu_n' = E\left[X^n\right] = \int_0^\infty x^n f(x)dx \qquad (7)$$

Considering *f(x)* to be the *pdf* of the Lindley-Lomax distribution as given in equation (6). Recall,

$$f(x)=\frac{\alpha\theta^{2}}{\beta(\theta+1)}\left[1+\left(\frac{x}{\beta}\right)\right]^{-(\alpha+1)}\left[1-\log\left(1+\left(\frac{x}{\beta}\right)\right)^{-\alpha}\right]\left(1+\left(\frac{x}{\beta}\right)\right)^{-\alpha(\theta-1)} \quad (8)$$

Before substituting (8) in (7), we perform the expansion and simplification and linear representation of the pdf as follows:

Note that for $\left|\left(1-\left[1+\left(\frac{x}{\beta}\right)\right]^{-\alpha}\right)\right|<1$, the following power series expansion holds, that is:

$$\log\left(1-\left(1-\left[1+\left(\frac{x}{\beta}\right)\right]^{-\alpha}\right)\right)=-\sum_{k=0}^{\infty}\frac{\left[1-\left[1+\left(\frac{x}{\beta}\right)\right]^{-\alpha}\right]^{K+1}}{k+1} \quad (9)$$

Making use of the result in (9) above and simplifying, equation (8) becomes

$$f(x)=\frac{\alpha\theta^{2}}{\beta(\theta+1)}\left[1+\left(\frac{x}{\beta}\right)\right]^{-\alpha\theta-1}\left[1+\sum_{k=0}^{\infty}\frac{\left[1-\left(1+\left(\frac{x}{\beta}\right)\right)^{-\alpha}\right]^{K+1}}{k+1}\right] \quad (10)$$

Now, if k is a positive non-integer, we can expand the last term in (10) as:

$$\left[1-\left(1+\left(\frac{x}{\beta}\right)\right)^{-\alpha}\right]^{k+1}=\sum_{m=0}^{\infty}(-1)^{m}\binom{k+1}{m}\left[\left(1+\left(\frac{x}{\beta}\right)\right)^{-\alpha}\right]^{m} \quad (11)$$

Making use of the result in (11) above in equation (10) and simplifying, we obtain:

$$f(x)=\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{\alpha\theta^{2}(-1)^{m}}{(k+1)\beta(\theta+1)}\binom{k+1}{m}\left(1+\left(\frac{x}{\beta}\right)\right)^{-\alpha\theta-\alpha m-1}+\frac{\alpha\theta^{2}}{\beta(\theta+1)}\left(1+\left(\frac{x}{\beta}\right)\right)^{-\alpha\theta-1} \quad (12)$$

Now, using the simplified pdf of the LinLomD in equation (12), the n^{th} ordinary moment of the LinLomD is derived as follows:

$$\mu_{n}^{'}=E\left[X^{n}\right]=\int_{0}^{\infty}x^{n}f(x)dx$$

Hence,

$$\mu_n^{'}=\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{\alpha\theta^2\left(-1\right)^m}{\left(k+1\right)\beta\left(\theta+1\right)}\binom{k+1}{m}\int_0^{\infty}x^n\left(1+\beta^{-1}x\right)^{-\alpha\theta-\alpha m-1}dx+\frac{\alpha\theta^2}{\beta\left(\theta+1\right)}\int_0^{\infty}x^n\left(1+\beta^{-1}x\right)^{-\alpha\theta-1}dx$$

(13)

Using integration by substitution in (13) above and substituting for x and dx in equation (13) and simplifying, we obtain

$$\mu_n^{'}=\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{\alpha\theta^2\left(-1\right)^{m+n}\beta^n}{\left(k+1\right)\left(\theta+1\right)}\binom{k+1}{m}\int_0^{\infty}y^{-\alpha(\theta+m)-1}\left(1-y\right)^{n+1-1}dy+\frac{\alpha\theta^2\left(-1\right)^n\beta^n}{\left(\theta+1\right)}\int_0^{\infty}y^{-\alpha\theta-1}\left(1-y\right)^{n+1-1}dy$$

(14)

Recall that:

$$B\left(x,y\right)=\int_0^{\infty}t^{x-1}\left(1-t\right)^{y-1}dt=\frac{\Gamma\left(x\right)\Gamma\left(y\right)}{\Gamma\left(x+y\right)}$$

Hence,

$$\mu_n^{'}=\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{\alpha\theta^2\left(-1\right)^{m+n}\beta^n}{\left(k+1\right)\left(\theta+1\right)}\binom{k+1}{m}\frac{\Gamma\left(-\alpha\theta-\alpha m\right)\Gamma\left(n+1\right)}{\Gamma\left(-\alpha\theta-\alpha m+n+1\right)}+\frac{\alpha\theta^2\left(-1\right)^n\beta^n}{\left(\theta+1\right)}\frac{\Gamma\left(-\alpha\theta\right)\Gamma\left(n+1\right)}{\Gamma\left(-\alpha\theta+n+1\right)} \quad (15)$$

The coefficient of variation, skewness and kurtosis can all be obtained from the non-central moments using the required relations.

3.2 Reliability analysis of the LinLomD.

The Survival function expresses likelihood that a system or an individual will not fail after a given time. It is mathematically given as:

$$S\left(x\right)=1-F\left(x\right) \quad (16)$$

Applying the *cdf* of the LinLomD in (5), the survival function for the LinLomD is obtained as:

$$S(x)=\left[1-\frac{\theta}{(\theta+1)}\left[\log\left(\left(1+\left(\frac{x}{\beta}\right)\right)^{-\alpha}\right)\right]\right]\left(1+\left(\frac{x}{\beta}\right)\right)^{-\alpha\theta} \quad (17)$$

The graph below shows different curves for the survival function of the LinLomD from different values of the parameters.

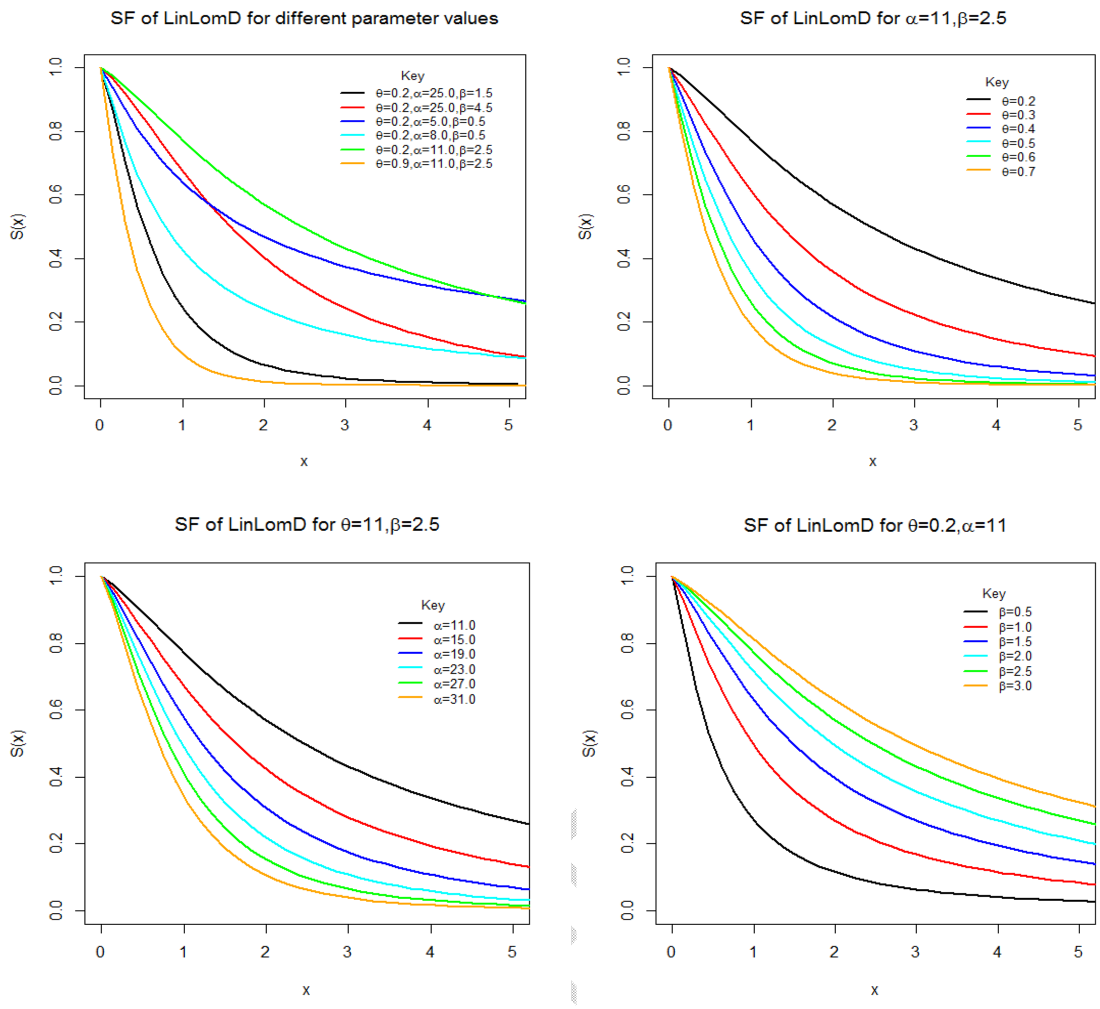

Figure 3: The survival function of the LinLomD.

From the figure above it can be seen that the probability of survival for any random variable following a Lindley-Lomax distribution decreases with time. This implies that the Lindley-Lomax distribution can applied in events with decreasing survival rates.

Hazard function is the probability that a component will fail or die for an interval of time. The hazard function is defined as;

$$h(x)=\frac{f(x)}{S(x)}=\frac{f(x)}{1-F(x)} \tag{18}$$

From the definition above, the hazard function of the LinLomD is given as;

$$h(x) = \frac{\frac{\alpha\theta^2}{\beta(\theta+1)}\left[1+\left(\frac{x}{\beta}\right)\right]^{-1}\left[1-\log\left(1+\left(\frac{x}{\beta}\right)\right)^{-\alpha}\right]}{\left[1-\frac{\theta}{(\theta+1)}\left[\log\left(\left(1+\left(\frac{x}{\beta}\right)\right)^{-\alpha}\right)\right]\right]} \qquad (19)$$

The figure below shows the shapes for the hazard function of the proposed distribution for different values of the model parameters.

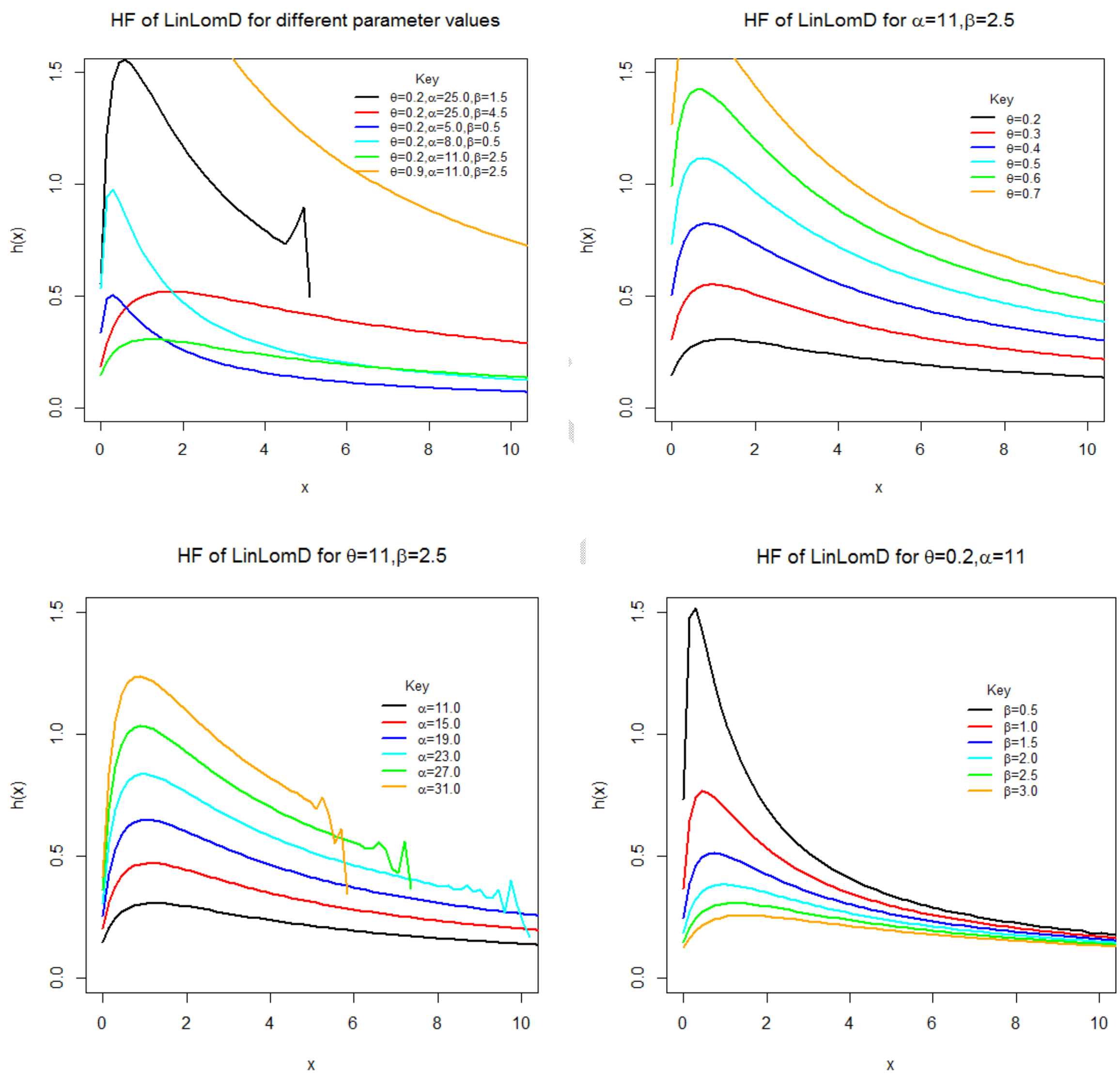

Figure 4: The hazard function of the LinLomD.

The figure above revealed that the probability of failure for any random variable following a Lindley-Lomax distribution increases with time, that is, as time passes, probability of death increases. This implies that the Lindley-Lomax distribution can be used to analyze random variables whose failure rate increases with time.

3.3 Order Statistics

Given that $X_1, X_2, \ldots\ldots, X_n$ is a random sample from a distribution with *pdf, f(x),* and $X_{1:n}, X_{2:n}, \ldots\ldots, X_{i:n}$ denote the corresponding order statistic derived from this sample. The *pdf*, $f_{i:n}(x)$ of the i^{th} order statistic can be defined as;

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} f(x) F(x)^{i-1} \left[1 - F(x)\right]^{n-i} \tag{20}$$

Taking *f(x)* and *F(x)* to be the *pdf* and *cdf* of the Lindley-Lomax distribution respectively and using (5) and (6), the *pdf* of the i^{th} order statistics $X_{i:n}$ for the LinLomD can be expressed from (20) as;

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} \sum_{k=0}^{n-i} (-1)^k \binom{n-i}{k} \left[\frac{\alpha\theta^2}{\beta(\theta+1)} \left[1 - \log\left(1+\left(\frac{x}{\beta}\right)\right)^{-\alpha}\right] \left(1+\left(\frac{x}{\beta}\right)\right)^{-\alpha(\theta+1)}\right] \left[1 - \left[1 - \frac{\theta}{(\theta+1)} \left[\log\left(\left(1+\left(\frac{x}{\beta}\right)\right)^{-\alpha}\right)\right]\right] \left(1+\left(\frac{x}{\beta}\right)\right)^{-\alpha\theta}\right]^{i+k-1} \tag{21}$$

Hence, the *pdf* of the minimum order statistic $X_{(1)}$ and maximum order statistic $X_{(n)}$ of the LinLomD are given by;

$$f_{1:n}(x) = n \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} \left[\frac{\alpha\theta^2}{\beta(\theta+1)} \left[1 - \log\left(1+\left(\frac{x}{\beta}\right)\right)^{-\alpha}\right] \left(1+\left(\frac{x}{\beta}\right)\right)^{-\alpha(\theta+1)}\right] \left[1 - \left[1 - \frac{\theta}{(\theta+1)} \left[\log\left(\left(1+\left(\frac{x}{\beta}\right)\right)^{-\alpha}\right)\right]\right] \left(1+\left(\frac{x}{\beta}\right)\right)^{-\alpha\theta}\right]^{k} \tag{22}$$

and

$$f_{n:n}(x) = n \left[\frac{\alpha\theta^2}{\beta(\theta+1)} \left[1 - \log\left(1+\left(\frac{x}{\beta}\right)\right)^{-\alpha}\right] \left(1+\left(\frac{x}{\beta}\right)\right)^{-\alpha(\theta+1)}\right] \left[1 - \left[1 - \frac{\theta}{(\theta+1)} \left[\log\left(\left(1+\left(\frac{x}{\beta}\right)\right)^{-\alpha}\right)\right]\right] \left(1+\left(\frac{x}{\beta}\right)\right)^{-\alpha\theta}\right]^{n-1} \tag{23}$$

respectively.

4 Estimation of Parameters of the Lindley-Lomax Distribution

Let $X_1, X_2, \ldots\ldots\ldots, X_n$ be a random sample of size 'n' independently and identically distributed random variables from the LinLomD with unknown parameters *α, β, a,* and *b* defined previously. The *pdf* of the LinLomD is given as

$$f(x) = \frac{\alpha\theta^2}{\beta(\theta+1)} \left[1+\left(\frac{x}{\beta}\right)\right]^{-(\alpha+1)} \left[1 - \log\left(1+\left(\frac{x}{\beta}\right)\right)^{-\alpha}\right] \left(1+\left(\frac{x}{\beta}\right)\right)^{-\alpha(\theta-1)}$$

The likelihood function is given by:

$$L\left(X/\alpha,\beta,\theta\right)=\left(\frac{\alpha\theta^{2}}{\beta\left(\theta+1\right)}\right)^{n}\prod_{i=1}^{n}\left\{\left[1+\left(\frac{x_i}{\beta}\right)\right]^{-(\alpha+1)}\left[1-\log\left(1+\left(\frac{x_i}{\beta}\right)\right)^{-\alpha}\right]\left(1+\left(\frac{x_i}{\beta}\right)\right)^{-\alpha(\theta-1)}\right\} \tag{24}$$

Let the log-likelihood function, $l=\log L\left(X/\alpha,\beta,\theta\right)$, therefore

$$l=n\log\alpha-n\log\beta+2n\log\theta-n\log(\theta+1)-(\alpha+1)\sum_{i=1}^{n}\log\left(1+\left(\frac{x_i}{\beta}\right)\right)-\alpha(\theta-1)\sum_{i=1}^{n}\log\left(1+\left(\frac{x_i}{\beta}\right)\right)+\sum_{i=1}^{n}\log\left[1-\log\left(1+\left(\frac{x_i}{\beta}\right)\right)^{-\alpha}\right]$$

(25)

Differentiating l partially with respect to α , β *and* θ respectively gives;

$$\frac{\partial l}{\partial\alpha}=\frac{n}{\alpha}-\sum_{i=1}^{n}\log\left(1+\left(\frac{x_i}{\beta}\right)\right)-(\theta-1)\sum_{i=1}^{n}\log\left(1+\left(\frac{x_i}{\beta}\right)\right)+\sum_{i=1}^{n}\left\{\frac{\left(1+\left(\frac{x_i}{\beta}\right)\right)^{-\alpha}\ln\left(1+\left(\frac{x_i}{\beta}\right)\right)}{\left[1-\log\left(1+\left(\frac{x_i}{\beta}\right)\right)^{-\alpha}\right]\left(1+\left(\frac{x_i}{\beta}\right)\right)^{-\alpha}}\right\} \tag{26}$$

$$\frac{\partial l}{\partial\beta}=-\frac{n}{\beta}+\frac{\alpha+1}{\beta^{2}}\sum_{i=1}^{n}x_i\left(1+\left(\frac{x_i}{\beta}\right)\right)^{-1}+\sum_{i=1}^{n}\left\{x_i\frac{\left(1+\left(\frac{x_i}{\beta}\right)\right)^{-\alpha-1}}{\left[1-\log\left(1+\left(\frac{x_i}{\beta}\right)\right)^{-\alpha}\right]\left(1+\left(\frac{x_i}{\beta}\right)\right)^{-\alpha}}\right\} \tag{27}$$

$$+\frac{\alpha(\theta-1)}{\beta^{2}}\sum_{i=1}^{n}x_i\left(1+\left(\frac{x_i}{\beta}\right)\right)^{-1}$$

$$\frac{\partial l}{\partial\theta}=\frac{2n}{\theta}-\frac{n}{(\theta+1)}-\alpha\sum_{i=1}^{n}\log\left(1+\left(\frac{x_i}{\beta}\right)\right) \tag{28}$$

Equating (26), (27) and (28) to zero and finding the values of the parameters from equations will produce the maximum likelihood estimators of parameters $\alpha,\beta-and-\theta$ respectively. But these estimators cannot be gotten analytically except numerically by using computer programmes like Python, R, SAS, e.t.c when data sets are available.

5. Application

This section presents a dataset, its' descriptive statistics, graphics and applications to some selected generalizations of the Lomax distribution. We have compared the adequacy of the LinLomD to other four generalizations of the Lomax model which include the Power Lomax distribution (PoLomD), transmuted Power Lomax distribution (TrPoLomD), Weibull-Lomax distribution (WeiLomD), and Lomax distribution (LomD).

To identify the most efficient or most fitted distribution to the MTCHIVTR dataset, the following model selection criteria were used which include the value of the log-likelihood function evaluated at the MLEs (ℓ), Akaike Information Criterion, *AIC*, Consistent Akaike Information Criterion, *CAIC*, Bayesian Information Criterion, *BIC*, Hannan Quin Information Criterion, *HQIC*, Anderson-Darling (A*), Cramèr-Von Mises (W*) and Kolmogorov-smirnov (K-S) statistics. More about the statistics A*, W* and K-S can be seen in [21]. Some of these statistics are computed using the following formulae:

$$AIC = -2\ell + 2k, \; BIC = -2\ell + k\log(n), \; CAIC = -2\ell + \frac{2kn}{(n-k-1)} \text{ and } HQIC = -2\ell + 2k\log\left[\log(n)\right]$$

Where ℓ denotes the value of log-likelihood function evaluated at the *MLEs*, *k* is the number of model parameters and *n* is the sample size. Decisively, the distribution with the lowest values of these criteria is considered to be the most fitted model to the dataset. Also, all the required computations are performed using the R package “AdequacyModel”.

5.1 Application to Mother-to-Child HIV Transmission Rate (MTCHIVTR)

This section presents a dataset on the rate of mother-to-child transmission of HIV (Human Immunodeficiency Virus) in Nigeria from the year 2000 to the year 2019. This data has been used by [22] and [23]. The descriptive statistics and graphical summary of the dataset is also presented.

The mother-to-child HIV transmission rate per 1,000 of population in Nigeria between 2000 and 2019 can be obtained from: www.data.unicef.org

The following table and figures present a critical exploration of the above dataset with some important discussions:

Table 1: Descriptive Statistics for the dataset

parameters	n	Minimum	Q_1	Median	Q_3	Mean	Maximum	Variance	Skewness	Kurtosis
Values	20	19.44	22.14	30.44	36.76	29.23	37.35	47.55	-0.18919	-1.55278

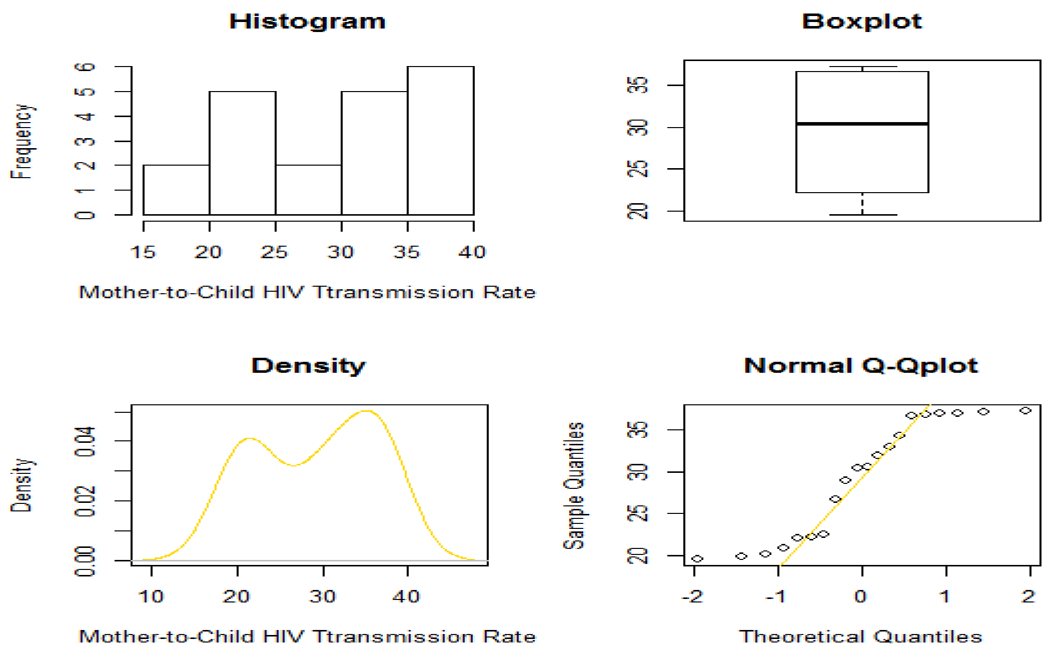

Figure 5.1: A graphical summary of the dataset

Following the summary of the descriptive statistics in table 1 and the histogram, box plot, density and normal Q-Q plot generally referred to as graphical summary in figure 5.1 above, it is seen that the rate of transmission of HIV from mother to child is bimodal and approximately normally distributed.

The following figure shows the trend in the rate of mother-to-child HIV transmission from 2000 to 2019 using a bar chart.

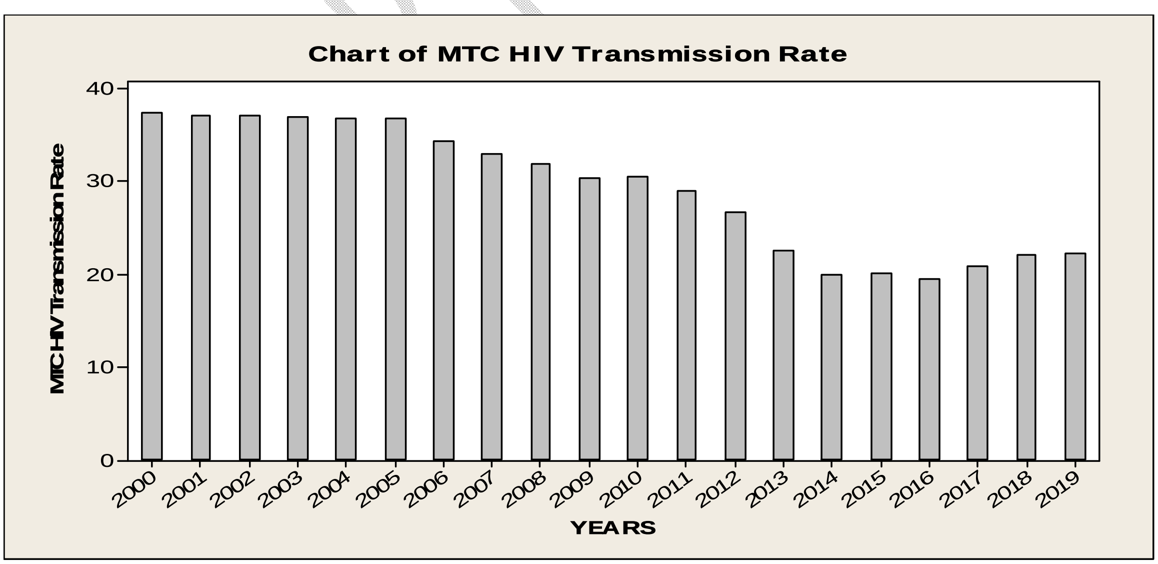

Figure 5.2: A Bar chart showing the Trend of Mother-to-child HIV Transmission Rate in Nigeria from 2000 to 2019

After checking the distribution of the dataset in figure 5.1, the bar chart in figure 5.2 above reveals the trend in the rate of mother-to-child transmission of HIV which indicates that mother-to-child HIV transmission was a very big problem from the year 2000 to 2005 with a non-decreasing rate. Meanwhile, there came a slightly decreasing trend in the rate of HIV transmission from mother to child as from the year 2006 to 2014, however, what we have from the year 2015 to 2019 is certainly an increasing pattern in the rate of mother-to-child transmission of HIV which suggests that more efforts need to be put in place to adequately reduce or eradicate the increasing rate of mother-to-child HIV transmission in Nigeria.

Considering the increasing rate of mother-to-child HIV transmission and the flexibility of the proposed distribution, this study fits the Lindley-Lomax distribution (LinLomD) to the above dataset in comparison with other existing probability distributions such as transmuted Power Lomax distribution (TrPoLomD), Weibull-Lomax distribution (WeiLomD), power Lomax distribution (POLomD) and the conventional Lomax distribution (LomD).

Note: In decision making, the model with the lowest values for these statistics would be chosen as the best fitted model.

Tables 2 lists the Maximum Likelihood Estimates of the model parameters, table 3 presents the statistics AIC, CAIC, BIC and HQIC while A*, W* and K-S for the fitted models are given in Table 4 as follows:

Table 2: Maximum Likelihood Parameter Estimates for the dataset

Distribution	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$
LinLomD	8.320432	8.312986	1.209011	-
WeiLomD	2.9761195	0.4075816	8.3461177	4.8447582
TrPoLomD	9.0904067	0.8106500	1.0968280	-0.9408202
PoLomD	8.1655634	0.6314789	0.9633156	-
LomD	-	0.7510835	9.8063883	-

Table 3: The statistics ℓ, AIC, CAIC, BIC and HQIC based on the dataset used

Distribution	$\hat{\ell}$	AIC	CAIC	BIC	HQIC	Ranks
LinLomD	68.7287	143.4574	144.9574	146.4446	144.0405	1st
WeiLomD	70.8238	149.6476	152.3143	153.6305	150.4251	2nd
TrPoLomD	93.35123	194.7025	197.3691	198.6854	195.48	3rd
PoLomD	100.4202	206.8404	208.3404	209.8276	207.4235	4th
LomD	99.22126	202.4425	203.1484	204.434	202.8313	5th

Table 4: The A^*, W^*, K-S statistic and P-values based on the dataset used.

Distribution	A^*	W^*	K-S	P-Value (K-S)	Ranks
LinLomD	1.0904	0.1658645	0.19006	0.4137	1^{st}
WeiLomD	0.9804542	0.1367811	0.32782	0.02048	2^{nd}
TrPoLomD	1.107228	0.1702136	0.45508	0.0002642	3^{rd}
PoLomD	1.099973	0.168381	0.51402	1.969e-05	4^{th}
LomD	1.099407	0.1682194	0.55989	1.951e-06	5^{th}

The following figure presents a histogram and estimated densities and cdfs of the fitted models to the dataset.

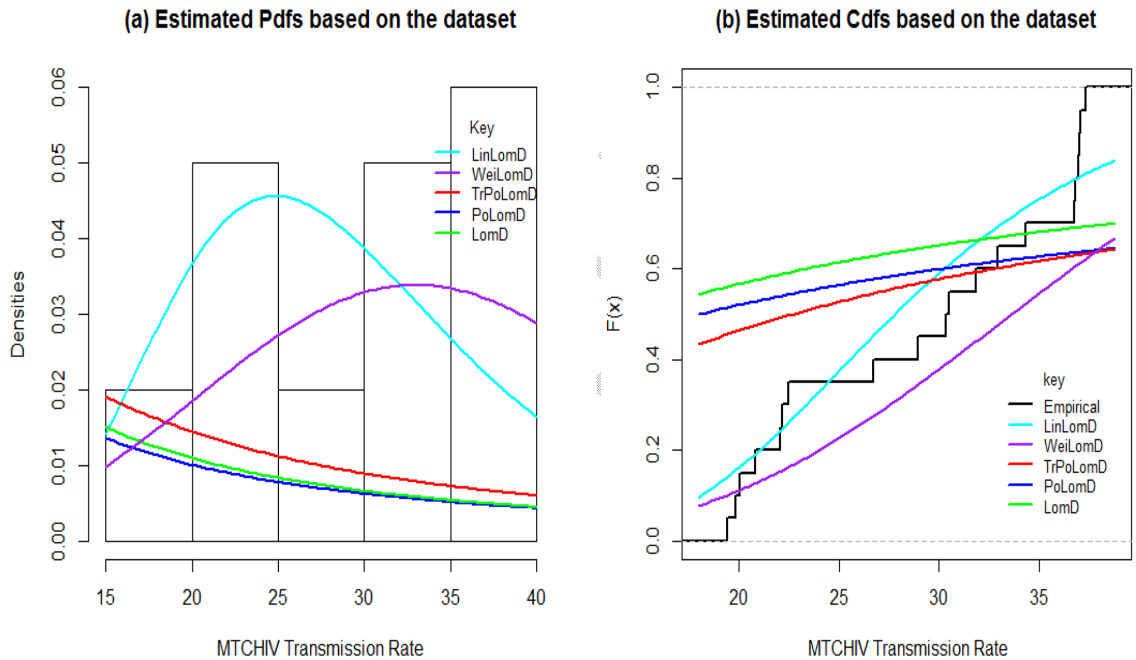

Figure 5.3: Histogram and plots of the estimated densities and cdfs of the fitted distributions to the dataset.

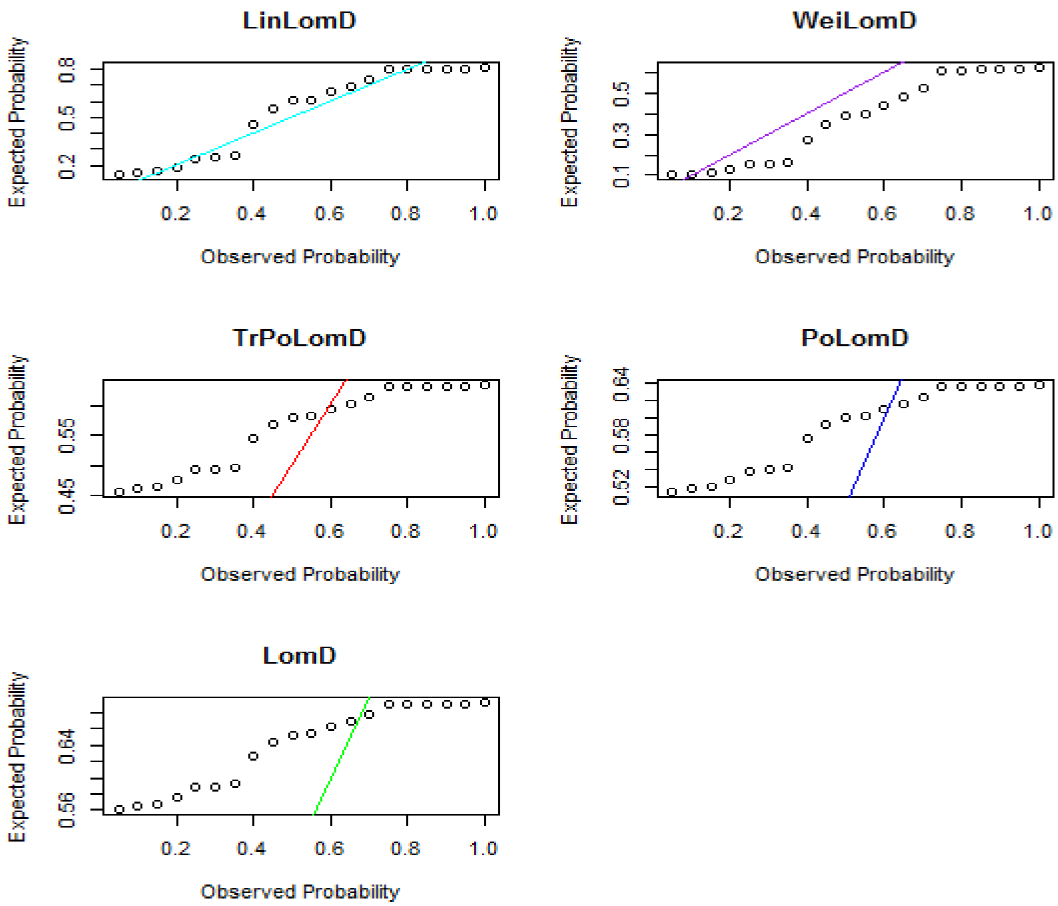

Figure 5.4: Probability plots for the six fitted distributions based on the MTCHIVTR dataset.

Table 2 lists the MLEs of the parameters for the fitted models to the mother-to-child HIV transmission rate dataset also referred to as the dataset. The values of the statistics AIC, CAIC and BIC for the fitted distributions are also listed in Tables 3 for mother-to-child HIV transmission rate data. Based on the dataset and the values of the model selection measures in table 3 and 4, it is clear that the proposed Lindley-Lomax distribution (LinLomD) with three parameters provides the best fit compared to the transmuted Power Lomax distribution *(TrPoLomD)*, Weibull-Lomax distribution (WeiLomD), power Lomax distribution (POLomD) and the conventional Lomax distribution (LomD). Also, the estimated pdfs and cdfs shown in figure 5.3 as well as the probability plots presented in figure 5.4 clearly support and confirm the results in Tables 3 and 4 as stated above.

6 Conclusion

This paper proposed a new extension of the Lomax distribution called Lindley-Lomax distribution. It proposed some mathematical properties of the distribution such as its ordinary moments, moment generating and characteristics functions. It also considered the reliability analysis of the distribution which as the survival and hazard functions as well as the distribution of ordered statistics of the new distribution. Some plots of the distribution revealed that it is a flexible and skewed distribution. The model parameters have been estimated using the method of maximum likelihood estimation. The implications of the plots for the survival function indicate that the Lindley-Lomax distribution could be used to model time or age-dependent events, where survival rate decreases with time or age. The performance of the new distribution is illustrated by an application of the model to a real life dataset on the rate of mother-to-child transmission of HIV in Nigeria. The results showed that the new distribution, Lindley-Lomax distribution performs better than the transmuted Power Lomax distribution *(TrPoLomD)*, Weibull-Lomax distribution (WeiLomD), power Lomax distribution (POLomD) and the conventional Lomax distribution (LomD) as shown in the analysis of the data in chapter four. This performance of our model is an indication that the proposed model will be useful for describing other real life situations.

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