Original Research Article

BAYESIAN ESTIMATION OF A SCALE PARAMETER OF THE GUMBEL-LOMAX DISTRIBUTION USING INFORMATIVE AND NON INFORMATIVE PRIORS

Abstract
This article aimed at estimating the scale parameter of the Gumbel-Lomax Distribution using Bayesian method of estimation and evaluating the estimators by assuming two non-informative prior distributions and one informative prior distribution. These estimators are obtained using the squared error loss function (*SELF*), Quadratic loss function (*QLF*) and precautionary loss function (*PLF*). The posterior distributions of the scale parameter of the Gumbel-Lomax distribution are derived and the Estimators are also obtained using the above mentioned priors and loss functions. Furthermore, a simulation using a package in R software is carried out to assess the performance of the estimators by making use of the Mean Squared Errors of the Estimators under the Bayesian approach and Maximum likelihood method. Our results show that Bayesian Method using precautionary Loss Function (*PLF*) under all priors produces the best estimators of the scale parameter compared to estimators using Maximum Likelihood method, Squared Error Loss Function (*SELF*) and Quadratic Loss Function (*QLF*) under all the priors irrespective of the values of the parameters and the different sample sizes. It is also discovered that the other parameters have no effect on the estimators of the scale parameter.

Keywords: Gumbel-Lomax distribution, Bayesian Method, Priors, Loss functions, MLE, Simulation, MSE.

1. Introduction

There are several standard probability distributions that have been used over the years for modeling real life datasets however research has shown that most of these distributions do not adequately model some of these heavily skewed datasets and therefore creating a problem in statistical theory and applications. Recently, numerous extended or compound probability distributions have proposed in the literature for modeling real life situations and these compound distributions are found to be skewed, flexible and much better in statistical modeling compared to their standard counterparts ([1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13]).
Sequel to the fact above, [14] proposed and studied an extension of the Lomax distribution called Gumbel-Lomax distribution (GuLD) with four parameters. Many important properties of this distribution were extensively derived and discussed with applications to a real life datasets. It was found that the Gumbel-Lomax distribution (GuLD) is flexible and fitted to the datasets much more better than some extensions of the Lomax distribution such as gamma-Lomax distribution [16], exponentiated-Lomax distribution [17], beta-Lomax distribution [18] and the conventional Lomax distributions based on some applications of the models to real life datasets [14].

For details of the general behavior of the different mathematical and statistical properties and applications of the Gumbel-Lomax distribution (GuLD) readers can check [14]. Due to the recorded performance of the Gumbel-Lomax distribution (GuLD) in real life applications, it is deemed very important for the authors of this research to investigate and consider the most appropriate approach(s) for estimating the scale parameter of this distribution (Gumbel-Lomax distribution (GuLD)) which will forever be useful during practical applications of this model.

There are two basic approaches to parameter estimation and these are the classical and the non classical methods. The classical theory of estimation involves a situation where the parameters are considered to be constant but unknown whereas the parameters are considered to be unknown and random just like variables under non classical approach. The most widely used method in classical theory is the method of maximum likelihood estimation while the Bayesian estimation method is used in the non classical theory. However, in most real life problems described by life time distributions, the parameters cannot be considered as constants in all the life testing period ([19], [20], [21]). Following this narrative, it becomes obvious that the classical (frequentist) approach can no longer handle adequately problems of parameter estimation in life time models and therefore the need for non classical or Bayesian estimation in life time models.

Estimation of parameters in a distribution differs by method from one parameter of the distribution to another and therefore this study aims at estimating one scale parameter of the Gumbel-Lomax distribution (GuLD) using Bayesian approach and making a comparison between the Bayesian approach and the method of maximum likelihood estimation.

The aim of this article is to estimate a shape parameter of the Gumbel-Lomax distribution (GuLD) using Bayesian approach assuming uniform prior, Jeffrey's prior and gamma prior distributions with three loss functions. Next to this introductory section are the remaining contents of this article presented as follows: in Section 2, maximum likelihood estimator (MLE) for the shape parameter is obtained. In Section 3, Bayesian estimators based on different loss functions by assuming uniform, Jeffrey's and gamma prior distributions are derived. The proposed estimators are compared in relation of their mean squared error (MSE) in Section 4. Lastly, the conclusion is provided in Section 5.

2. Maximum Likelihood Estimation

Let $X_1, X_2, ..., X_n$ be a random sample from a population X of size 'n' independently and identically distributed random variables with probability density function $f(x)$, . The likelihood is the joint probability function of the data, but viewed as a function of the parameters, treating the observed data as fixed quantities. Given that the values, $\underline{x} = \left(x_1, x_2, ..., x_n\right)$ are obtained independently from a Gumbel-Lomax distribution (GuLD) with unknown parameters α , β , θ and λ , the likelihood function is given by:

$$L\left(\underline{x} \mid \alpha, \beta, \theta, \lambda\right) = P\left(x_1, x_2, ..., x_n \mid \alpha, \beta, \theta, \lambda\right) = \prod_{i=1}^{n} P\left(\underline{x} \mid \alpha, \beta, \theta, \lambda\right) \quad (1)$$

The likelihood function, $L\left(\underline{x} \mid \alpha, \beta, \theta, \lambda\right)$ based on the pdf of Gumbel-Lomax distribution (GuLD) is defined to be the joint density of the random variables $x_1, x_2, ..., x_n$ and it is given as:

$$L\left(X \mid \alpha, \beta, \theta, \lambda\right) \propto \left(\frac{\alpha\lambda}{\beta\theta}\right)^n \prod_{i=1}^{n}\left(\left(1+\frac{x_i}{\beta}\right)^{-\frac{\alpha}{\theta}-1}\left[1-\left(1+\frac{x_i}{\beta}\right)^{\alpha}\right]^{-\left(\frac{1}{\theta}+1\right)}\right)\exp\left\{-\lambda\sum_{i=1}^{n}\left[\left(1+\frac{x_i}{\beta}\right)^{\alpha}-1\right]^{-\frac{1}{\theta}}\right\} \quad (2)$$

For the scale parameter of the Gumbel-Lomax distribution (GuLD), λ, the likelihood function is given by;

$$L(X \mid \lambda) \propto \eta \lambda^{n} \exp\left\{ -\lambda \sum_{i=1}^{n} \left[\left(1+\frac{x_i}{\beta}\right)^{\alpha} - 1 \right]^{-\frac{1}{\theta}} \right\}$$

$$L(X \mid \lambda) \propto \eta \lambda^{n} \exp\left\{ -\lambda \sum_{i=1}^{n} \left[\left(1+\frac{x_i}{\beta}\right)^{\alpha} - 1 \right]^{-\frac{1}{\theta}} \right\} \tag{3}$$

Where $\eta = \left(\frac{\alpha}{\beta\theta}\right)^{n} \prod_{i=1}^{n} \left(\left(1+\frac{x_i}{\beta}\right)^{-\frac{\alpha}{\theta}-1} \left[1 - \left(1+\frac{x_i}{\beta}\right)^{\alpha} \right]^{-\left(\frac{1}{\theta}+1\right)} \right)$ is a constant which is independent of the scale parameter, λ.

Let the log-likelihood function, $l = \log L(\underline{x} \mid \lambda)$, therefore

$$l = \log L(X \mid \lambda) = n \log \lambda - \lambda \sum_{i=1}^{n} \left[\left(1+\frac{x_i}{\beta}\right)^{\alpha} - 1 \right]^{-\frac{1}{\theta}} \tag{4}$$

Differentiating ***l*** partially with respect to λ respectively gives;

$$\frac{\partial l}{\partial \lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} \left[\left(1+\frac{x_i}{\beta}\right)^{\alpha} - 1 \right]^{-\frac{1}{\theta}}$$

And solving for $\hat{\lambda}$ gives;

$$\Rightarrow \hat{\lambda} = n \left(\sum_{i=1}^{n} \left[\left(1+\frac{x_i}{\beta}\right)^{\alpha} - 1 \right]^{-\frac{1}{\theta}} \right)^{-1} \tag{5}$$

where $\hat{\lambda}$ is the maximum likelihood estimator of the scale parameter, λ. Details concerning the maximum likelihood estimators of the other three parameters of the Gumbel-Lomax distribution (GuLD) can be found in [14].

3. Bayesian Estimation

The Bayesian inference requires appropriate choice of prior(s) for the parameter(s). From the Bayesian viewpoint, there is no clear cut way from which one can conclude that one prior is better than the other. Nevertheless, very often priors are chosen according to one's subjective knowledge and beliefs. However, if one has adequate information about the parameter(s), it is better to choose informative prior(s); otherwise, it is preferable to use non-informative prior(s).

In this study, two non-informative priors (uniform and Jeffrey) and an informative prior (gamma) will be considered for estimating the scale parameter of the Gumbel-Lomax distribution (GuLD). These assumed prior distributions have been used widely by several authors including [22], [23], [24], [25], [26], [27], [28], [29]. This study also considers three loss functions including square error, quadratic and precautionary loss functions which have also been used previously by some researchers such as [30], [31], [32], [33], [34], [35], [36], [37], [38], [39] and [40] etc. The stated prior distributions and loss functions are defined as follows:

a. The uniform prior is defined as:

$$p(\lambda) \propto 1; 0 < \lambda < \infty \tag{6}$$

b. Also, the Jeffrey's prior is defined as:

$$p(\lambda) \propto \frac{1}{\lambda}; 0 < \lambda < \infty \tag{7}$$

c. Also, the gamma prior is defined as:

$$P(\lambda) = \frac{a^b}{\Gamma(b)} \lambda^{b-1} e^{-a\lambda} \tag{8}$$

i. Squared Error Loss Function (*SELF*)

The squared error loss function relating to the scale parameter λ is defined as:

$$L(\lambda, \lambda_{SELF}) = (\lambda - \lambda_{SELF})^2 \tag{9}$$

where λ_{SELF} is the estimator of the parameter λ under *SELF*.

ii. Quadratic Loss Function (QLF)

The quadratic loss function is defined from [41] as

$$L(\lambda, \lambda_{QLF}) = \left(\frac{\lambda - \lambda_{QLF}}{\lambda}\right)^2 \tag{10}$$

where λ_{QLF} is the estimator of the parameter λ under *QLF*.

iii. Precautionary Loss Function (*PLF*)

The precautionary loss function (*PLF*) introduced by [42] is an asymmetric loss function and is defined as

$$L(\lambda_{PLF}, \lambda) = \frac{(\lambda_{PLF} - \lambda)^2}{\lambda_{PLF}} \tag{11}$$

where λ_{PLF} is the estimator of the scale parameter λ under *PLF*.

The posterior distribution of a parameter is the distribution of the parameter after observing the available data and it is obtained by using Bayes' theorem in relation to the scale parameter λ, likelihood function and prior distribution as follows:

$$P(\lambda \mid \underline{x}) = \frac{P(\lambda, \underline{x})}{P(\underline{x})} = \frac{P(\underline{x} \mid \lambda) P(\lambda)}{P(\underline{x})} = \frac{P(\underline{x} \mid \lambda) P(\lambda)}{\int P(\underline{x} \mid \lambda) P(\lambda) d\lambda} = \frac{L(\underline{x} \mid \lambda) P(\lambda)}{\int L(\underline{x} \mid \lambda) P(\lambda) d\lambda} \tag{12}$$

where $P(\underline{x})$ is the marginal distribution of X and $P(\underline{x})=\sum_{x}^{\infty} p(\lambda)L(\underline{x}\mid\lambda)$ when the prior distribution of λ is discrete and $P(\underline{x})=\int_{-\infty}^{\infty} p(\lambda)L(\underline{x}\mid\lambda)d\lambda$ when the prior distribution of λ is continuous. Also note that $p(\lambda)$ and $L(\underline{x}\mid\lambda)$ are the prior distribution and the Likelihood function respectively.

3.1 Bayesian Analysis under Uniform Prior with Three Loss Functions

The posterior distribution of the scale parameter λ assuming a uniform prior distribution is obtained from equation (12) using integration by substitution method as:

$$P(\lambda\mid\underline{x})=\frac{\lambda^{n}\left[\sum_{i=1}^{n}\left[\left(1+\frac{x_i}{\beta}\right)^{\alpha}-1\right]^{-\frac{1}{\theta}}\right]^{n+1}}{\Gamma(n+1)\,e^{\lambda\sum_{i=1}^{n}\left[\left(1+\frac{x_i}{\beta}\right)^{\alpha}-1\right]^{-\frac{1}{\theta}}}} \tag{13}$$

Now the Bayes estimators under uniform prior using *SELF*, *QLF* and *PLF* are given respectively as:

$$\lambda_{SELF}=(n+1)\left[\sum_{i=1}^{n}\left[\left(1+\frac{x_i}{\beta}\right)^{\alpha}-1\right]^{-\frac{1}{\theta}}\right]^{-1} \tag{14}$$

$$\lambda_{QLF}=\frac{(n-1)}{\left[\sum_{i=1}^{n}\left[\left(1+\frac{x_i}{\beta}\right)^{\alpha}-1\right]^{-\frac{1}{\theta}}\right]} \tag{15}$$

and

$$\lambda_{PLF}=\left[(n+1)(n+2)\right]^{\frac{1}{2}}\left[\sum_{i=1}^{n}\left[\left(1+\frac{x_i}{\beta}\right)^{\alpha}-1\right]^{-\frac{1}{\theta}}\right]^{-1} \tag{16}$$

3.2 Bayesian Analysis under Jeffrey's Prior with Three Loss Functions

The posterior distribution of the scale parameter λ for a given data assuming a Jeffrey's prior distribution is obtained from equation (12) using integration by substitution method as:

$$P(\lambda\mid\underline{x})=\frac{\left[\sum_{i=1}^{n}\left[\left(1+\frac{x_i}{\beta}\right)^{\alpha}-1\right]^{-\frac{1}{\theta}}\right]^{n}\lambda^{n-1}e^{-\lambda\sum_{i=1}^{n}\left[\left(1+\frac{x_i}{\beta}\right)^{\alpha}-1\right]^{-\frac{1}{\theta}}}}{\Gamma(n)} \tag{17}$$

Again the Bayes estimators under Jeffrey's prior using *SELF*, *QLF* and *PLF* are given respectively as:

$$\lambda_{SELF} = E\left(\lambda \mid \underline{x}\right) = n\left[\sum_{i=1}^{n}\left[\left(1+\frac{x_i}{\beta}\right)^{\alpha}-1\right]^{-\frac{1}{\theta}}\right]^{-1} \tag{18}$$

$$\lambda_{QLF} = \frac{\left(n-2\right)}{\sum_{i=1}^{n}\left[\left(1+\frac{x_i}{\beta}\right)^{\alpha}-1\right]^{-\frac{1}{\theta}}} \tag{19}$$

and

$$\lambda_{PLF} = \left[n\left(n+1\right)\right]^{\frac{1}{2}}\left[\sum_{i=1}^{n}\left[\left(1+\frac{x_i}{\beta}\right)^{\alpha}-1\right]^{-\frac{1}{\theta}}\right]^{-1} \tag{20}$$

3.3 Bayesian Analysis under Gamma Prior with Three Loss Functions

The posterior distribution of the scale parameter λ for a given data assuming a gamma prior distribution is obtained from equation (12) using integration by substitution method as

$$p\left(\lambda \mid \underline{x}\right) = \frac{\lambda^{n+b-1}\left(a+\sum_{i=1}^{n}\left[\left(1+\frac{x_i}{\beta}\right)^{\alpha}-1\right]^{-\frac{1}{\theta}}\right)^{n+b} \mathrm{e}^{-\lambda\left(a+\sum_{i=1}^{n}\left[\left(1+\frac{x_i}{\beta}\right)^{\alpha}-1\right]^{-\frac{1}{\theta}}\right)}}{\Gamma\left(n+b\right)} \tag{21}$$

Also the Bayes estimators under gamma prior using *SELF, QLF* and *PLF* are given respectively as:

$$\lambda_{SELF} = E\left(\lambda \mid \underline{x}\right) = \left(n+b\right)\left[a+\sum_{i=1}^{n}\left[\left(1+\frac{x_i}{\beta}\right)^{\alpha}-1\right]^{-\frac{1}{\theta}}\right]^{-1} \tag{22}$$

$$\lambda_{QLF} = \frac{\left(n+b-2\right)}{a+\sum_{i=1}^{n}\left[\left(1+\frac{x_i}{\beta}\right)^{\alpha}-1\right]^{-\frac{1}{\theta}}} \tag{23}$$

and

$$\lambda_{PLF} = \left[\left(n+b\right)\left(n+b+1\right)\right]^{\frac{1}{2}}\left[a+\sum_{i=1}^{n}\left[\left(1+\frac{x_i}{\beta}\right)^{\alpha}-1\right]^{-\frac{1}{\theta}}\right]^{-1} \tag{24}$$

4. Results and Discussions

In this section, Monte Carlo simulation with R software under 10,000 replications is considered to generate random samples of sizes n = (23, 77, 126, 200) from Gumbel-Lomax distribution (GuLD) using the quantile function (inverse transformation method of simulation) under the following combination of parameter values: $\alpha = 0.8, \beta = 0.5, \theta = 0.4, \lambda = 0.9, a = 1.0$ and $b = 1.0$;

$\alpha = 1.8, \beta = 0.5, \theta = 0.4, \lambda = 0.9, a = 1.0$ and $b = 1.0$; $\alpha = 0.8, \beta = 1.5, \theta = 0.4, \lambda = 0.9, a = 1.0$ and $b = 1.0$; $\alpha = 0.8, \beta = 0.5, \theta = 1.4, \lambda = 0.9, a = 1.0$ and $b = 1.0$ $\alpha = 0.8, \beta = 0.5, \theta = 0.4, \lambda = 0.9, a = 2.5$ and $b = 1.0$ and $\alpha = 0.8, \beta = 0.5, \theta = 0.4, \lambda = 0.9, a = 1.0$ and $b = 2.5$. The following tables present the results of our simulation study by listing the average estimates of the shape parameter with their respective Mean Square Errors (MSEs) under the appropriate estimation methods which include the Maximum Likelihood Estimation (*MLE*), Squared Error Loss Function (*SELF*), Quadratic Loss Function (*QLF*), and Precautionary Loss Function (*PLF*) under Uniform Jeffrey and gamma priors respectively. The criterion for evaluating the performance of the estimators in this study is the Mean Square Error (MSE): $MSE = \frac{1}{n} E\left(\hat{\lambda} - \lambda\right)^2$.

Table 1: Average Estimates (Estimates) and Mean Squared Errors (MSEs) of the estimated scale parameter ($\hat{\lambda}$) for $\alpha = 0.8, \beta = 0.5, \theta = 0.4, \lambda = 0.9, a = 1.0$ and $b = 1.0$ under three different priors and loss functions with varying sample sizes.

n	**Measures**	**MLE**	**Uniform Prior**			**Jeffrey's Prior**			**Gamma Prior**		
			SELF	QLF	PLF	SELF	QLF	PLF	SELF	QLF	PLF
23	Estimate	0.7213	0.7527	0.6899	0.7682	0.7213	0.6586	0.7368	0.7292	0.6685	0.7443
	MSE	0.0449	0.0358	0.0560	0.0320	0.0449	0.0691	0.0401	0.0416	0.0641	0.0372
77	Estimate	0.7077	0.7168	0.6985	0.7214	0.7077	0.6893	0.7122	0.7103	0.6920	0.7148
	MSE	0.0417	0.0384	0.0452	0.0368	0.0417	0.0489	0.0401	0.0407	0.0477	0.0390
126	Estimate	0.7056	0.7112	0.700	0.7140	0.7056	0.6944	0.7084	0.7073	0.6961	0.7100
	MSE	0.0408	0.0387	0.043	0.0377	0.0408	0.0452	0.0398	0.0402	0.0445	0.0392
200	Estimate	0.7043	0.7078	0.7007	0.7095	0.7043	0.6972	0.7060	0.7053	0.6983	0.7070
	MSE	0.0404	0.0390	0.0418	0.0384	0.0404	0.0431	0.0397	0.0400	0.0427	0.0393

MLE=Maximum likelihood estimator, SELF=Square error loss function, QLF= Quadratic loss function, PLF= Precautionary loss function.

From table 1, we can see that the estimator using *PLF* is better than the other estimators under both Uniform and Jeffrey priors with smaller values of MSE irrespective of the variation in the samples. Hence, we can say that Bayesian estimation (using *PLF* under Uniform and Jeffrey prior) for this parameter is better than Method of Maximum Likelihood estimation (*MLE*) for the chosen parameter values irrespective of small, medium or large sample sizes.

Table 2: Average Estimates (Estimates) and Mean Squared Errors (MSEs) of the estimated scale parameter ($\hat{\lambda}$) for $\alpha = 1.8, \beta = 0.5, \theta = 0.4, \lambda = 0.9, a = 1.0$ and $b = 1.0$ under three different priors and loss functions with varying sample sizes.

n	**Measures**	**MLE**	**Uniform Prior**			**Jeffrey's Prior**			**Gamma Prior**		
			SELF	QLF	PLF	SELF	QLF	PLF	SELF	QLF	PLF
23	Estimate	0.7213	0.7527	0.6899	0.7682	0.7213	0.6586	0.7368	0.7292	0.6685	0.7443
	MSE	0.0449	0.0358	0.0560	0.0320	0.0449	0.0691	0.0401	0.0416	0.0641	0.0372
77	Estimate	0.7077	0.7168	0.6985	0.7214	0.7077	0.6893	0.7122	0.7103	0.6920	0.7148
	MSE	0.0417	0.0384	0.0452	0.0368	0.0417	0.0489	0.0401	0.0407	0.0477	0.0390

n	Measures	MLE	SELF	QLF	PLF	SELF	QLF	PLF	SELF	QLF	PLF
126	Estimate	0.7056	0.7112	0.700	0.7140	0.7056	0.6944	0.7084	0.7073	0.6961	0.7100
	MSE	0.0408	0.0387	0.043	0.0377	0.0408	0.0452	0.0398	0.0402	0.0445	0.0392
200	Estimate	0.7043	0.7078	0.7007	0.7095	0.7043	0.6972	0.7060	0.7053	0.6983	0.7070
	MSE	0.0404	0.0390	0.0418	0.0384	0.0404	0.0431	0.0397	0.0400	0.0427	0.0393

MLE=Maximum likelihood estimator, SELF=Square error loss function, QLF= Quadratic loss function, PLF= Precautionary loss function.

Table 2 also gives a similar pattern of the result found in table 1 with the same lower values of MSE for the estimators using PLF under all the priors after increasing the value of α from 0.8 to 1.8. This result indicates that changing the value of α does not affect the estimate of the scale parameter, λ.

Table 3: Average Estimates (Estimates) and Mean Squared Errors (MSEs) of the estimated scale parameter ($\hat{\lambda}$) for $\alpha = 0.8, \beta = 1.5, \theta = 0.4, \lambda = 0.9, a = 1.0$ and $b = 1.0$ under three different priors and loss functions with varying sample sizes.

n	**Measures**	**MLE**	**Uniform Prior**			**Jeffrey's Prior**			**Gamma Prior**		
			SELF	QLF	PLF	SELF	QLF	PLF	SELF	QLF	PLF
23	Estimate	0.7213	0.7527	0.6899	0.7682	0.7213	0.6586	0.7368	0.7292	0.6685	0.7443
	MSE	0.0449	0.0358	0.0560	0.0320	0.0449	0.0691	0.0401	0.0416	0.0641	0.0372
77	Estimate	0.7077	0.7168	0.6985	0.7214	0.7077	0.6893	0.7122	0.7103	0.6920	0.7148
	MSE	0.0417	0.0384	0.0452	0.0368	0.0417	0.0489	0.0401	0.0407	0.0477	0.0390
126	Estimate	0.7056	0.7112	0.700	0.7140	0.7056	0.6944	0.7084	0.7073	0.6961	0.7100
	MSE	0.0408	0.0387	0.043	0.0377	0.0408	0.0452	0.0398	0.0402	0.0445	0.0392
200	Estimate	0.7043	0.7078	0.7007	0.7095	0.7043	0.6972	0.7060	0.7053	0.6983	0.7070
	MSE	0.0404	0.0390	0.0418	0.0384	0.0404	0.0431	0.0397	0.0400	0.0427	0.0393

MLE=Maximum likelihood estimator, SELF=Square error loss function, QLF= Quadratic loss function, PLF= Precautionary loss function.

Table 3 also gives a similar pattern of the result found in table 1 and 2 with the same lower values of MSE for the estimators using PLF under all the priors after increasing the value of β from 0.5 to 1.5. This again indicates that changing the value of β does not affect the estimates of the scale parameter, λ as shown in Fig. 1.

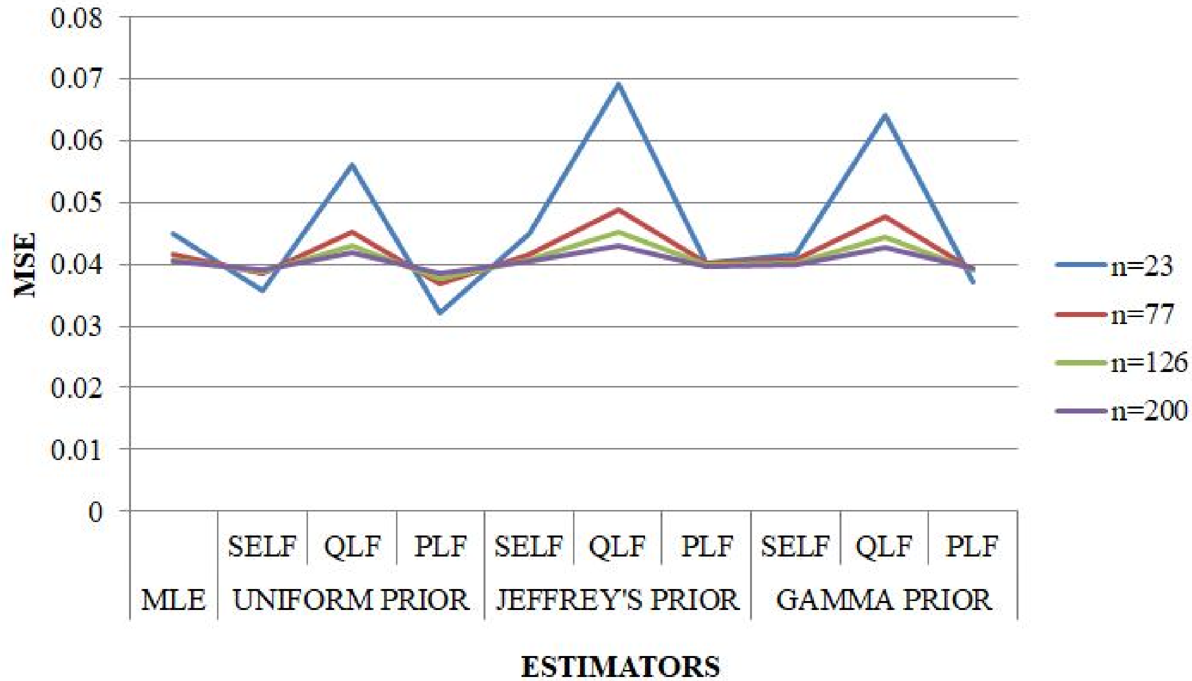

Figure 1: A graph of MSE versus the estimators from Table 1, 2 and 3.

Table 4: Average Estimates (Estimates) and Mean Squared Errors (MSEs) of the estimated scale parameter $(\hat{\lambda})$ for $\alpha = 0.8, \beta = 0.5, \theta = 1.4, \lambda = 0.9, a = 1.0$ and $b = 1.0$ under three different priors and loss functions with varying sample sizes.

n	Measures	MLE	Uniform Prior			Jeffrey's Prior			Gamma Prior		
			SELF	QLF	PLF	SELF	QLF	PLF	SELF	QLF	PLF
23	Estimate	0.1287	0.1343	0.1231	0.1371	0.1287	0.1175	0.1315	0.1329	0.1218	0.1357
	MSE	0.6096	0.6023	0.6170	0.5988	0.6096	0.6246	0.6060	0.6039	0.6185	0.6003
77	Estimate	0.0668	0.0677	0.0660	0.0681	0.0668	0.0651	0.0673	0.0676	0.0659	0.0680
	MSE	0.6975	0.6962	0.6989	0.6955	0.6975	0.7003	0.6969	0.6963	0.6991	0.6957
126	Estimate	0.0522	0.0526	0.0517	0.0528	0.0522	0.0513	0.0524	0.0525	0.0517	0.0527
	MSE	0.7207	0.7200	0.7214	0.7197	0.7207	0.7221	0.7204	0.7201	0.7214	0.7198
200	Estimate	0.0419	0.0422	0.0417	0.0423	0.0419	0.0415	0.0420	0.0421	0.0417	0.0422
	MSE	0.7374	0.7371	0.7378	0.7369	0.7374	0.7381	0.7372	0.7371	0.7378	0.7369

MLE=Maximum likelihood estimator, SELF=Square error loss function, QLF= Quadratic loss function, PLF= Precautionary loss function.

The above table also shows that the PLF gives us the most efficient estimators for the scale parameter, and looking at all the results presented in the tables above, we can conclude that Bayes estimators using precautionary loss function (*PLF*) under uniform, Jeffrey and gamma priors are associated with minimum MSE when compared to those obtained using *MLE*, *SELF* and QLF under Jeffrey prior, gamma prior and Uniform prior irrespective of the parametric values as well as the allocated sample sizes of n=23, 77, 126 and 200.

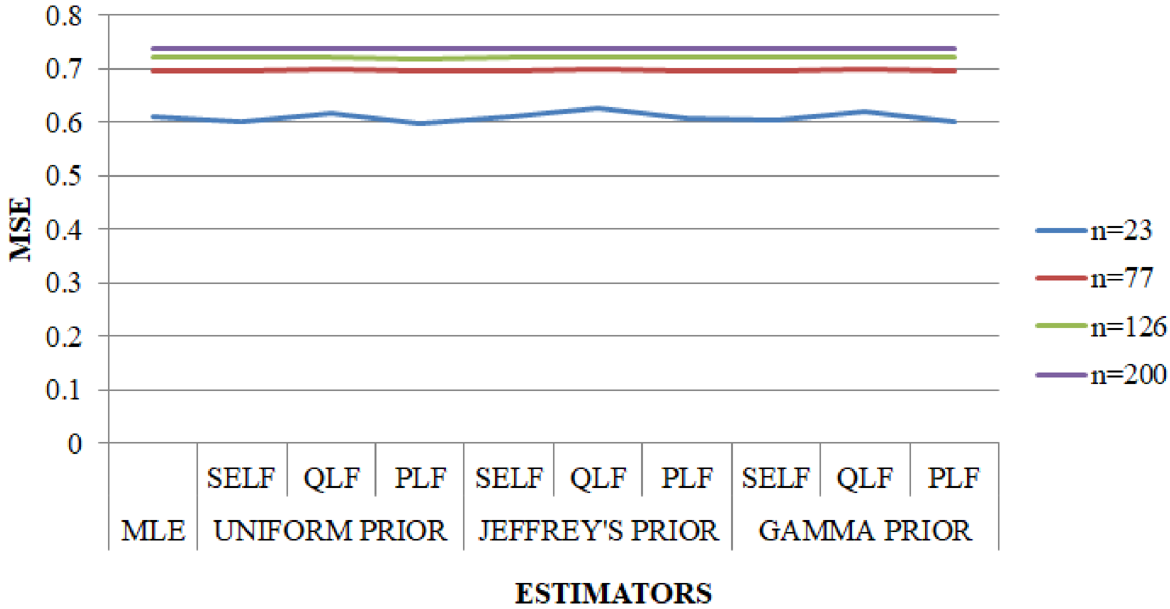

Figure 2: A graph of MSE versus the estimators from Table 4

Table 5: Average Estimates (Estimates) and Mean Squared Errors (MSEs) of the estimated scale parameter ($\hat{\lambda}$) for $\alpha = 0.8, \beta = 0.5, \theta = 0.4, \lambda = 0.9, a = 2.5$ and $b = 1.0$ under three different priors and loss functions with varying sample sizes.

n	Measures	MLE	Uniform Prior			Jeffrey's Prior			Gamma Prior		
			SELF	QLF	PLF	SELF	QLF	PLF	SELF	QLF	PLF
23	Estimate	0.7213	0.7527	0.6899	0.7682	0.7213	0.6586	0.7368	0.6968	0.6387	0.7111
	MSE	0.0449	0.0358	0.0560	0.0320	0.0449	0.0691	0.0401	0.0518	0.0771	0.0466
77	Estimate	0.7077	0.7168	0.6985	0.7214	0.7077	0.6893	0.7122	0.7006	0.6826	0.7051
	MSE	0.0417	0.0384	0.0452	0.0368	0.0417	0.0489	0.0401	0.0442	0.0515	0.0425
126	Estimate	0.7056	0.7112	0.700	0.7140	0.7056	0.6944	0.7084	0.7014	0.6903	0.7041
	MSE	0.0408	0.0387	0.043	0.0377	0.0408	0.0452	0.0398	0.0424	0.0468	0.0413
200	Estimate	0.7043	0.7078	0.7007	0.7095	0.7043	0.6972	0.7060	0.7016	0.6946	0.7033
	MSE	0.0404	0.0390	0.0418	0.0384	0.0404	0.0431	0.0397	0.0414	0.0442	0.0407

MLE=Maximum likelihood estimator, SELF=Square error loss function, QLF= Quadratic loss function, PLF= Precautionary loss function.

Table 6: Average Estimates (Estimates) and Mean Squared Errors (MSEs) of the estimated scale parameter ($\hat{\lambda}$) for $\alpha = 0.8, \beta = 0.5, \theta = 0.4, \lambda = 0.9, a = 1.0$ and $b = 2.5$ under three different priors and loss functions with varying sample sizes.

n	Measur	MLE	Uniform Prior	Jeffrey's Prior	Gamma Prior

	es		SELF	QLF	PLF	SELF	QLF	PLF	SELF	QLF	PLF
23	Estimate	0.7213	0.7527	0.6899	0.7682	0.7213	0.6586	0.7368	0.7748	0.7140	0.7899
	MSE	0.0449	0.0358	0.0560	0.0320	0.0449	0.0691	0.0401	0.0298	0.0465	0.0268
77	Estimate	0.7077	0.7168	0.6985	0.7214	0.7077	0.6893	0.7122	0.7239	0.7057	0.7285
	MSE	0.0417	0.0384	0.0452	0.0368	0.0417	0.0489	0.0401	0.0359	0.0424	0.0344
126	Estimate	0.7056	0.7112	0.700	0.7140	0.7056	0.6944	0.7084	0.7156	0.7045	0.7184
	MSE	0.0408	0.0387	0.043	0.0377	0.0408	0.0452	0.0398	0.0371	0.0412	0.0361
200	Estimate	0.7043	0.7078	0.7007	0.7095	0.7043	0.6972	0.7060	0.7106	0.7035	0.7123
	MSE	0.0404	0.0390	0.0418	0.0384	0.0404	0.0431	0.0397	0.0380	0.0407	0.0373

MLE=Maximum likelihood estimator, SELF=Square error loss function, QLF= Quadratic loss function, PLF= Precautionary loss function.

From table 4 and 5 where a and b are increased respectively, it was discovered that uniform prior with PLF gives the most efficient estimators for the scale parameter, and looking at all the results presented in the tables above, we can conclude that Bayes estimators using precautionary loss function (*PLF*) under uniform prior are more better than estimators using *MLE*, *SELF* and QLF under Jeffrey prior, uniform prior and gamma priors irrespective of the parametric values as well as the allocated sample sizes of n=23, 77, 126 and 200.

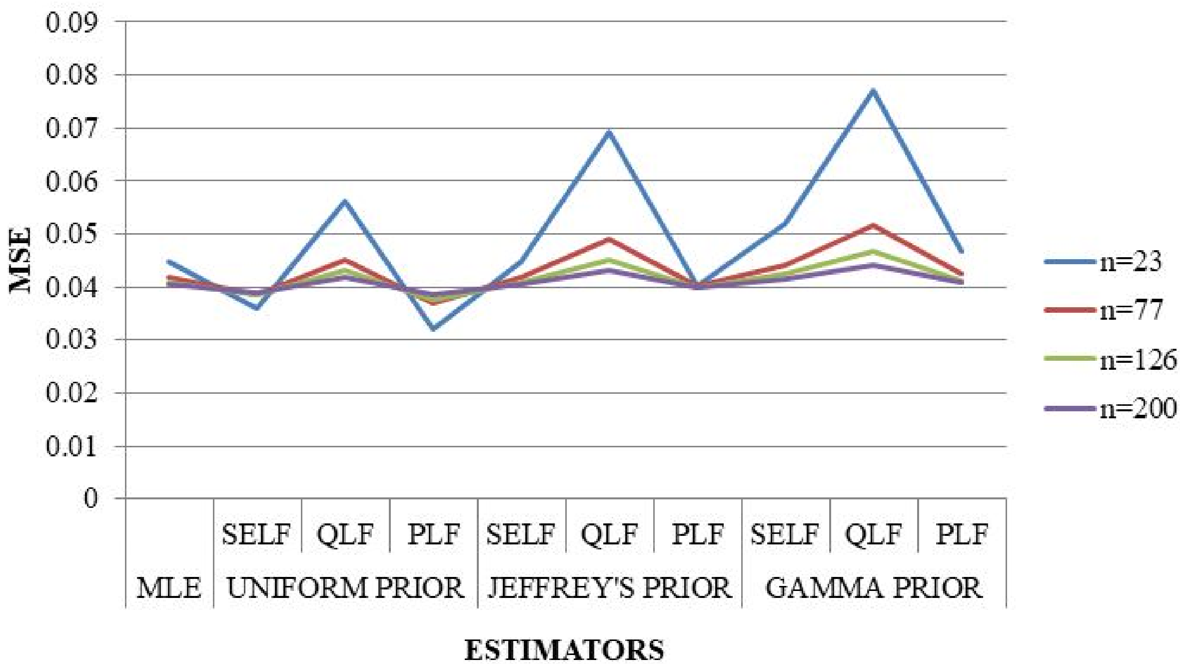

Figure 3: A graph of MSE versus the estimators from Table 4, 5

5. Conclusion

This study is aimed at estimating a scale parameter of the Gumbel-Lomax distribution (GuLD) using the Bayesian method of estimation and evaluating the estimator with the assumption of two non-informative priors and one informative prior distributions namely; Uniform, Jeffrey and gamma prior distributions. These estimators were obtained under the squared error loss function

(*SELF*), Quadratic loss function (*QLF*) and precautionary loss function (*PLF*). The posterior distributions associated with the scale parameter of the Gumbel-Lomax distribution (GuLD) were derived and also the Estimators were also obtained using the above mentioned priors and loss functions. Furthermore, we carried out Monte-Carlo simulation using a package in R software to assess the performance of the proposed estimators by making use of the associated *MSEs* of the Estimators under the Bayesian approach and Maximum likelihood method.

The performance of these estimators is assessed on the basis of their mean square errors. Monte Carlo Simulations are used to compare the performance of the estimators. It is discovered that using the *PLF* (under uniform prior) produces the least measures of MSE, followed by the *PLF* (under Jeffrey's prior and gamma prior) then the *SELF,MLE* and lastly the *QLF* under both Uniform, Jeffrey and gamma priors irrespective of the parameter values and different in sample sizes. Most importantly, we found that Bayesian Method using Precautionary Loss Function (*PLF*) under all the priors produces the best estimators of the scale parameter compared to estimators using Maximum Likelihood method, Squared Error Loss Function (*SELF*) and Quadratic Loss Function (*QLF*) under both Uniform and Jeffrey priors irrespective of the values of the parameters and the different sample sizes. It is also discovered that the values of the other parameters have no effect on the estimators of the scale parameter. It is also discovered that the values of the other parameters have no effects on the estimators of the scale parameter because changing the values of the other parameters alone does not change the direction of the result or the mean squared errors (MSEs).

Based on our findings from the results of this study, we recommend that;
Bayesian method using Precautionary Loss Function should be used under uniform prior for the estimation of the scale parameter of the Gumbel-Lomax distribution (GuLD) irrespective of the parametric values or the sample size. When estimating the scale parameter in question, the researcher should also consider Precautionary Loss Function under Jeffrey's prior and gamma priors.

REFERENCES

[1] Cordeiro GM, Afify AZ, Ortega EMM, Suzuki AK & Mead ME. The odd Lomax generator of distributions: Properties, estimation and applications. *J. of Comp. and Appl. Math.*, 2019; 347: 222–237.

[2] Cordeiro GM, Ortega EMM, Popovic BV & Pescim RR. The Lomax generator of distributions: Properties, minification process and regression model. *Appl. Math. and Comp.*, 2014; 247: 465-486.

[3] Abdullahi UK & Ieren TG. On the inferences and applications of transmuted exponential Lomax distribution, *Int. J. of Adv. Prob. and Stat.*, 2018; 6(1): 30-36.

[4] Yahaya A & Ieren TG. On Transmuted Weibull-Exponential Distribution: Its Properties and Applications, *Nigerian J. of Sci. Res.*, 2017; 16(3): 289-*297*.

[5] Afify MZ, Yousof HM, Cordeiro GM, Ortega EMM. & Nofal ZM. The Weibull Frechet Distribution and Its Applications. *J. of Appl. Stat.*, 2016; 1-22.

[6] Ieren TG & Kuhe AD. On the Properties and Applications of Lomax-Exponential Distribution. *Asian J. of Prob. and Stat.*, 2018; 1(4): 1-13.

[7] Tahir MH, Zubair M, Mansoor M, Cordeiro GM & Alizadeh M. A New Weibull-G family of distributions. *Hacettepe Journal of Mathematics and Statistics*, 2016; 45(2), 629-647.

[8] Ieren TG & Yahaya A. The Weimal Distribution: its properties and applications. *J. of the Nigeria Ass. of Math. Physics,* 2017; 39: 135-148.

[9] Bourguignon M, Silva RB & Cordeiro G. M. The weibull-G family of probability distributions. *J. of Data Sci.*, 2014; 12: 53-68.

[10] Gomes-Silva F, Percontini A, De Brito E, Ramos MW, Venancio R & Cordeiro GM. The Odd Lindley-G Family of Distributions. *Austrian J. of Stat.*, 2017; 46, 65-87.

[11] Ieren TG, Koleoso PO, Chama AF, Eraikhuemen IB & Yakubu N. A Lomax-inverse Lindley Distribution: Model, Properties and Applications to Lifetime Data. *J. of Adv. in Math. and Comp. Sci.*, 2019; 34(3-4): 1-28.

[12] Alzaatreh A, Famoye F & Lee C. A new method for generating families of continuous distributions. *Metron,* 2013; 71: 63–79.

[13] Ieren TG, Kromtit FM, Agbor BU, Eraikhuemen IB & Koleoso PO. A Power Gompertz Distribution: Model, Properties and Application to Bladder Cancer Data. *Asian Res. J. of Math.*, 2019; 15(2): 1-14.

[14] Tahir, M. H., Hussain, M. A., Cordeiro, G. M., Hamedani, G. G., Mansoor, M and Zubair, M. (2016). The Gumbel-Lomax Distribution: Properties and Applications. *Journal of Statistical Theory and Applications*, 15(1): 61-79.

[15] Yahaya A & Lawal A. A study on Lomax-Lindley distribution with applications to simulated and real life data, a paper Presented at the 2nd Int. Conf. Nig. Stat. Society, held at the CES Building, University of Calabar, Calabar, Nigeria. 10th -12th April, 2018.

[16] Shanker R, Kamlesh KK & Fesshaye H. A Two Parameter Lindley Distribution: Its Properties and Applications. *Biostat Biometrics Open Acc J.*, 2017; 1(4): 555570. DOI: 10.19080/BBOAJ.2017.01.555570.

[17] Merovci C. Transmuted Lindley distribution. Int. J. Open Problems Compt. Math., 2013; 6(2): 63-74.

[18] Lindley DV. Fiducial distributions and Bayes' theorem, *J. Royal Stat. Soc. Series B*, 1958; 20: 102-107.

[19] Martz HF & Waller RA. *Bayesian reliability analysis*. New York, NY: John Wiley, 1982.

[20] Ibrahim JG, Chen MH & Sinha D. *Bayesian survival analy.*. New York, NY: Springer-Verlag, 2001.

[21] Singpurwalla ND. *Reliability and risk: A Bayesian perspective*. Chichester: John Wiley, 2006.

[22] Dey S. Bayesian estimation of the shape parameter of the generalized exponential distribution under different loss functions. *Pakistan J. of Stat. and Operat. Res.*, 2010; *6*(2): 163-174.

[23] Ahmed AOM, Ibrahim NA, Arasan J & Adam MB. Extension of Jeffreys' prior estimate for weibull censored data using Lindley's approximation. *Austr. J. of B. and Appl. Sci.*, 2011; 5(12): 884–889.

[24] Pandey BN, Dwividi N, Pulastya B. Comparison between Bayesian and maximum likelihood estimation of the scale parameter in Weibull distribution with known shape under linex loss function, *J. of Sci. Resr.*, 2011; 55: 163–172.

[25] Mabur TM, Omale A, Lawal A, Dewu MM & Mohammed S. Bayesian Estimation of the Scale Parameter of the Weimal Distribution. *Asian J. of Prob. and Stat.*, 2018; 2(4): 1-9.

[26] Aliyu Y & Yahaya A. Bayesian estimation of the shape parameter of generalized Rayleigh distribution under non-informative prior. *Int. J. of Adv. Stat. and Prob.*, 2016; 4 (1):1-10.

[27] Ahmad SP & Ahmad K. Bayesian analysis of weibull distribution using R software, *Austr. J. of B. and Appl. Sc.*, 2013; 7(9), 156–164.

[28] Ahmad K, Ahmad SP & Ahmed A. On parameter estimation of erlang distribution using bayesian method under different loss functions, *Proc. of Int. Conf. on Adv. in Computers, Com., and Electronic Eng.*, 2015; 200–206.

[29] Ieren TG, Chama AF, Bamigbala OA, Joel J, Kromtit FM & Eraikhuemen IB. On A Shape Parameter of Gompertz Inverse Exponential Distribution Using Classical and Non Classical Methods of Estimation. *J. of Sci. Res. & Rep.*, 2020; 25(6): 1-10.

[30] Ieren TG & Oguntunde PE. A Comparison between Maximum Likelihood and Bayesian Estimation Methods for a Shape Parameter of the Weibull-Exponential Distribution. *Asian J. of Prob. and Stat.*, 2018; 1(1): 1-12.

[31] Ieren TG & Chukwu AU. Bayesian Estimation of a Shape Parameter of the Weibull-Frechet Distribution. *Asian J. of Prob. and Stat.*, 2018; 2(1):1-19, 2018.

[32] Krishna H & Goel N. Maximum Likelihood and Bayes Estimation in Randomly Censored Geometric Distribution. *J. of Prob. and Stat.*, 2017. 12 pages.

[33] Gupta PK & Singh AK. Classical and Bayesian estimation of Weibull distribution in presence of outliers. *Cogent Math.*, 2017; 4: 1300975.

[34] Gupta I. Estimation of Parameter And Reliability Function of Exponentiated Inverted Weibull Distribution using Classical and Bayesian Approach. *Int. J. of Recent Sci. Res.*, 2017; 8(7): 18117-18819.

[35] Ahmad K, Ahmad SP & Ahmed A. Classical and Bayesian Approach in Estimation of Scale Parameter of Nakagami Distribution. *J. of Prob. and Stat.*, Article ID 7581918, 8, 2016. pages http://dx.doi.org/10.1155/2016/7581918.

[36] Preda V, Eugenia P & Alina C. Bayes Estimators of Modified-Weibull Distribution parameters using Lindley's approximation. *WSEAS TRANSACTIONS on MATHEMATICS*, 2010; 9 (7), 539-549.

[37] Abbas K, Abbasi NY, Ali A, Khan SA, Manzoor S, Khalil A, Khalil U, Khan DM, Hussain Z & Altaf M. Bayesian Analysis of Three-Parameter Frechet Distribution with Medical

Applications. *Comput. and Math. Meth. in Medicine*, Volume 2019, Article ID 9089856, 8 pages, https://doi.org/10.1155/2019/9089856.

[38] Ramos PL, Louzada F & Ramos E. Posterior Properties of the Nakagami-m Distribution Using Noninformative Priors and Applications in Reliability. *IEEE Transactions on reliability*, 67(1): 105-117, 2017.

[39] Ramos PL & Louzada F. Bayesian reference analysis for the Generalized Gamma distribution. *IEEE Communications Letters*, 2018; 22(9), 1950-1953.

[40] Tomazella VLD, de Jesus SR, Louzada F, Nadarajah S & Ramos PL. Reference Bayesian analysis for the generalized lognormal distribution with application to survival data. *Stat. and Its Interface*, 2020; 13(1): 139-149.

[41] Azam Z, & Ahmad SA. Bayesian Approach in Estimation of Scale Parameter of Nakagami Distribution. *Pakistan J. of Stat. and Operat. Res.*, 2014; 10(2), 217-228.

[42] Norstrom JG. The use of precautionary loss functions in risk analysis. IEEE Trans. Reliab., 1996; 45(3): 400–403, 1996. http://dx.doi.org/10.1109/24.536992.