Original Research Article

ON THE NUMBER OF CENTER POINTS FOR RESPONSE SURFACE EXPLORATION USING THE SMALL BOX – BEHNKEN DESIGN

Abstract

Industrial exploration using the Box-Behnken Design (BBD) has been faced with a serious setback. The design runs size increase rapidly as the number of factors increase. This, therefore, discourages researchers and practitioners from using it.The Small Box-Behnken designs (SBBD) which achieve the research goal of BBD were proposed to overcome the setback. This paper, aimed at recommending an appropriate number of center points suitable for response surface exploration using the SBBD. The method adopted for assessing the center points is the G-efficiency optimality criterion, which is a prediction variance-based optimality criterion. The range of design factors, k, considered is 3 to 11, while comparing the designs at0 - 5 number of center points. For each of the design factors considered, the result showed that increasing the center point, decreases the G-efficiency value.The implication of this finding suggests that increasing the center point does not contribute significantly to the prediction variance capability of the designs considered. However, for the estimation of pure error and test of model lack of fit which is very important in experimental design analysis, this study recommends that at most two runs (center points) be replicated at the center since with this number, approximately 90% G-efficiency can be achieved for response surface exploration using the Small Box-Behnkendesigns.

Keywords: Small Box-Behnken design, Center Point, Prediction Variance, G-Efficiency, Optimality

1. Introduction

[1], developed the SDDB as an alternative to the Box-Behnken Design (BBD) which has a major setback of the rapid increase in the run size as the number of factors, k, increases. This discourages many practitioners from using BBD for experimental and industrial explorations. The SDDB possesses reasonably high D-efficiency, a much smaller run size compared to BBD, and preserved the original orthogonality property of the BBD. Hence, the designs are preferred to BBD for fitting the second-order response surface model. [2] investigated the percentage rotatability for evaluating the SBBD using the measure of rotatability introduced by Khuri (1988). They observed that factors k, the SBBD is rotatable for k = 3 factors, near rotatable for k = 4, 7 factors and not rotatable for k = 5, 6, 8, 9, 10 and 11 factors. However, no work has been done on recommending the appropriate number of center points suitable for evaluating the SBBD. The center point(s) is very important and plays vital role in the estimation of pure error and test of model lack of fit in experimental design analysis.This is the basics for this study.

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Considering that all variables in the response surface model are assumed to be measurable, the expressions in (1) and (2) for the first-order response surface model and second-order response surface model respectively, are used to quantify the relationship between the controllable input parameters and the obtained response surfaces in Response Surface Methodology as shown in [3].

$$T = \tau_0 + \sum_{i=1}^{k} \tau_i x_i + \xi \qquad (1)$$

$$T = \tau_0 + \sum_{i=1}^{k} \tau_i x_i + \sum_{i<j}\sum \tau_{ij} x_i x_j + \sum_{i=1}^{k} \tau_{ii} x_i^2 + \xi \qquad (2)$$

where T, is a $N \times 1$, vector of responses, x, is an $N \times P$, of the design matrix. The τ ξ y consist of coefficients of the design factors under consideration and errors term, with dimension $P \times 1$ and $N \times 1$ respectively.

2. Optimality criteria

It is important to note the fact that a design performs better than other designs under certain optimality criteria does not always guarantee that it will retain such performance when considered by other optimality criteria. Hence, to choose a design, attention will be on the choice of design evaluation criteria used. The common optimality criteria used in design evaluation are A-, D-, E-, and G- optimality criteria, [4]. [5] in a comparative study of five varieties of CCDs - Central Composite designs (SCCD, RCCD, OCCD, Slope-R, FCC) in RSM, evaluated the performances of the designs using the D-, A-, G- and IV-optimality criteria. The results showed a reduction in the D- and G-optimality criteria, and an increase in the A-optimality criterion while the IV-optimality remain relatively the same for the CCDs in all the factors that were considered at a different number of center points. [6], evaluate and compare the performances of three classes of Central Composite Design CCDs (CCCD, CCFD, and CCID) using the A-, D-, and G- efficiencies for factors, k, that range from 3 to 10, with 0-5 center points. It was shown from the results that, for the three CCDs compared, the G-efficiency outperformed other efficiency criteria employed for all the factors and center points considered. [7], applied the D- and G-optimal criteria on non-pure blends slope designs to study the second-order Kronecker model on Equally Weighted Simplex Centroid Axial Design and Un-equally Weighted Simplex Centroid Axial Design. It was shown from the result that the D- and G-optimality values performed better on both centroids compared.

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3. Center Points

The center points are observations collected at the center of all factor ranges, $x_i = 0, (i = 1,2,...,k)$. These replicated points at the center of all factors are among other things, used to calculate the pure error of second-order models, to check for curvature, and to provide a center point adds to the estimation of the coefficients of quadratic terms and are used to identify curvature measure of process stability and inherent variability: see [8].According to [9], in the response.[10], opined that in other to avoid singularity in the information matrix of a design, an effort should be to add center runs to the design so as to maintain favorable design qualities such as good prediction variance. [11], tested the effect of varying the number of center points on

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parameters estimation by employing the optimality criteria A-, D-, and E. [12] used the integrated variance criterion to determine an appropriate number of center points for response surface designs; he concluded that fewer center points are appropriate. [13] recommended a different number of the center point, ranging from 3 to 12 for the Box-Behnken designs. [14], examined the contributions of center points on prediction variance performances on CCDs using the G-optimal, I optimal, and FDS plots. It was discovered that the designs perform better with or without replication (center points).

4. Evaluation of the Appropriate Number of Center Points

In this section, center points ranging from 0 to 5 will be compared for design factors, k, ranging from 3 to 11, using the G-efficiency criteria. Let, n_o, indicate the number of center points and N, the number of design runs.

The expanded design matrix for SDDB for k = 3, with n_o= 0 is;

$$X = \begin{bmatrix} 1 & -1 & -1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 & -1 & 0 & 0 & 1 & 1 & 0 \\ 1 & -1 & 1 & 0 & -1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & -1 & 0 & -1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & -1 & 0 & -1 & 0 & 1 & 0 & 1 \\ 1 & -1 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & -1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & -1 & 0 & 0 & -1 & 0 & 1 & 1 \\ 1 & 0 & -1 & 1 & 0 & 0 & -1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

$$(X'X)^{-1} = \begin{bmatrix} 1.28674E+15 & 0 & 0 & 0 & 0 & 0 & 0 & -6.43371E+14 & -6.43371E+14 & -6.43371E+14 \\ 0 & 0.125 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.125 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.125 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.125 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.125 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.125 & 0 & 0 & 0 \\ -6.43371E+14 & 0 & 0 & 0 & 0 & 0 & 0 & 3.21686E+14 & 3.21686E+14 & 3.21686E+14 \\ -6.43371E+14 & 0 & 0 & 0 & 0 & 0 & 0 & 3.21686E+14 & 3.21686E+14 & 3.21686E+14 \\ -6.43371E+14 & 0 & 0 & 0 & 0 & 0 & 0 & 3.21686E+14 & 3.21686E+14 & 3.21686E+14 \end{bmatrix}$$

$$x = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

$x'(X'X)x$ = 0.7500.

The G-efficiency is obtained by $100\frac{P/N}{\sigma^2_{\max}}$, where, $\sigma^2_{\max}$ is the maximum prediction variance, N is the number of design runs, and P is the number of parameters for each of the designs considered.

The same procedure was used to obtain the G-efficiencies for all the factors at different center points. The results for the G-efficiency for 0 - 5 center points for each of the factors under consideration are shown in Table 1.

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Table 1: Result of G-efficiency values for 0 – 5 center points.

K	**P**	**N**	$Max[x'(X'X)x]$	n_{+0}	n_{+1}	n_{+2}	n_{+3}	n_{+4}	n_{+5}
3	10	12	0.7500	111.11	102.56	95.24	88.89	83.33	78.43
4	15	16	1	93.75	88.24	83.33	78.94	75.00	71.43
5	21	24	0.8889	98.44	94.50	90.86	87.50	84.37	81.46
6	28	32	0.9167	95.45	92.56	89.84	87.27	84.85	82.55
7	36	40	1	90.00	87.81	85.71	83.72	81.82	80.00
8	45	56	0.9670	83.10	81.64	80.23	78.87	77.56	76.29
9	55	60	0.9333	98.22	96.61	95.05	93.54	92.08	90.66
10	66	76	0.9153	94.88	93.65	92.45	91.28	90.13	89.02
11	78	96	0.8967	90.61	89.68	88.76	87.86	86.99	86.12

From Table 1, it could be seen that additional center points to each of the designs under consideration rather decreases the G-efficiency, hence, there is no need to increase the number of center points. However, for error estimation which is very important in experimental analysis, one or two center points are recommended since, with this number, one can still achieve approximately 90% G-efficiency.

5. Fig 1: Graphical assessment

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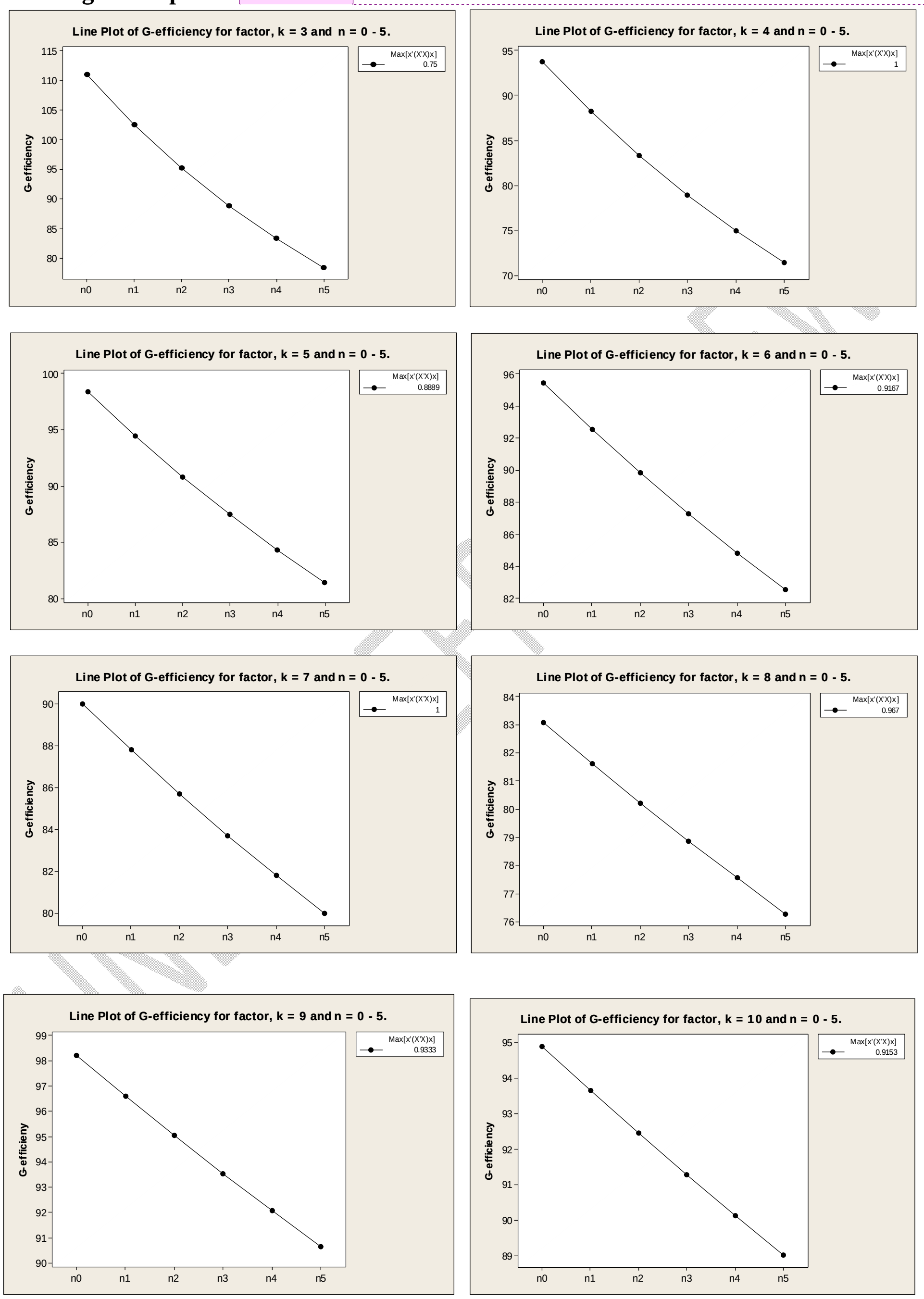

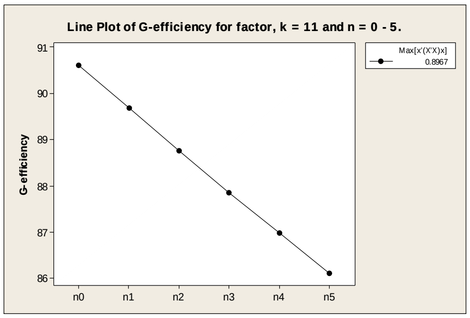

The graphical assessment of the G-efficiency at 0 to 5 center point, shows a decrease in the G-efficiency value as the center point rises and this result was consistent for all the factors under consideration.

6. Conclusion

This study has examined the SBBD at a different number of center points in other to recommend the appropriate number of center points required for response surface exploration using design. It can be seen from the result that, as the number of center points rises from 0 to 5, the G-efficiency decreases. This finding is in agreement with the findings of [6], where the G-efficiency performed better than other alphabet-based optimality criteria, and in particular, the G-efficiency decreases as the number of center pointers increases for CCFD. The implication of this finding suggests that increasing the center point does not contribute significantly to the prediction variance capability of the designs considered. However, for the estimation of pure error and test of model lack of fit, this study recommends that at most two runs (center points) be replicated for response surface exploration using the Small Box-Behnken designs.

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