Original Research Article

BAYESIAN ESTIMATION OF A SCALE PARAMETER OF THE GUMBEL-LOMAX DISTRIBUTION USING INFORMATIVE AND NON INFORMATIVE PRIORS

Abstract

Estimating the scale parameter of the Gumbel-Lomax Distribution using the Bayesian method of estimation and evaluating the estimators by assuming two non-informative prior distributions and one informative prior distribution is very important for the general application of the Gumbel-Lomax distribution. These estimators are obtained using the squared error loss function (*SELF*), Quadratic loss function (*QLF*) and precautionary loss function (*PLF*). The posterior distributions of the scale parameter of the Gumbel-Lomax distribution are derived and the Estimators are also obtained using the above mentioned priors and loss functions. Furthermore, a simulation using a package in R software is carried out to assess the performance of the estimators by making use of the Mean Squared Errors of the Estimators under the Bayesian approach and Maximum likelihood method. Our results show that Bayesian Method using *PLF* under all priors produces the best estimators of the scale parameter compared to estimators using the Maximum Likelihood method, *SELF* and *QLF* under all the priors irrespective of the values of the parameters and the different sample sizes. It is also discovered that the other parameters have no effect on the estimators of the scale parameter.

Keywords: Gumbel-Lomax distribution, Bayesian Method, Priors, Loss functions, MLE, Simulation, MSE.

1. Introduction

Research has shown that there are many classical probability distributions proposed and applied in the previous decades for modeling real life datasets however it has been discovered that some of these distributions do not analyze some of these skewed datasets appropriately and hence generating a problem in statistical theory and applications. Over the past years, several compound probability distributions have been studied in the literature for modeling real life situations and most of these compound distributions are found to be skewed, flexible and perform better in statistical modeling compared to the classical distributions ([1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13]).

Considering the inventions above, [14] worked on an extension of the Lomax distribution known as Gumbel-Lomax distribution (GuLD) with four parameters. Some useful properties of this distribution have been discussed with applications to a real life datasets. The authors found that the GuLD is flexible and fitted the datasets better than other extensions of the Lomax distribution such as gamma-Lomax distribution [16], exponentiated-Lomax distribution [17], beta-Lomax distribution [18] and the conventional Lomax distributions based on the research study [14].

Other features of the GuLD could be obtained from [14]. Following the importance of the GuLD in modeling real life situations, it is therefore necessary for us to research and find the best method in estimating the scale parameter of the GuLD which will remain useful for practical applications of this model.

The two basic methods of parameter estimation are the classical and the non classical methods. The classical method of estimation considers the parameters to be constant but unknown whereas the parameters are taken to be unknown and random just like variables under non classical approach. Maximum likelihood estimation is the most widely used classical method while the Bayesian estimation method is a non classical theory. However, in most real life situations modeled by life time distributions, the model parameters cannot be referred to as fixed in the entire duration of life testing ([19], [20], [21]). With the above facts, it is clear that the classical approach cannot adequately solve problems of parameter estimation in life time models and hence the need for Bayesian estimation in life time models.

Estimating the parameters of model is very important and each parameter in a model is best obtained by a particular method and hence, this study aims at estimating one scale parameter of the GuLD using Bayesian approach and evaluating the estimators in comparison with estimators obtained by the method of maximum likelihood estimation.

The remaining sections of this article are arranged as follows: maximum likelihood estimation for the scale parameter is presented in section 2. Bayesian estimation of the scale parameter based on the three loss functions and assumption of three prior distributions (uniform, Jeffrey's and gamma) is done in section 3. The estimators in section 2 and 3 are evaluated in relation to their mean squared error (MSE) in Section 4. Lastly, the conclusion is provided in Section 5.

2. Maximum Likelihood Estimation

Given that $X_1, X_2, ..., X_n$ is a random sample from a population X of size 'n' independently and identically distributed random variables with probability density function $f(x)$. Given that the values, $\underline{x} = (x_1, x_2, ..., x_n)$ are obtained independently from a GuLD with unknown parameters α, β, θ and λ, the likelihood function is given by:

$$L(\underline{x} \mid \alpha, \beta, \theta, \lambda) = P(x_1, x_2, ..., x_n \mid \alpha, \beta, \theta, \lambda) = \prod_{i=1}^{n} P(\underline{x} \mid \alpha, \beta, \theta, \lambda) \tag{1}$$

The likelihood function, $L(\underline{x} \mid \alpha, \beta, \theta, \lambda)$ based on the pdf of GuLD is defined to be the joint density of the random variables $x_1, x_2, ..., x_n$ and it is given as:

$$L(X \mid \alpha, \beta, \theta, \lambda) \propto \left(\frac{\alpha\lambda}{\beta\theta}\right)^n \prod_{i=1}^{n}\left(\left(1+\frac{x_i}{\beta}\right)^{-\frac{\alpha}{\theta}-1}\left[1-\left(1+\frac{x_i}{\beta}\right)^{\alpha}\right]^{-\left(\frac{1}{\theta}+1\right)}\right)\exp\left\{-\lambda\sum_{i=1}^{n}\left[\left(1+\frac{x_i}{\beta}\right)^{\alpha}-1\right]^{-\frac{1}{\theta}}\right\} \tag{2}$$

For the scale parameter of the GuLD, λ, the likelihood function is given by;

$$L(X \mid \lambda) \propto \eta\lambda^n \exp\left\{-\lambda\sum_{i=1}^{n}\left[\left(1+\frac{x_i}{\beta}\right)^{\alpha}-1\right]^{-\frac{1}{\theta}}\right\}$$

$$L\left(X\mid\lambda\right)\propto\eta\lambda^{n}\exp\left\{-\lambda\sum_{i=1}^{n}\left[\left(1+\frac{x_i}{\beta}\right)^{\alpha}-1\right]^{-\frac{1}{\theta}}\right\} \tag{3}$$

Where $\eta=\left(\frac{\alpha}{\beta\theta}\right)^{n}\prod_{i=1}^{n}\left(\left(1+\frac{x_i}{\beta}\right)^{-\frac{\alpha}{\theta}-1}\left[1-\left(1+\frac{x_i}{\beta}\right)^{\alpha}\right]^{-\left(\frac{1}{\theta}+1\right)}\right)$ is a constant which is independent of the scale parameter, λ .

Let the log-likelihood function, $l=\log L\left(\underline{x}\mid\lambda\right)$, therefore

$$l=\log L\left(\underline{x}\mid\lambda\right)=n\log\lambda-\lambda\sum_{i=1}^{n}\left[\left(1+\frac{x_i}{\beta}\right)^{\alpha}-1\right]^{-\frac{1}{\theta}} \tag{4}$$

Taking the partial derivative of $\boldsymbol{l}$ with respect to λ gives;

$$\frac{\partial l}{\partial\lambda}=\frac{n}{\lambda}-\sum_{i=1}^{n}\left[\left(1+\frac{x_i}{\beta}\right)^{\alpha}-1\right]^{-\frac{1}{\theta}}$$

Simplifying and working for $\hat{\lambda}$ gives the result below;

$$\Rightarrow\hat{\lambda}=n\left(\sum_{i=1}^{n}\left[\left(1+\frac{x_i}{\beta}\right)^{\alpha}-1\right]^{-\frac{1}{\theta}}\right)^{-1} \tag{5}$$

where $\hat{\lambda}$ is the maximum likelihood estimator of the scale parameter, λ. More information about the estimation of the other three parameters of the GuLD can be seen in [14].

3. Bayesian Estimation

Bayesian estimation uses appropriate choice of prior(s) for estimating each parameter. According to the Bayesian estimation theory, no prior distribution for a parameter is considered the best until it is tested and validated. Also, most prior distributions are selected according to one's subjective knowledge and beliefs. Hence, if one has enough knowledge of the parameter(s), it is wise to select an informative prior(s); otherwise, it is better to consider non-informative prior(s). For this research, we have selected two non-informative priors (uniform and Jeffrey) and an informative prior (gamma). These assumed prior distributions have been used widely by several authors such as, [22], [23], [24], [25], [26], [27], [28], [29]. This research has considered three loss functions which are squared error, quadratic and precautionary loss functions. These loss functions have also been used by other authors including; [30], [31], [32], [33], [34], [35], [36], [37], [38], [39] and [40] etc. The definitions of the above listed loss functions is presented as follows:

a. The uniform prior is defined as:

$$p(\lambda) \propto 1; 0 < \lambda < \infty \quad (6)$$

b. Also, the Jeffrey's prior is defined as:

$$p(\lambda) \propto \frac{1}{\lambda}; 0 < \lambda < \infty \quad (7)$$

c. Also, the gamma prior is defined as:

$$P(\lambda) = \frac{a^b}{\Gamma(b)} \lambda^{b-1} e^{-a\lambda} \quad (8)$$

i. Squared Error Loss Function

The squared error loss function relating to the scale parameter λ is defined as:

$$L(\lambda, \lambda_{SELF}) = (\lambda - \lambda_{SELF})^2 \quad (9)$$

where λ_{SELF} is the estimator of the parameter λ under *SELF*.

ii. Quadratic Loss Function

The quadratic loss function is defined from [41] as

$$L(\lambda, \lambda_{QLF}) = \left(\frac{\lambda - \lambda_{QLF}}{\lambda} \right)^2 \quad (10)$$

where λ_{QLF} is the estimator of the parameter λ under *QLF*.

iii. Precautionary Loss Function

The *PLF* introduced by [42] is an asymmetric loss function and is defined as

$$L(\lambda_{PLF}, \lambda) = \frac{(\lambda_{PLF} - \lambda)^2}{\lambda_{PLF}} \quad (11)$$

where λ_{PLF} is the estimator of the scale parameter λ under *PLF*.

The posterior distribution of a parameter is obtained as follows:

$$P(\lambda \mid \underline{x}) = \frac{P(\lambda, \underline{x})}{P(\underline{x})} = \frac{P(\underline{x} \mid \lambda) P(\lambda)}{P(\underline{x})} = \frac{P(\underline{x} \mid \lambda) P(\lambda)}{\int P(\underline{x} \mid \lambda) P(\lambda) d\lambda} = \frac{L(\underline{x} \mid \lambda) P(\lambda)}{\int L(\underline{x} \mid \lambda) P(\lambda) d\lambda} \quad (12)$$

where $P(\underline{x})$ is the marginal distribution of X and $P(\underline{x}) = \sum_{x}^{\infty} p(\lambda) L(\underline{x} \mid \lambda)$ when the prior distribution of λ is discrete and $P(\underline{x}) = \int_{-\infty}^{\infty} p(\lambda) L(\underline{x} \mid \lambda) d\lambda$ when the prior distribution of λ is continuous. Also note that $p(\lambda)$ and $L(\underline{x} \mid \lambda)$ are the prior distribution and the Likelihood function respectively.

3.1 Bayesian Analysis under Uniform Prior with Three Loss Functions

He we present a derivation of the posterior distribution for the scale parameter λ for a sample of observations by assuming a uniform prior distribution. This is obtained from equation (12) using integration by substitution as follows:

$$P\left(\lambda\,|\,\underline{x}\right)=\frac{\lambda^{n}\left[\sum_{i=1}^{n}\left[\left(1+\frac{x_i}{\beta}\right)^{\alpha}-1\right]^{-\frac{1}{\theta}}\right]^{n+1}}{\Gamma\left(n+1\right)\mathrm{e}^{\lambda\sum_{i=1}^{n}\left[\left(1+\frac{x_i}{\beta}\right)^{\alpha}-1\right]^{-\frac{1}{\theta}}}} \tag{13}$$

Based on calculations and algebra, the Bayes estimators with uniform prior using *SELF*, *QLF* and *PLF* are derived respectively as:

$$\lambda_{SELF}=\left(n+1\right)\left[\sum_{i=1}^{n}\left[\left(1+\frac{x_i}{\beta}\right)^{\alpha}-1\right]^{-\frac{1}{\theta}}\right]^{-1} \tag{14}$$

$$\lambda_{QLF}=\frac{\left(n-1\right)}{\left[\sum_{i=1}^{n}\left[\left(1+\frac{x_i}{\beta}\right)^{\alpha}-1\right]^{-\frac{1}{\theta}}\right]} \tag{15}$$

and

$$\lambda_{PLF}=\left[\left(n+1\right)\left(n+2\right)\right]^{\frac{1}{2}}\left[\sum_{i=1}^{n}\left[\left(1+\frac{x_i}{\beta}\right)^{\alpha}-1\right]^{-\frac{1}{\theta}}\right]^{-1} \tag{16}$$

3.2 Bayesian Analysis under Jeffrey's Prior with Three Loss Functions

He we present a derivation of the posterior distribution for the scale parameter λ for a sample of observations by assuming a Jeffrey's prior distribution. This is obtained from equation (12) using integration by substitution as follows::

$$P\left(\lambda\,|\,\underline{x}\right)=\frac{\left[\sum_{i=1}^{n}\left[\left(1+\frac{x_i}{\beta}\right)^{\alpha}-1\right]^{-\frac{1}{\theta}}\right]^{n}\lambda^{n-1}\mathrm{e}^{-\lambda\sum_{i=1}^{n}\left[\left(1+\frac{x_i}{\beta}\right)^{\alpha}-1\right]^{-\frac{1}{\theta}}}}{\Gamma\left(n\right)} \tag{17}$$

Again the Bayes estimators under Jeffrey's prior using *SELF*, *QLF* and *PLF* are given respectively as:

$$\lambda_{SELF}=E\left(\lambda\,|\,\underline{x}\right)=n\left[\sum_{i=1}^{n}\left[\left(1+\frac{x_i}{\beta}\right)^{\alpha}-1\right]^{-\frac{1}{\theta}}\right]^{-1} \tag{18}$$

$$\lambda_{QLF}=\frac{\left(n-2\right)}{\sum_{i=1}^{n}\left[\left(1+\frac{x_i}{\beta}\right)^{\alpha}-1\right]^{-\frac{1}{\theta}}} \tag{19}$$

and

$$\lambda_{PLF} = \left[n(n+1)\right]^{\frac{1}{2}} \left[\sum_{i=1}^{n} \left[\left(1+\frac{x_i}{\beta}\right)^{\alpha} - 1 \right]^{-\frac{1}{\theta}} \right]^{-1} \tag{20}$$

3.3 Bayesian Analysis under Gamma Prior with Three Loss Functions

He we present a derivation of the posterior distribution for the scale parameter λ for a sample of observations by assuming a gamma prior distribution. This is obtained from equation (12) using integration by substitution as follows:

$$p(\lambda \mid \underline{x}) = \frac{\lambda^{n+b-1} \left(a + \sum_{i=1}^{n} \left[\left(1+\frac{x_i}{\beta}\right)^{\alpha} - 1 \right]^{-\frac{1}{\theta}} \right)^{n+b} \mathrm{e}^{-\lambda \left(a + \sum_{i=1}^{n} \left[\left(1+\frac{x_i}{\beta}\right)^{\alpha} - 1 \right]^{-\frac{1}{\theta}} \right)}}{\Gamma(n+b)} \tag{21}$$

Also the Bayes estimators under gamma prior using *SELF*, *QLF* and *PLF* are given respectively as:

$$\lambda_{SELF} = E(\lambda \mid \underline{x}) = (n+b) \left[a + \sum_{i=1}^{n} \left[\left(1+\frac{x_i}{\beta}\right)^{\alpha} - 1 \right]^{-\frac{1}{\theta}} \right]^{-1} \tag{22}$$

$$\lambda_{QLF} = \frac{(n+b-2)}{a + \sum_{i=1}^{n} \left[\left(1+\frac{x_i}{\beta}\right)^{\alpha} - 1 \right]^{-\frac{1}{\theta}}} \tag{23}$$

and

$$\lambda_{PLF} = \left[(n+b)(n+b+1)\right]^{\frac{1}{2}} \left[a + \sum_{i=1}^{n} \left[\left(1+\frac{x_i}{\beta}\right)^{\alpha} - 1 \right]^{-\frac{1}{\theta}} \right]^{-1} \tag{24}$$

4. Results and Discussions

This section presents the Monte Carlo simulation study using R software with 10,000 replications and random samples of sizes n = (23, 77, 126, 200) based on the GuLD. The research has considered the inverse transformation method of simulation using the quantile function under the following combination of parameter values: $\alpha = 0.8, \beta = 0.5, \theta = 0.4, \lambda = 0.9, a = 1.0$ and $b = 1.0$; $\alpha = 1.8, \beta = 0.5, \theta = 0.4, \lambda = 0.9, a = 1.0$ and $b = 1.0$; $\alpha = 0.8, \beta = 1.5, \theta = 0.4, \lambda = 0.9, a = 1.0$ and $b = 1.0$; $\alpha = 0.8, \beta = 0.5, \theta = 1.4, \lambda = 0.9, a = 1.0$ and $b = 1.0$ $\alpha = 0.8, \beta = 0.5, \theta = 0.4, \lambda = 0.9, a = 2.5$ and $b = 1.0$ and $\alpha = 0.8, \beta = 0.5, \theta = 0.4, \lambda = 0.9, a = 1.0$ and $b = 2.5$. The following tables present the results of our simulation study by listing the average estimates of the shape parameter with their respective Mean Square Errors (MSEs) derived from the considered methods of estimation which include the Maximum Likelihood Estimation, *SELF*, *QLF*, and *PLF* with Uniform, Jeffrey and gamma priors respectively. The criterion for evaluating the performance of the estimators in this study is the Mean Square Error (MSE): $MSE = \frac{1}{n} E\left(\hat{\lambda} - \lambda\right)^2$.

Table 1: Average Estimates (Estimates) and Mean Squared Errors (MSEs) of the estimated scale parameter ($\hat{\lambda}$) for $\alpha = 0.8, \beta = 0.5, \theta = 0.4, \lambda = 0.9, a = 1.0$ and $b = 1.0$ under three different priors and loss functions with varying sample sizes.

n	Measures	MLE	Uniform Prior			Jeffrey's Prior			Gamma Prior		
			SELF	QLF	PLF	SELF	QLF	PLF	SELF	QLF	PLF
23	Estimate	0.7213	0.7527	0.6899	0.7682	0.7213	0.6586	0.7368	0.7292	0.6685	0.7443
	MSE	0.0449	0.0358	0.0560	0.0320	0.0449	0.0691	0.0401	0.0416	0.0641	0.0372
77	Estimate	0.7077	0.7168	0.6985	0.7214	0.7077	0.6893	0.7122	0.7103	0.6920	0.7148
	MSE	0.0417	0.0384	0.0452	0.0368	0.0417	0.0489	0.0401	0.0407	0.0477	0.0390
126	Estimate	0.7056	0.7112	0.700	0.7140	0.7056	0.6944	0.7084	0.7073	0.6961	0.7100
	MSE	0.0408	0.0387	0.043	0.0377	0.0408	0.0452	0.0398	0.0402	0.0445	0.0392
200	Estimate	0.7043	0.7078	0.7007	0.7095	0.7043	0.6972	0.7060	0.7053	0.6983	0.7070
	MSE	0.0404	0.0390	0.0418	0.0384	0.0404	0.0431	0.0397	0.0400	0.0427	0.0393

The results in Table 1 above indicates that the estimators obtained by *PLF* are better than the ones obtained from the other estimators with both Uniform and Jeffrey priors having lower values of MSE even when the sample sizes are changed. Therefore one can state that the Bayesian estimation (with *PLF* for both the Uniform and Jeffrey prior) of this parameter is better than Method of Maximum Likelihood estimation (*MLE*) for the selected parameter values despite the variations in the sample sizes.

Table 2: Average Estimates (Estimates) and Mean Squared Errors (MSEs) of the estimated scale parameter ($\hat{\lambda}$) for $\alpha = 1.8, \beta = 0.5, \theta = 0.4, \lambda = 0.9, a = 1.0$ and $b = 1.0$ under three different priors and loss functions with varying sample sizes.

n	Measures	MLE	Uniform Prior			Jeffrey's Prior			Gamma Prior		
			SELF	QLF	PLF	SELF	QLF	PLF	SELF	QLF	PLF
23	Estimate	0.7213	0.7527	0.6899	0.7682	0.7213	0.6586	0.7368	0.7292	0.6685	0.7443
	MSE	0.0449	0.0358	0.0560	0.0320	0.0449	0.0691	0.0401	0.0416	0.0641	0.0372
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	MSE	0.0417	0.0384	0.0452	0.0368	0.0417	0.0489	0.0401	0.0407	0.0477	0.0390
126	Estimate	0.7056	0.7112	0.700	0.7140	0.7056	0.6944	0.7084	0.7073	0.6961	0.7100
	MSE	0.0408	0.0387	0.043	0.0377	0.0408	0.0452	0.0398	0.0402	0.0445	0.0392
200	Estimate	0.7043	0.7078	0.7007	0.7095	0.7043	0.6972	0.7060	0.7053	0.6983	0.7070
	MSE	0.0404	0.0390	0.0418	0.0384	0.0404	0.0431	0.0397	0.0400	0.0427	0.0393

Based on the values in Table 2 which are related to those in Table 1 with the same lower values of MSE for the estimators using PLF under all the priors after increasing the value of α from 0.8

to 1.8. This result indicates that changing the value of α does not affect the estimate of the scale parameter, λ .

Table 3: Average Estimates (Estimates) and Mean Squared Errors (MSEs) of the estimated scale parameter ($\hat{\lambda}$) for $\alpha = 0.8, \beta = 1.5, \theta = 0.4, \lambda = 0.9, a = 1.0$ and $b = 1.0$ under three different priors and loss functions with varying sample sizes.

n	**Measures**	**MLE**	**Uniform Prior**			**Jeffrey's Prior**			**Gamma Prior**		
			SELF	QLF	PLF	SELF	QLF	PLF	SELF	QLF	PLF
23	Estimate	0.7213	0.7527	0.6899	0.7682	0.7213	0.6586	0.7368	0.7292	0.6685	0.7443
	MSE	0.0449	0.0358	0.0560	0.0320	0.0449	0.0691	0.0401	0.0416	0.0641	0.0372
77	Estimate	0.7077	0.7168	0.6985	0.7214	0.7077	0.6893	0.7122	0.7103	0.6920	0.7148
	MSE	0.0417	0.0384	0.0452	0.0368	0.0417	0.0489	0.0401	0.0407	0.0477	0.0390
126	Estimate	0.7056	0.7112	0.700	0.7140	0.7056	0.6944	0.7084	0.7073	0.6961	0.7100
	MSE	0.0408	0.0387	0.043	0.0377	0.0408	0.0452	0.0398	0.0402	0.0445	0.0392
200	Estimate	0.7043	0.7078	0.7007	0.7095	0.7043	0.6972	0.7060	0.7053	0.6983	0.7070
	MSE	0.0404	0.0390	0.0418	0.0384	0.0404	0.0431	0.0397	0.0400	0.0427	0.0393

From Table 3, we have the same pattern of the result found in Table 1 and 2 with the same lower values of MSE for the estimators using PLF under all the priors after increasing the value of β from 0.5 to 1.5. This again indicates that changing the value of β does not affect the estimates of the scale parameter, λ as shown in Fig. 1.

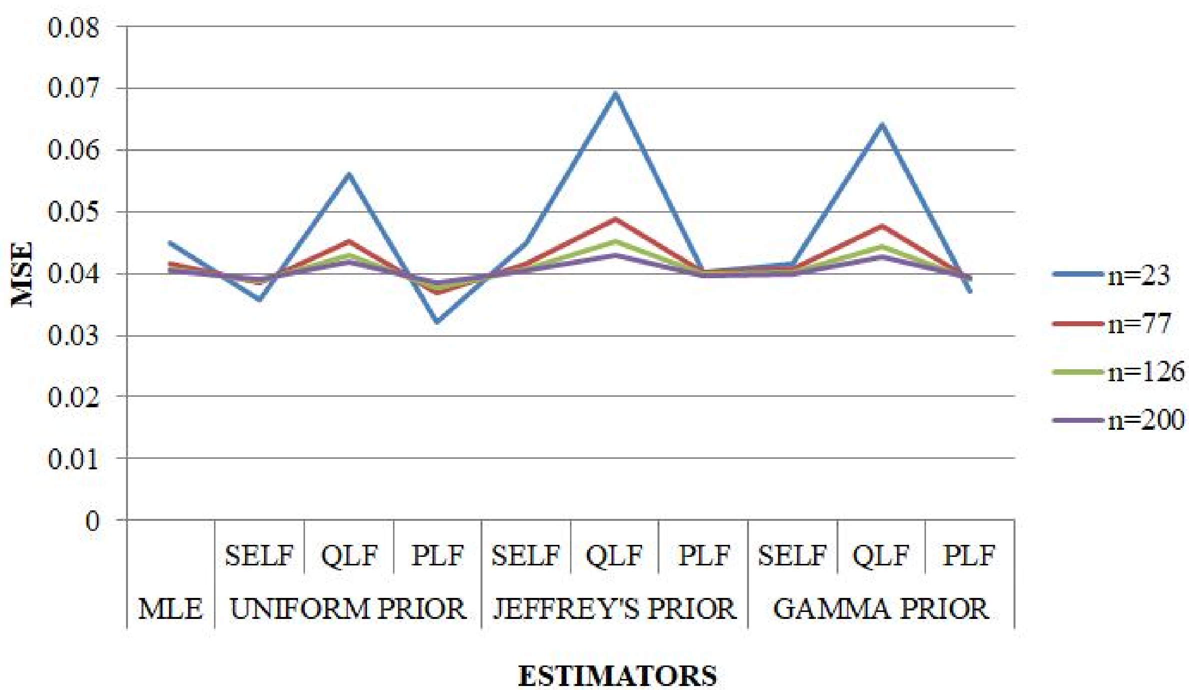

Figure 1: A graph of MSE versus the estimators from Table 1, 2 and 3.

Table 4: Average Estimates (Estimates) and Mean Squared Errors (MSEs) of the estimated scale parameter ($\hat{\lambda}$) for $\alpha = 0.8, \beta = 0.5, \theta = 1.4, \lambda = 0.9, a = 1.0$ and $b = 1.0$ under three different priors and loss functions with varying sample sizes.

n	**Measures**	**MLE**	**Uniform Prior**			**Jeffrey's Prior**			**Gamma Prior**		
			SELF	QLF	PLF	SELF	QLF	PLF	SELF	QLF	PLF
23	Estimate	0.1287	0.1343	0.1231	0.1371	0.1287	0.1175	0.1315	0.1329	0.1218	0.1357
	MSE	0.6096	0.6023	0.6170	0.5988	0.6096	0.6246	0.6060	0.6039	0.6185	0.6003
77	Estimate	0.0668	0.0677	0.0660	0.0681	0.0668	0.0651	0.0673	0.0676	0.0659	0.0680
	MSE	0.6975	0.6962	0.6989	0.6955	0.6975	0.7003	0.6969	0.6963	0.6991	0.6957
126	Estimate	0.0522	0.0526	0.0517	0.0528	0.0522	0.0513	0.0524	0.0525	0.0517	0.0527
	MSE	0.7207	0.7200	0.7214	0.7197	0.7207	0.7221	0.7204	0.7201	0.7214	0.7198
200	Estimate	0.0419	0.0422	0.0417	0.0423	0.0419	0.0415	0.0420	0.0421	0.0417	0.0422
	MSE	0.7374	0.7371	0.7378	0.7369	0.7374	0.7381	0.7372	0.7371	0.7378	0.7369

It can be seen from the table above that the PLF gives us the most efficient estimators for the scale parameter, and looking at all the results presented in the tables above, we can conclude that Bayes estimators using precautionary loss function (*PLF*) under uniform, Jeffrey and gamma priors are associated with minimum MSE when compared to those obtained using *MLE*, *SELF* and QLF under Jeffrey prior, gamma prior and Uniform prior irrespective of the parametric values as well as the allocated sample sizes of n=23, 77, 126 and 200.

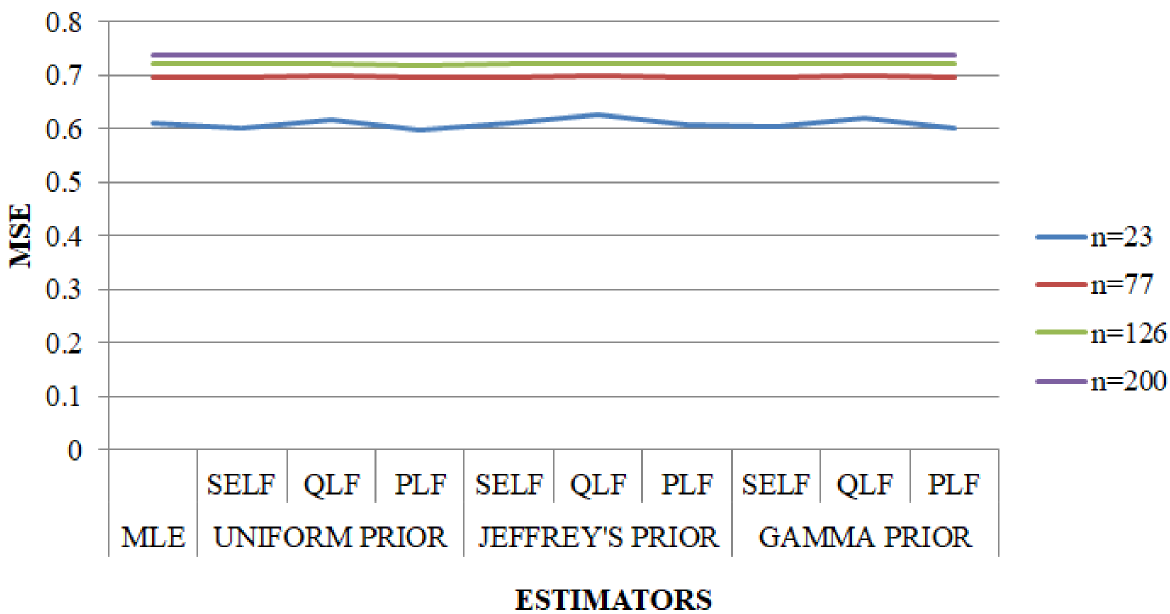

Figure 2: A graph of MSE versus the estimators from Table 4

Table 5: Average Estimates (Estimates) and Mean Squared Errors (MSEs) of the estimated scale parameter ($\hat{\lambda}$) for $\alpha = 0.8, \beta = 0.5, \theta = 0.4, \lambda = 0.9, a = 2.5$ and $b = 1.0$ under three different priors and loss functions with varying sample sizes.

n	**Measures**	**MLE**	**Uniform Prior**			**Jeffrey's Prior**			**Gamma Prior**		
			SELF	QLF	PLF	SELF	QLF	PLF	SELF	QLF	PLF
23	Estimate	0.7213	0.7527	0.6899	0.7682	0.7213	0.6586	0.7368	0.6968	0.6387	0.7111
	MSE	0.0449	0.0358	0.0560	0.0320	0.0449	0.0691	0.0401	0.0518	0.0771	0.0466
77	Estimate	0.7077	0.7168	0.6985	0.7214	0.7077	0.6893	0.7122	0.7006	0.6826	0.7051
	MSE	0.0417	0.0384	0.0452	0.0368	0.0417	0.0489	0.0401	0.0442	0.0515	0.0425
126	Estimate	0.7056	0.7112	0.700	0.7140	0.7056	0.6944	0.7084	0.7014	0.6903	0.7041
	MSE	0.0408	0.0387	0.043	0.0377	0.0408	0.0452	0.0398	0.0424	0.0468	0.0413
200	Estimate	0.7043	0.7078	0.7007	0.7095	0.7043	0.6972	0.7060	0.7016	0.6946	0.7033
	MSE	0.0404	0.0390	0.0418	0.0384	0.0404	0.0431	0.0397	0.0414	0.0442	0.0407

Table 6: Average Estimates (Estimates) and Mean Squared Errors (MSEs) of the estimated scale parameter ($\hat{\lambda}$) for $\alpha = 0.8, \beta = 0.5, \theta = 0.4, \lambda = 0.9, a = 1.0$ and $b = 2.5$ under three different priors and loss functions with varying sample sizes.

n	**Measures**	**MLE**	**Uniform Prior**			**Jeffrey's Prior**			**Gamma Prior**		
			SELF	QLF	PLF	SELF	QLF	PLF	SELF	QLF	PLF
23	Estimate	0.7213	0.7527	0.6899	0.7682	0.7213	0.6586	0.7368	0.7748	0.7140	0.7899
	MSE	0.0449	0.0358	0.0560	0.0320	0.0449	0.0691	0.0401	0.0298	0.0465	0.0268
77	Estimate	0.7077	0.7168	0.6985	0.7214	0.7077	0.6893	0.7122	0.7239	0.7057	0.7285
	MSE	0.0417	0.0384	0.0452	0.0368	0.0417	0.0489	0.0401	0.0359	0.0424	0.0344
126	Estimate	0.7056	0.7112	0.700	0.7140	0.7056	0.6944	0.7084	0.7156	0.7045	0.7184
	MSE	0.0408	0.0387	0.043	0.0377	0.0408	0.0452	0.0398	0.0371	0.0412	0.0361
200	Estimate	0.7043	0.7078	0.7007	0.7095	0.7043	0.6972	0.7060	0.7106	0.7035	0.7123
	MSE	0.0404	0.0390	0.0418	0.0384	0.0404	0.0431	0.0397	0.0380	0.0407	0.0373

From Table 4 and 5 where a and b are increased respectively, it was discovered that uniform prior with PLF gives the most efficient estimators for the scale parameter, and looking at all the results presented in the tables above, we can conclude that Bayes estimators using *PLF* under uniform prior are more better than estimators using *MLE*, *SELF* and QLF under Jeffrey prior, uniform prior and gamma priors irrespective of the parametric values as well as the allocated sample sizes of n=23, 77, 126 and 200.

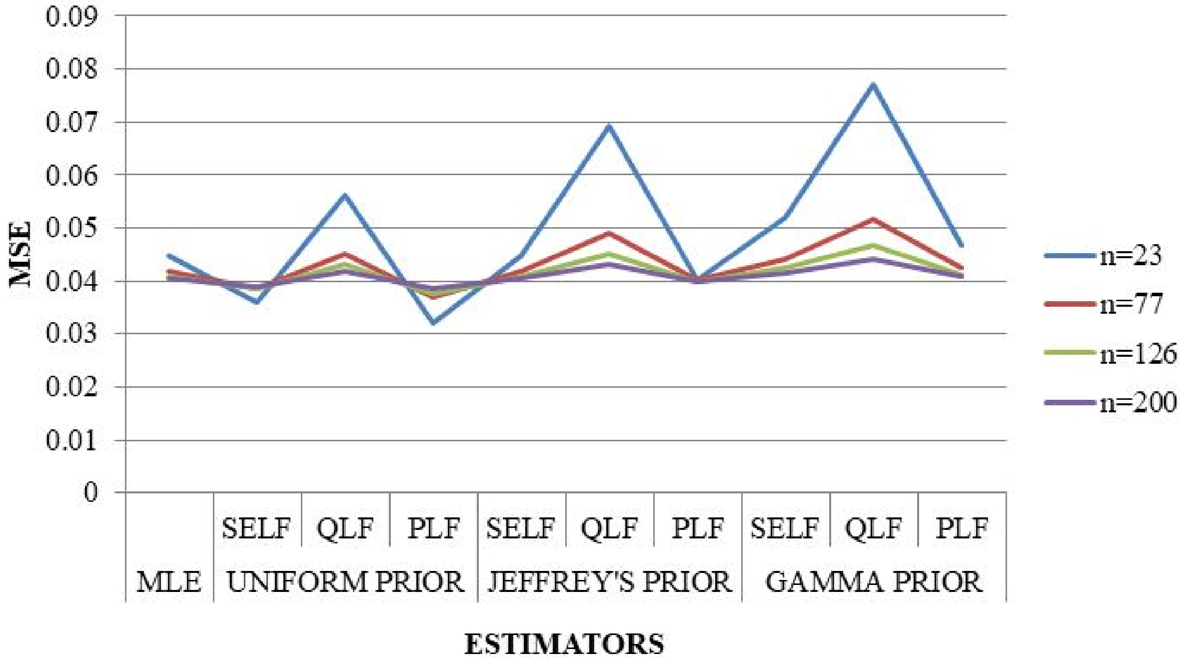

Figure 3: A graph of MSE versus the estimators from Table 4, 5

5. Conclusion

This study is aimed at estimating a scale parameter of the GuLD using the Bayesian method of estimation and evaluating the estimator with the assumption of two non-informative priors and one informative prior distributions namely; Uniform, Jeffrey and gamma prior distributions. These estimators were obtained under the *SELF*, *QLF* and *PLF*. The posterior distributions associated with the scale parameter of the GuLD were derived and also the Estimators were also obtained using the above mentioned priors and loss functions. Furthermore, we carried out Monte-Carlo simulation using a package in R software to assess the performance of the proposed estimators by making use of the associated *MSEs* of the Estimators under the Bayesian approach and the Maximum likelihood method.

The performance of these estimators is assessed on the basis of their mean square errors. Monte Carlo Simulations are used to compare the performance of the estimators. It is discovered that using the *PLF* (under uniform prior) produces the least measures of MSE, followed by the *PLF* (under Jeffrey's prior and gamma prior) then the *SELF*,*MLE* and lastly the *QLF* under both Uniform, Jeffrey and gamma priors irrespective of the parameter values and different in sample sizes. Most importantly, we found that Bayesian Method using *PLF* under all the priors produces the best estimators of the scale parameter compared to estimators using Maximum Likelihood method, *SELF* and *QLF* under both Uniform and Jeffrey priors irrespective of the values of the parameters and the different sample sizes. It is also discovered that the values of the other parameters have no effect on the estimators of the scale parameter. It is also discovered that the values of the other parameters have no effects on the estimators of the scale parameter because changing the values of the other parameters alone does not change the direction of the result or the MSEs.

Based on our findings from the results of this study, we recommend that;
Bayesian method using Precautionary Loss Function should be used under uniform prior for the estimation of the scale parameter of the GuLD irrespective of the parametric values or the sample size. When estimating the scale parameter in question, the researcher should also consider Precautionary Loss Function under Jeffrey's prior and gamma priors.

REFERENCES

[1] Cordeiro GM, Afify AZ, Ortega EMM, Suzuki AK & Mead ME. The odd Lomax generator of distributions: Properties, estimation and applications. *J. of Comp. and Appl. Math.*, 2019; 347: 222–237.

[2] Cordeiro GM, Ortega EMM, Popovic BV & Pescim RR. The Lomax generator of distributions: Properties, minification process and regression model. *Appl. Math. and Comp.,* 2014; 247: 465-486.

[3] Abdullahi UK & Ieren TG. On the inferences and applications of transmuted exponential Lomax distribution, *Int. J. of Adv. Prob. and Stat.*, 2018; 6(1): 30-36.

[4] Yahaya A & Ieren TG. On Transmuted Weibull-Exponential Distribution: Its Properties and Applications, *Nigerian J. of Sci. Res.,* 2017; 16(3): 289-*297*.

[5] Afify MZ, Yousof HM, Cordeiro GM, Ortega EMM. & Nofal ZM. The Weibull Frechet Distribution and Its Applications. *J. of Appl. Stat.*, 2016; 1-22.

[6] Ieren TG & Kuhe AD. On the Properties and Applications of Lomax-Exponential Distribution. *Asian J. of Prob. and Stat.,* 2018; 1(4): 1-13.

[7] Tahir MH, Zubair M, Mansoor M, Cordeiro GM & Alizadeh M. A New Weibull-G family of distributions. *Hacettepe Journal of Mathematics and Statistics*, 2016; 45(2), 629-647.

[8] Ieren TG & Yahaya A. The Weimal Distribution: its properties and applications. *J. of the Nigeria Ass. of Math. Physics,* 2017; 39: 135-148.

[9] Bourguignon M, Silva RB & Cordeiro G. M. The weibull-G family of probability distributions. *J. of Data Sci.*, 2014; 12: 53-68.

[10] Gomes-Silva F, Percontini A, De Brito E, Ramos MW, Venancio R & Cordeiro GM. The Odd Lindley-G Family of Distributions. *Austrian J. of Stat.,* 2017; 46, 65-87.

[11] Ieren TG, Koleoso PO, Chama AF, Eraikhuemen IB & Yakubu N. A Lomax-inverse Lindley Distribution: Model, Properties and Applications to Lifetime Data. *J. of Adv. in Math. and Comp. Sci.*, 2019; 34(3-4): 1-28.

[12] Alzaatreh A, Famoye F & Lee C. A new method for generating families of continuous distributions. *Metron,* 2013; 71: 63–79.

[13] Ieren TG, Kromtit FM, Agbor BU, Eraikhuemen IB & Koleoso PO. A Power Gompertz Distribution: Model, Properties and Application to Bladder Cancer Data. *Asian Res. J. of Math.*, 2019; 15(2): 1-14.

[14] Tahir, M. H., Hussain, M. A., Cordeiro, G. M., Hamedani, G. G., Mansoor, M and Zubair, M. (2016). The Gumbel-Lomax Distribution: Properties and Applications. *Journal of Statistical Theory and Applications*, 15(1): 61-79.

[15] Yahaya A & Lawal A. A study on Lomax-Lindley distribution with applications to simulated and real life data, a paper Presented at the 2nd Int. Conf. Nig. Stat. Society, held at the CES Building, University of Calabar, Calabar, Nigeria. 10th -12th April, 2018.

[16] Shanker R, Kamlesh KK & Fesshaye H. A Two Parameter Lindley Distribution: Its Properties and Applications. *Biostat Biometrics Open Acc J.*, 2017; 1(4): 555570. DOI: 10.19080/BBOAJ.2017.01.555570.

[17] Merovci C. Transmuted Lindley distribution. Int. J. Open Problems Compt. Math., 2013; 6(2): 63-74.

[18] Lindley DV. Fiducial distributions and Bayes' theorem, *J. Royal Stat. Soc. Series B*, 1958; 20: 102-107.

[19] Martz HF & Waller RA. *Bayesian reliability analysis*. New York, NY: John Wiley, 1982.

[20] Ibrahim JG, Chen MH & Sinha D. *Bayesian survival analy.*. New York, NY: Springer-Verlag, 2001.

[21] Singpurwalla ND. *Reliability and risk: A Bayesian perspective*. Chichester: John Wiley, 2006.

[22] Dey S. Bayesian estimation of the shape parameter of the generalized exponential distribution under different loss functions. *Pakistan J. of Stat. and Operat. Res.*, 2010; *6*(2): 163-174.

[23] Ahmed AOM, Ibrahim NA, Arasan J & Adam MB. Extension of Jeffreys' prior estimate for weibull censored data using Lindley's approximation. *Austr. J. of B. and Appl. Sci.*, 2011; 5(12): 884–889.

[24] Pandey BN, Dwividi N, Pulastya B. Comparison between Bayesian and maximum likelihood estimation of the scale parameter in Weibull distribution with known shape under linex loss function, *J. of Sci. Resr.*, 2011; 55: 163–172.

[25] Mabur TM, Omale A, Lawal A, Dewu MM & Mohammed S. Bayesian Estimation of the Scale Parameter of the Weimal Distribution. *Asian J. of Prob. and Stat.*, 2018; 2(4): 1-9.

[26] Aliyu Y & Yahaya A. Bayesian estimation of the shape parameter of generalized Rayleigh distribution under non-informative prior. *Int. J. of Adv. Stat. and Prob.*, 2016; 4 (1):1-10.

[27] Ahmad SP & Ahmad K. Bayesian analysis of weibull distribution using R software, *Austr. J. of B. and Appl. Sc.*, 2013; 7(9), 156–164.

[28] Ahmad K, Ahmad SP & Ahmed A. On parameter estimation of erlang distribution using bayesian method under different loss functions, *Proc. of Int. Conf. on Adv. in Computers, Com., and Electronic Eng.*, 2015; 200–206.

[29] Ieren TG, Chama AF, Bamigbala OA, Joel J, Kromtit FM & Eraikhuemen IB. On A Shape Parameter of Gompertz Inverse Exponential Distribution Using Classical and Non Classical Methods of Estimation. *J. of Sci. Res. & Rep.*, 2020; 25(6): 1-10.

[30] Ieren TG & Oguntunde PE. A Comparison between Maximum Likelihood and Bayesian Estimation Methods for a Shape Parameter of the Weibull-Exponential Distribution. *Asian J. of Prob. and Stat.,* 2018; 1(1): 1-12.

[31] Ieren TG & Chukwu AU. Bayesian Estimation of a Shape Parameter of the Weibull-Frechet Distribution. *Asian J. of Prob. and Stat.*, 2018; 2(1):1-19, 2018.

[32] Krishna H & Goel N. Maximum Likelihood and Bayes Estimation in Randomly Censored Geometric Distribution. *J. of Prob. and Stat.*, 2017. 12 pages.

[33] Gupta PK & Singh AK. Classical and Bayesian estimation of Weibull distribution in presence of outliers. *Cogent Math.*, 2017; 4: 1300975.

[34] Gupta I. Estimation of Parameter And Reliability Function of Exponentiated Inverted Weibull Distribution using Classical and Bayesian Approach. *Int. J. of Recent Sci. Res.*, 2017; 8(7): 18117-18819.

[35] Ahmad K, Ahmad SP & Ahmed A. Classical and Bayesian Approach in Estimation of Scale Parameter of Nakagami Distribution. *J. of Prob. and Stat.*, Article ID 7581918, 8, 2016. pages http://dx.doi.org/10.1155/2016/7581918.

[36] Preda V, Eugenia P & Alina C. Bayes Estimators of Modified-Weibull Distribution parameters using Lindley's approximation. *WSEAS TRANSACTIONS on MATHEMATICS*, 2010; 9 (7), 539-549.

[37] Abbas K, Abbasi NY, Ali A, Khan SA, Manzoor S, Khalil A, Khalil U, Khan DM, Hussain Z & Altaf M. Bayesian Analysis of Three-Parameter Frechet Distribution with Medical Applications. *Comput. and Math. Meth. in Medicine*, Volume 2019, Article ID 9089856, 8 pages, https://doi.org/10.1155/2019/9089856.

[38] Ramos PL, Louzada F & Ramos E. Posterior Properties of the Nakagami-m Distribution Using Noninformative Priors and Applications in Reliability. *IEEE Transactions on reliability*, 67(1): 105-117, 2017.

[39] Ramos PL & Louzada F. Bayesian reference analysis for the Generalized Gamma distribution. *IEEE Communications Letters*, 2018; 22(9), 1950-1953.

[40] Tomazella VLD, de Jesus SR, Louzada F, Nadarajah S & Ramos PL. Reference Bayesian analysis for the generalized lognormal distribution with application to survival data. *Stat. and Its Interface*, 2020; 13(1): 139-149.

[41] Azam Z, & Ahmad SA. Bayesian Approach in Estimation of Scale Parameter of Nakagami Distribution. *Pakistan J. of Stat. and Operat. Res.*, 2014; 10(2), 217-228.

[42] Norstrom JG. The use of precautionary loss functions in risk analysis. IEEE Trans. Reliab., 1996; 45(3): 400–403, 1996. http://dx.doi.org/10.1109/24.536992.