Magic Polygons and Combinatorial Algorithms

Original Research Article

Abstract

In this work, we study the Magic Polygons of order 3 ($P(n,2)$) and we introduce some properties that was useful to build an algorithm to find all possibilities for all possibles regular polygons up to 24 sides. The concepts of Equivalents Magic Polygons and Derivatives Magic Polygons which allowed to classify and avoid ambiguity about the representations of such elements are also introduced.

Keywords: Magic Polygons, Combinatorial Algorithms, Symmetric Group, Dihedral Group;
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1 Introduction

Magic Polygons are introduced in [1] where its properties are introduced, with existence conditions and is provided a way to built a Magic Polygon for all regular polygons of n sides, if n is even greater than six. In [2] a generalization of Magic Polygons is made and is presented a new class of Magic Polygons called “Degenerated Magic Polygons”. However, about the Magic Polygons, is known that a Magic Polygon exists for all regular polygons of n with the restriction cited, but is not known how many they are, even for $n = 6$. The main goal of this work, is to show how many Magic Polygon exists for all polygon ip to 24 sides. Here is showed the properties on what the algorithm relys. This way, is known how many Magic Polygos exists for regular polygons up to 24 sides. Is also introduced the concepts of Equivalents Magic Polygons and Derivatives Magic Polygos.

2 Magic Polygons - Definition and Properties

Definition 2.1. Let Ω be a set of $\frac{k}{2}$ regular polygons on plane with n sides and corresponding parallel sides and centered in a central point C.

A magic polygon $P(n,k)$ of n sides and order $k+1$ is a set of $\frac{k^2n}{2}+1$ points satisfying the following conditions:

(i) Points of magic polygon are labeled by distict values from 1 to $\frac{k^2 n}{2}+1$;

(ii) One point of a magic polygon is the central point C;

(iii) $\frac{kn}{2}$ points of magic polygon are vertices of the $\frac{k}{2}$ regular polygons of Ω;

(iv) The magic polygon has $k-1$ intermediate points on each edge of regular polygons in Ω, which gives a total of $\dfrac{(k-1)nk}{2}$ intermediate points.

(v) Segments with diametrically opposite ends of the larger polygon of Ω intersecting the central vertex contain $k+1$ points of the magic polygon;

(vi) Segments with ends at two adjacent vertices of a polygon of Ω contains $k+1$ points of the magic polygon;

(vii) The sum of values corresponding to the $k+1$ points on each segment defined in (iv) and (v) is a fixed value u, called of **magic sum.**

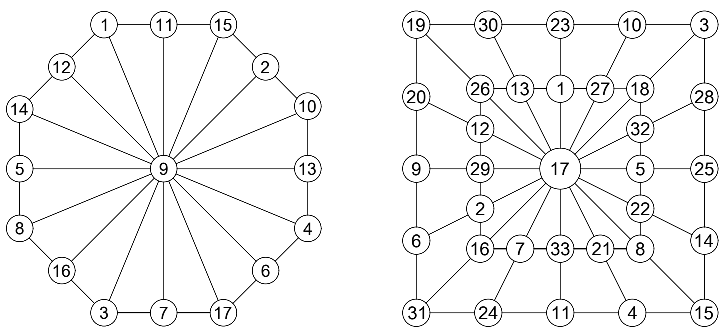

Figure 1: Examples of Magic Polygons $P(8,2)$ and $P(4,4)$ respectively

Theorem 2.1. *A Magic Polygon $P(n,k)$ has the following properties:*

(i) the magic sum is: $(k+1)\dfrac{k^2n+4}{4}$;

(ii) the value of the central point is: $\dfrac{k^2n+4}{4}$

(iii) the sum S_j of the values representing to $j-th$ points partitioning each edge on magic polygon chosen clockwise is:

$$S_j = \frac{kn(k^2n+4)}{8}$$

These properties are prooved in [2].

If we take $k=$, Magic Polygons $P(n,2)$ are formed by one regular polygon, the vertex, and the edges middle points and its geometric center. Therefore has $2n+1$ elements. Its magic sum is $3(n+3)$, the root vertex has the value $n+1$ and the sum S_j of all elements in the vertex is $n(n+1)$ (the same for the sum of the elements in the edges middle points). We note that the magic sum u, is three time the root vertex c ($u = 3(n+1)$).

In [1, 2] is prooved that a Magic Polygon $P(n,2)$ exist if, and only if, the regular polygon has a number of sides even and greater than six. All approach made in this work is valid only for the Magic Polygons of order 3 ($P(n,2)$).

Is easy to see that Magic Polygons has a discrete and finite framework, therefore we can adopt an algebraic representation for such objects in order to be aided by algebraics concepts to explore its properties. Once the geometric framework of magic polygons has the symmetric properties natives in the regular polygons, and its elements are positives integers numbers along of the perimeter, the concepts of *dihedral group*, *finite groups* $\mathbb{Z}/n\mathbb{Z}$ and *symmetric groups*, shall be very importants.

As said before, a Magic Polygon is about the natural numbers $1, 2, 3...2n+1$ along the perimeter of the regular polygon, so it is nothing beyond a permutation of such elements, therefore a Magic Polygon can be algebraically represented in the well known notation for symmetric groups:

$$\begin{pmatrix} 1 & 2 & \dots & n & n+1 & n+2 & \dots & 2n & 2n+1 \\ f(1) & f(2) & \dots & f(n) & n+1 & f(n+2) & \dots & f(2n) & f(2n+1) \end{pmatrix}$$

where the numeration direction is counter-clockwise, and he first element is by convention the right vertex nearest the symmetric vertical axis of the polygon, and above the horizontal symmetric axis. As we can see, the image of $n+1$ is $n+1$, this is in fact the root vertex that stands in the polygon geometric center.

Definition 2.2. An element of the symmetric group of $(2n+1)!$ order, is an algebraically representation of the Magic Polygon of n if the following conditions are satisfied:

$$\begin{cases} i) & f(n+1) = n+1; \\ ii) & \forall i, 1 \le i \le n, f(i) + f(i+(n+1)) = 2c; \\ iii) & \forall i \text{ odd }, 1 \le i < n-1, f(i) + f(i+1) + f(i+2) = u; \\ iv) & f(n-1) + f(n) + f(n+2) = u; \\ v) & \forall i \text{ even }, n+2 \le i < 2n-1, f(i) + f(i+1) + f(i+2) = u; \\ vi) & f(2n) + f(2n+1) + f(1) = u; \end{cases}$$

The first condition is due to vertex root of the Magic Polygon, which value is fixed with $n+1$, the second is due to the opposite elements in the perimeter of the polygon, once both stands in the same symmetric axis that splites the polygon in two, they are the elements $f(i)$ and $f(i+n+1)$, because the central point is the element $f(n+1)$, thus $f(i) + f(i+n+1) + c = u$, once $u = 3c$ we have $f(i) + f(i+n+1) = 2c$. The next condition is due to the sum of the elements along the edge of regular polygon, by definition, their sum must match u. Once $f(1)$ is by definition a vertex, for all i odd less than $n-1$ it follows that $f(i) + f(i+1) + f(i+2) = u$. We have, $f(n+1)$ as central point, thus $f(n-1)$,$f(n)$ e $f(n+2)$ are in the same edge and their sum shall be the magic sum. From the element $n+2$ beyond, if i is even, so $f(i)$, $f(i+1)$ shall $f(i+2)$ be in the same edge, and therefore their sum must be the magic sum. It give us the fifth condition. By last, the sixth condition refers the fact that the elements $f(2n)$, $f(2n+1)$ and $f(1)$ stands in the same edge which close the polygon.

By adopting this convention, we can represent the Magic Polygon $P(8, 2)$ on the Figure 1 by the following way:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \mathbf{9} & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 \\ 15 & 11 & 1 & 12 & 14 & 5 & 8 & 16 & \mathbf{9} & 3 & 7 & 17 & 6 & 4 & 13 & 10 & 2 \end{pmatrix}$$

It follows that the Magic Polygons are elements of the symmetric group, but as a subset, they didn't make a subgroup, because it's obvious that the identity function, which is the neutral element in the symmetric group, does not represent a Magic Polygon. From this representation, to find all Magic Polygons for a given n, one option is by declaring $2n$ variables, and visiting all permutations of the set:

$$\{1, 2, 3 \dots 2n, 2n+1\} - \{n+1\}$$

and verify, which ones satisfies the the conditions listed in Definition 2.2 However, such algorithm would return the same permutation many times, and this is not good for the computation.

Based in some Magic Polygons properties that shall be here approached, will be possible find all Magic Polygons for a given n by declaring only n variables, and reducing how many times the same permutation is returned. Simplifying and fastering the computational process.

Theorem 2.2. *The algebraic representation of a Magic Polygon is well defined by the values* $f(1)$, $f(2)$, $\dots$ $f(n)$. *Puting in other words, a Magic Polygon can be represented by a* $n-$*permutations of the set* $\{1, 2, 3, \dots 2n+1\}$.

Proof. Let $f : \{1, 2, \dots n\} \mapsto \{1, 2, 3, \dots 2n+1\}$ an one-to-one function which has the following properties:

i) $f(i) \neq n+1 \; \forall \; i \in \{1, 2, \dots n\}$;
ii) $f(i) + f(i+1) + f(i+2) = u \; \forall \; i$ odd $\in \{1, 2, \dots n-2\}$;
iii) $f(i) + f(j) \neq 2c \; \forall \; i, j \in \{1, 2, \dots n\}$;
iv) $f(n-1) + f(n) - f(1) = c$;

ie, f defines a n-permutation of $\{1, 2, 3, \dots 2n+1\}$.

Let's extend the function f for the set $\{1, 2, 3 \dots 2n+1\}$ in order to generate an element belonging the symmetric group of order $(2n+1)!$ this way:

- $f(n+1) = n+1$;
- $\forall i \in \{1, 2 \dots n\}, f(i+n+1) = 2c - f(i)$

so we have the σ element:

$$\sigma = \begin{pmatrix} 1 & 2 & \dots & n & n+1 & n+2 & \dots & 2n & 2n+1 \\ f(1) & f(2) & \dots & f(n) & n+1 & f(n+2) & \dots & f(2n) & f(2n+1) \end{pmatrix} \tag{2.1}$$

Affirmation: σ represents a Magic Polygon.

The injectivity of original function f with the properties i and iii ensures the bijectivity of the extended function for the set $\{1, 2, \dots 2n+1\}$. By propertie ii, fora all i odd $1 \leq i < n-1 \Rightarrow f(i) + f(i+1) + f(i+2) = u$ thus:

$$\begin{gathered} f(i+n+1) + f(i+n+2) + f(i+n+3) = \\ (2c - f(i)) + (2c - f(i+1)) + (2c - f(i+2)) = \\ 6c - (f(i) + f(i+1) + f(i+2)) = \\ 6c - 3c = 3c = u \end{gathered}$$

this way, for all i even $n+2 \leq i \leq 2n-2$, $f(i) + f(i+1) + f(i+2) = u$. By propertie iv, $f(n-1) + f(n) - f(1) = c$ as $f(i+n+1) = 2c - f(i) \Rightarrow -f(1) = 2c - f(n+2)$ therefore:

$$\begin{gathered} f(n-1) + f(n) - 2c + f(n+2) = c \\ f(n-1) + f(n) + f(n+2) = 3c = u \end{gathered}$$

this way sum of the elements $f(n-1)$, $f(n)$ and $f(n+2)$ is the Magic Sum:

Now we have:

$$\begin{gathered} f(2n) = 2c - f(n-1) \\ f(2n+1) = 2c - f(n) \\ f(1) = 2c - f(n+2) \end{gathered}$$

As seen before, $f(n-1) + f(n) + f(n+2) = u$, the sum $f(2n) + f(2n+1) + f(1)$ shall be:

$$2c - f(n-1) + 2c - f(n) + 2c - f(n+2) =$$
$$6c - (f(n-1) + f(n) + f(n+2)) =$$
$$6c - 3c = 3c = u$$

Therefore, in conformity with the Definition 2.2 the sum of all elements in the edges of Regular Polygons satisfies the magic sum.

Now, it's necessary verify if the sums of the elements on the symmetric axis also satisfies the magic sum.

The sums in the symmetric axis, match the pattern $f(i) + c + f(i+n+1)$, where $1 \leq i \leq$. Once $f(i+n+1) = 2c - f(i)$ the result follows.

□

The Definition 2.2 introduces the Magic Polygons as a permutation of the set $\{1, 2, \ldots 2n+1\}$ which satisfies a few properties. In other words, the process of find Magic Polygons is a combinatorial issue. And based in the Theorem 2.2 is enough to declare only n variables in the algorithm.

3 Equivalents Magic Polygons

It's easy to see geometrically that rotations based on the geometric center of the polygon, and reflection on the symmetric axis, provide us an different way to represent what is the same Magic Polygon. A first look may induce to think that is another Magic Polygon, but in fact, is the same, represented by a distinct way.

This encourage us to define Equivalent Magic Polygons, distinct representations that we can obtain from a Magic Polygon by applying two operations, the rotation (r_o) and the reflection (r_e).

In this paragraph, in the one following, we adopt the notation:

$a \leftrightarrow b$ means that the value a and the value b switches places;

$a \rightarrow b$ means that the value a is replaced by the value b;

Definition 3.1. Let σ an element of symmetric group that represents a Magic Polygon. We say that the element σ' of S_{2n+1} is an Equivalent Magic Polygon σ if σ' can be obtained from σ applying the following operations a finite number of times:

ii) Rotation in the geometric center of polygon:

$r_o(\sigma)$:

$$\begin{pmatrix} 1 & 2 & \text{3} & .. & \text{n-1} & n & n+1 & n+2 & .. & 2n & 2n+1 \\ f(2n) & f(2n+1) & \text{f(1)} & .. & \text{f(n-3)} & f(n-2) & n+1 & f(n) & .. & f(2n-2) & f(2n-1) \end{pmatrix}$$

ii) Reflection applied in vertex:

$r_{ev}(\sigma)$:

$$\begin{pmatrix} 1 & 2 & \text{3} & .. & \text{n-1} & n & n+1 & n+2 & .. & 2n & 2n+1 \\ f(1) & f(2n+1) & \text{f(2n)} & .. & \text{f(n+4)} & f(n+3) & n+1 & f(n+2) & .. & f(3) & f(2) \end{pmatrix}$$

If $i \oplus j = 2$ on group $\mathbb{Z}/(2n+1)\mathbb{Z} \Rightarrow f(i) \leftrightarrow f(j)$

*iii)*Reflection applied in the Edge Middle Point:

$r_{em}(\sigma)$:

$$\begin{pmatrix} 1 & 2 & \text{3} & .. & \text{n-1} & n & n+1 & n+2 & .. & 2n & 2n+1 \\ f(2n) & f(2n-1) & \text{f(2n-2)} & .. & \text{f(n+2)} & f(n) & n+1 & f(n-1) & .. & f(2) & f(2n+1) \end{pmatrix}$$

If $i \oplus j = 0$ on group $\mathbb{Z}/(2n+1)\mathbb{Z} \Rightarrow f(i) \leftrightarrow f(j)$

In reflection operation applied on vertex, the elements $f(n+2)$ e $f(n+1)$ would switch their places, however, we know that the element $f(n+1)$ is a fixed element, thus, both remain without changes. In case $i = 1$, $f(i)$ would switch place with itself, for $1 + 1 = 2 \in \mathbb{Z}/(2n+1)\mathbb{Z}$, therefore it also remain without change. Geometrically, the line that contain the elements $f(1)$, $f(n+1)$ e $f(n+2)$ defines the symmetric axis by which we reflect the polygon, and, as expect, such elements does not change its positions.

In reflection operation applied edge middle point, something similar occurs, the elements $f(n+2)$ e $f(n+1)$ would switch their places, however, once again $f(n+1)$ is fixed, therefore both remain without change. And the element $f(2n+1)$ would switch with itself, therefore remain with no change. Again such elements stands in the symmetric axis by which the polygon is reflected.

Such operations are due to the Dihedral Group (D_n) that acts in the regular polygon where is built the Magic Polygon. We note that in a Magic Polygon, what really matters, is the adjacent elements, and the elements that lies in the end of the lines that defines the symmetric axis. In a Dihedral Group, such setup is not affected. Therefore, every element of a Dihedral Group gives us a distinct mode to represent exactly the same Magic Polygon. So we have the following proposition:

Proposition 3.1. *Every Magic Polygon of n sides can be represented by $2n$ distincts ways.*

Considering the way fo the numeration, defined above, and the definition of the r_o operation, we can see that such operation occurs in the clockwise way. And the reflection is applied only in two vertex, one of them is defined by two vertex, and the another by two edge middle points. However is known that a regular polygon has n symmetric axis, but there's no loss of generality here. Because the definition takes two axis that are like the "representants" of the others, thus, if a composition of operations is made, we can reflect the Magic Polygon in any symmetric axis, by the following way:

$$r_o^{n-k}(r_e(r_o^k(\sigma))) \star$$

onde r^k means the r_o applied k times, where k is the numbers of times necessary for the choosen axis match one of those given in the definition. And r_o^{n-k} make the axis back to its original position. Because $r_o^{n-k}(r_o^k(\sigma)) = r_o^n = e$ (neutral element). Note that the Dihedral Group is not commutative, so the order of the operations in $\star$ cannot be changed.

Taking as example the Polygon $P(8,2)$ showed in the Figure 1, if we applied the r_o operation we shall have:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \mathbf{9} & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 \\ 15 & 11 & 1 & 12 & 14 & 5 & 8 & 16 & \mathbf{9} & 3 & 7 & 17 & 6 & 4 & 13 & 10 & 2 \end{pmatrix}$$

$$r_o(\sigma) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \mathbf{9} & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 \\ 10 & 2 & 15 & 11 & 1 & 12 & 14 & 5 & \mathbf{9} & 8 & 16 & 3 & 7 & 17 & 6 & 4 & 13 \end{pmatrix}$$

The Figure 2 show both representations:

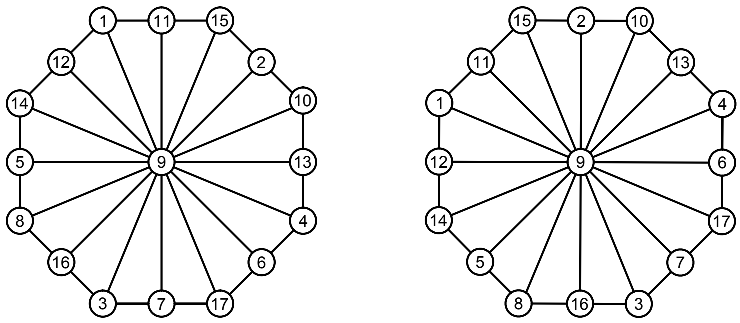

Figure 2: To left σ, to right $r_o(\sigma)$

Again, taking the Magic Polygon $P(8,2)$, if we applying in it the reflection in the vertex, we have:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \mathbf{9} & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 \\ 15 & 11 & 1 & 12 & 14 & 5 & 8 & 16 & \mathbf{9} & 3 & 7 & 17 & 6 & 4 & 13 & 10 & 2 \end{pmatrix}$$

$$r_{ev}(\sigma) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \mathbf{9} & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 \\ 15 & 2 & 10 & 13 & 4 & 6 & 17 & 7 & \mathbf{9} & 3 & 16 & 8 & 5 & 14 & 12 & 1 & 11 \end{pmatrix}$$

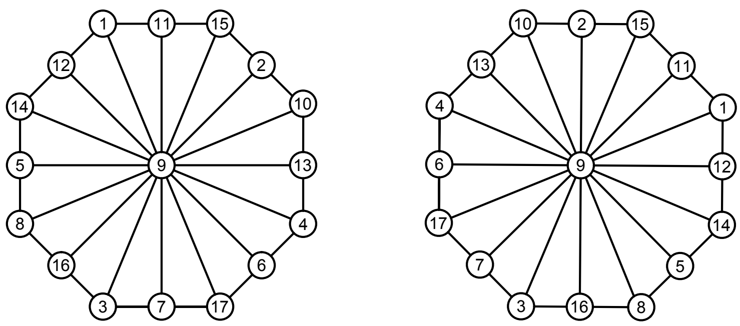

Figure 3: To left σ to right $r_{ev}(\sigma)$

4 Derivatives Magic Polygons

By definition, the Magic Polygons, are built with the natural sequence from 1 to $2n+1$, and the value $n+1$ is fixed in the geometric center of the polygon. but is easy to see that if an constant k is added to every values in the Magic Polygon, the sum keeps constant, because, once $x+y+z=u$ so: $(x+k)+(y+k)+(z+k)=u+3k$ and $k \in \mathbb{Z}$. Technically this is not a Magic Polygon, because according the definition, a Magic Polygon is strictly build with the values $1, 2, \ldots, 2n+1$.

An particular case is the null magic sum, this can be done by subtracting $n+1$ from every element from a Magic Polygon, the Figure Below ilustrates that.

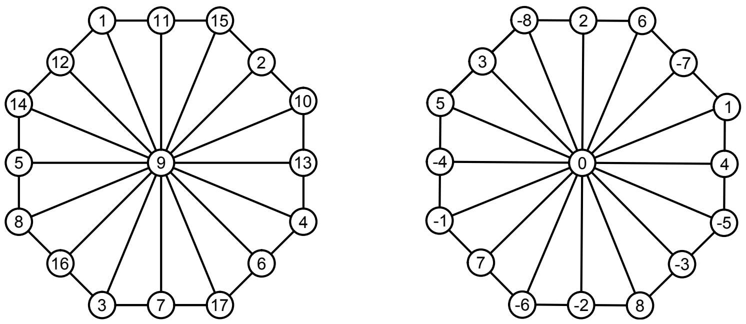

Figure 4: Null Magic Sum

More generally, it is possible replace every value x in the Magic Polygon by $x-th$ value from a arithmetic progression that the sum keeps constant, this is prooved in the Theorem below.

Theorem 4.1. *Let σ an element of the symmetric group that represents a Magic Polygon, and be $a_1, a_2, a_3 \ldots, a_{2n}, a_{2n+1}$ an arithmetic progression where the first value is a_0 and common diference r. The replace of every $f(i)$ por $a_{f(i)}$, change the magic sum to $3(a_1+nr)$ and the value of the root vertex to a_1+nr.*

Proof. Is known that if a_1 is the first term of an arithmetic progression, the $n-th$ is: $a_1+(n-1)r$

Let $f(i)+c+f(i+n+1)=u$ any sum along the symmetric axis of the polygon, due the rotation (r_o), let's suppose, without loss of generality $f(i) > f(i+n+1)$, where $1 < i < n$. So we have $f(i) = c+k$ for some k positive, and $f(i+n+1) = c-k$, because as prooved before, $f(i)+f(i+n+1) = 2c$. Once we have $c = n+1$ so:

$$\begin{aligned} f(i) &= c+k = n+1+k \\ f(i+n+1) &= c-k = n+1-k \end{aligned}$$

Therefore, the replaces shall be:

$$\begin{aligned} f(i) &\rightarrow a_1+((n+1+k)-1)r \\ f(i+n+1) &\rightarrow a_1+((n+1-k)-1)r \\ c &\rightarrow a_1+((n+1)-1)r \end{aligned}$$

ie $c = a_1+nr$, and the new sum shall be:

$$a_1+((n+1)-1)r+a_1+((n+1+k)-1)r+a_1+((n+1-k)-1)r =$$

$$3a_1+nr+(n+k)r+(n-k)r =$$

$$3a_1 + (3n + k - k)r =$$
$$3a_1 + 3nr = 3(a_1 + nr)$$

Now, we have to verify the sum along the edges. Sejam $i, j, k \in \{1, 2, 3 \dots 2n + 1\} - \{n + 1\}$ where $f(i)$, $f(j)$ e $f(k)$ lies in the same edge, so: $f(i) + f(j) + f(k) = u$.

$$\begin{array}{rcl} f(i) & \rightarrow & a_1 + (f(i) - 1)r \\ f(j) & \rightarrow & a_1 + (f(j) - 1)r \\ f(k) & \rightarrow & a_1 + (f(k) - 1)r \end{array}$$

$$a_1 + (f(i) - 1)r + a_1 + (f(j) - 1)r + a_1 + (f(k) - 1)r =$$
$$3a_1 + (f(i) + f(j) + f(k) - 3)r =$$

By hypothesis: $f(i) + f(j) + f(k) = u = 3n + 3$, so we have:

$$3a_1 + (3n + 3 - 3)r = 3(a_1 + nr).$$

□

If $a_1 = 1$ and $r = 1$ we get the magic sum $3n + 3$ for the natural sequence $\{1, 2 \dots 2n + 1\}$ as we know.

Definition 4.1. The Magic Polygons built from an arithmetic progression that is not the natural, as procedure described in the Theorem 4.1, are called *Derivatives Magic Polygons*.

Corollary 4.1. *Let, m, c' e r integers where $m = 3k$ for some $k \in \mathbb{Z}$ and $r \neq 0$, and be $n \geq 6$ even, so:*

i) Exists a Magic Polygon of n sides built by an arithmetic progression of common diference r and magic sum m;

ii) Exists a Magic Polygon of n sides built by an arithmetic progression of common diference r, and root vertex c' and magic sum $3c'$;

Proof. Based on the Theorem 2.1 the Magic Sum is $u = 3c \Rightarrow c = k$ and $u = 3k$, therefore according Theorem 4.1 is enough to get an arithmetic progression $a_1, a_2, \dots a_{2n+1}$ of common difference r where $a_{n+1} = k$. So we have:

$$a_{n+1} = a_1 + ((n + 1) - 1)r$$
$$k = a_1 + nr \Rightarrow a_1 = k - nr$$

This way, we have the common, r and the initial term a_1 given by the relation above, now we just need to replace every value in a known Magic Polygon. The second case is the same, we just need to change k by c'.

□

5 Finding Magic Polygons

Based in the Theorem 2.2, it was implemented in the *C Programming Language* using loops and simple conditionals, programs to find all Magic Polygons up to 24 sides.

For each Magic Polygon of n sides, the algorithm has been implemented by declaring n integers variables in an array, (in the C language: *int* a[n];), and aided by the for loop, the value of each a[i] variable was changed from 1 to $2n + 1$ and with the $if()$ conditional, it was verifyed if the current permutation attended the restrictions showed in the Theorem 2.2. If so, the permutation was returned by the program and write in a plain text file. An auxiliar variable was initiated with value 0 and was increased by one each time a permutation was found, this way the number of magic polygons was counted.

Once each Magic Polygon can be represented in distinct ways, in virtue of the operation r_e and r_o here decribed, it would be desirable that the programs does not returned the same solutions many times.

By this way, the number 1 has been fixed as image of $f(1)$, ie as a vertex, and a second time as image of $f(2)$, ie as a middle point of a edge. This way, a permutation σ is returned, but the the equivalents:

$$r_o(\sigma),\ r_o(r_o(\sigma)),\ r_o(r_o(r_o(\sigma))),\ \ldots\ r_o^{n-1}(\sigma)$$

would not, for the value 1 wont't be the image of the others vertex, and they are: $f(3), \ldots f(n-1), f(n+2) \ldots f(2n)$. This way, all Magic Polygons where 1 is vertex are found. However, 1 can be middle point of an edge as well, thus running the program one more time, with the value 1 as image of $f(2)$, all Magic Polygons where 1 is middle point are found. Once 1 is necessarily a vertex or a edge middle point, all possible Magic Polygons for the current n are found.

However such procedure does not avoid the represetations $r_{ev}(\sigma)$ $r_{em}(\sigma)$ being returned, this way, all Magic Polygons are returned by the programs twice, being necessary to divide by 2 the quantity of Magic Polygons founded by the programs to know the real number. By fixing 1 first as vertex and second as a edge middle point, is avoided that n Magic Polygons Equivalents are returned, because we can rotate the regular polygon n times until it back the original position.

An example, for hexagon, the distint solution are in number of 4, however, we can represent every by $2n$ distinct modes, as mentioned before due the action of dihedral group. If no element is fixed, so the program would return 48 permutations, with this optimization, only 8 is returned. This difference for polygons with a major number of sides becames relevant, because as we shall see, the number of possibles Magic Possibles grows fast.

In order to provide examples, the tables below show all possibles Magic Polygons for the Magic Hexagon and the Magic Octogon, for both, they are in numbe of four. For the others Polygons, due the high number of possibles Magic Polygons, it would not be suitable to represent all in tables.

Table1: POSSIBILITIES FOR MAGIC HEXAGON

#	$f(1)$	$f(2)$	$f(3)$	$f(4)$	$f(5)$	$f(6)$	$f(7)$	$f(8)$	$f(9)$	$f10$	$f(11)$	$f(12)$	$f(13)$
1	1	8	12	4	5	3	**7**	13	6	2	10	9	11
2	1	9	11	4	6	2	**7**	13	5	3	10	8	12
3	8	1	12	5	4	11	**7**	6	13	2	9	10	3
4	9	1	11	6	4	12	**7**	5	13	3	8	10	2

Table 2: POSSIBILITIES FOR MAGIC OCTOGON

#	$f(1)$	$f(2)$	$f(3)$	$f(4)$	$f(5)$	$f(6)$	$f(7)$	$f(8)$
1	1	10	16	5	6	14	7	3
2	1	11	15	2	10	13	4	6
3	14	1	12	10	5	15	7	16
4	16	1	10	14	3	11	13	12

According to Theorem 2.2, the possibilitis for Magic Octogon are represented by a $8-$permutation of the set $\{1, 2, 3 \ldots 16, 17\}$.

The table below show the quantities for all Magic Polygons up to 24 sides. According Theorem 4.1 we can obtain infinities Magic Polygons, but they shall be just rellabelling of these built with the sequence {1,2,3...2n+1}.

The column $f(1) = 1$ show all possibilities when 1 is a vertex, and the column show all possibilities when 1 is a edge middle point.

Table 3: Quantities for all Magic Polygons up to 24 sides

n	$f(1) = 1$	$f(2) = 1$	Total	**n**	$f(1) = 1$	$f(2) = 1$	Total
6	2	2	4	**16**	2.034	2.950	4.984
8	2	2	4	**18**	20.056	25.218	45.274
10	4	8	12	**20**	203.265	257.563	460.828
12	87	79	166	**22**	1.908.120	2.585.799	4.493.919
14	375	487	862	**24**	22.062.915	29.268.446	51.331.361

These programs has been executed ind a Personal Computer in the Operational System *Debian GNU/Linux 11* with a 7.6 GB of RAM and the processor Intel Core $i5$-6200U 2.3 GHz Quad Core.

6 Conclusion

Basics but no trivials optimizations has been used in order to improve the algorithm. However, further properties of the Magic Polygons can be helpful to create algorithm with even better performance. By introducing the Magic Polygons as elements of the *symmetric groups*, some possibilities appears, like the aid of the concepts in group group theory and others algebraic structures to understand better such objects. However, such approach is still in an embryonic stage.

References

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