Original Research Article

Analysis of a Cournot-Bertrand Duoploy Game with Differentiated Products: Stability, Bifurcation and Control

Abstract

In this work, we investigate a Cournot-Bertrand duopoly model with product differentiation between two different firms, where one company uses price as a decision variable and the other uses quantity. The stability of the Nash equilibrium point is investigated, and the role of the parameters on the model stability is studied. It is shown that the system experiences three different types of bifurcations when the parameters go through the different boundary curves of the stability region. Furthermore, the model's bifurcation is controlled using a hybrid control strategy.

Keywords: Cournot-Bertrand; duopoly game; stability; bifurcation; chaos control

2010 Mathematics Subject Classification: 39A28; 39A30; 92D25

1 Introduction

Cournot and Bertrand were the pioneers of oligopoly research. The products in the Cournot model [10] are homogenous, and companies compete on the quantity of outputs. Firms compete on pricing in the Bertrand model [8]. There have been a few papers written in recent years about Cournot-Bertrand competition [7, 14, 15, 19]. This type of competition is characterized by the fact that the market can be subdivided into two groups of firms: the first of which optimally adjusts prices, and the other of which optimally adjusts their output to ensure maximum profit. There are examples of Cournot-Bertrand mixed models in the real world economy. For instance, in a market characterized by duopoly, one company competes in a dominating position and selects output as its decision variable, whilst the other company, which is in a disadvantageous position, selects price as its decision variable in an effort to increase its share of the market. In general, the upstream companies in the supply chain prefer output competition, while the downstream firms prefer price wars.

We consider a differentiated product market with an inverse demand function. It is assumed that the market promotes differentiated products to give consumers certain preferences and their main interests take advantage of differentiated products. The first enterprise produces quantity q_1 at price p_1 and the second enterprise produces quantity q_2 at price p_2. Here, the competitive profile is $p = (p_1, p_2)$, where each firm wants to maximize profits based on the following:

$$Max_{p_i}\pi_i(p_i, p_{-i}) = q_i p_i - C_i(q_i) \tag{1.1}$$

where p_{-i} refers to the price of some other enterprise distinct from enterprise i,and $C_i(q_i)$ is the cost function. Using the consumer preferences introduced in [18], the following utility function is given:

$$U(q_1, q_2) = a(q_1 + q_2) - \frac{1}{2}(q_1^2 + q_2^2 + 2bq_1q_2), \tag{1.2}$$

where q_1, q_2, a are positive and $0 < b < 1$. The parameter b is an index of product differentiation. When $b = 1$, firms produce homogeneous goods. When $b = 0$, products are unrelated and each firm is a monopolist. In this illustration, we consider the duopoly case with differentiation, $0 < b < 1$. This utility function gives rise to a linear demand structure. Inverse demands are given by $p_i = \frac{\partial U}{\partial q_i}, = 1, 2$ which are

$$\begin{cases} p_1 = a - q_1 - bq_2, \\ p_2 = a - q_2 - bq_1. \end{cases} \tag{1.3}$$

p_1, q_2 is expressed as a function of q_1, p_2 and the competitive model of q_1, p_2 is studied as follows:

$$\begin{cases} p_1 = a(1-b) - (1-b^2)q_1 + bp_2, \\ q_2 = a - p_2 - bq_1. \end{cases} \tag{1.4}$$

We consider a nonlinear cost function:

$$C_i(q_i) = c_i q_i + d_i q_i q_j, \ \ i, j = 1, 2, \ \ i \neq j.$$

Here c_i represent marginal cost of quantity q_i which are positive and $c_i \in (0, a)$, and $d_i > 0, \ i = 1, 2$ is a control parameter of cost.

Net benefit is expressed as follows:

$$\begin{cases} \pi_1(q_1, p_2) = p_1q_1 - C_1 = [a(1-b) - (1-b^2)q_1 + bp_2]q_1 - c_1q_1 - d_1q_1(a - p_2 - bq_1) \\ \pi_2(q_1, p_2) = p_2q_2 - C_2 = p_2(a - p_2 - bq_1) - c_2(a - p_2 - bq_1) - d_2q_1(a - p_2 - bq_1) \end{cases} \tag{1.5}$$

We assume that the firms maximize the relative profit, which is denoted by

$$\begin{cases} \Pi_1(q_1, p_2) = \pi_1(q_1, p_2) - \pi_2(q_1, p_2) \\ \Pi_2(q_1, p_2) = \pi_2(q_1, p_2) - \pi_1(q_1, p_2) \end{cases} \tag{1.6}$$

Substituting (1.5) into(1.6), after some modifications, we get

$$\begin{cases} \Pi_1(q_1, p_2) = Gq_1 - Hp_2 + Eq_1^2 + p_2^2 + Fq_1p_2 \\ \Pi_2(q_1, p_2) = -Gq_1 + Hp_2 - Eq_1^2 - p_2^2 - Fq_1p_2 \end{cases} \tag{1.7}$$

where,

$$E = (b^2 - 1) + b(d_1 - d_2), \ F = 2b + d_1 - d_2,$$
$$G = a(1-b) - (c_1 + bc_2) - a(d_1 - d_2), \ H = a + c_2$$

According to the relative profit maximization conditions $\frac{\partial \Pi_1}{\partial q_1} = 0$, $\frac{\partial \Pi_2}{\partial p_2}$, substituting equation (1.7) into them,we can get

$$\begin{cases} \frac{\partial \Pi_1}{\partial q_1} = G + 2Eq_1 + Fp_2 = 0 \\ \frac{\partial \Pi_2}{\partial p_2} = H - 2p_2 - Fq_1 = 0 \end{cases} \tag{1.8}$$

We assume that both firms are acting in a rational way in order to build and study the dynamics of this game. Rational behavior means that firms use short-sighted adjustment mechanisms that require them to know if their relative profits are going up or down. Using this mechanism, we build a dynamic system to explain the game process as follows:

$$\begin{cases} q_1(t+1) = q_1(t) + \alpha_1 q_1(t)\frac{\partial \Pi_1}{\partial q_1} = q_1(t) + \alpha_1 q_1(t)(G + 2Eq_1(t) + Fp_2(t)), \\ p_2(t+1) = p_2(t) + \alpha_2 p_2(t)\frac{\partial \Pi_2}{\partial p_2} = p_2(t) + \alpha_2 p_2(t)(H - 2p_2(t) - Fq_1(t)), \end{cases} \tag{1.9}$$

where,

$$E = (b^2 - 1) + b(d_1 - d_2),\ F = 2b + d_1 - d_2,$$
$$G = a(1-b) - (c_1 + bc_2) - a(d_1 - d_2),\ H = a + c_2$$

and $\alpha_i > 0, i = 1, 2$ are speed adjustment parameters.

The majority of our findings relate the stability and instability of the model's fixed points, particularly the paths that lead to the various forms of bifurcations. We direct the attention of the readers to [1–6, 13, 20] for a more in-depth consideration of stability, bifurcation, and chaos control.

2 Existence and stability of equilibrium points

Using the Jury condition, this section will explore the existence of fixed points as well as their stability.

We consider the set

$\Gamma = \{(a, b, c_1, c_2, d_1, d_2, \alpha_1, \alpha_2) \in \mathbb{R}^8$: all parameters are positive and $0 < b < 1, 0 < c_1 < a, 0 < c_2 < a\}$

The equilibrium points $(\bar{q_1}, \bar{p_2})$ of the system (1.9) are the solutions of the following system:

$$\begin{cases} \bar{q_1} = \bar{q_1} + \alpha_1 \bar{q_1}(G + 2E\bar{q_1} + F\bar{p_2}), \\ \bar{p_2} = \bar{p_2} + \alpha_2 \bar{p_2}(H - 2\bar{p_2} - F\bar{q_1}), \end{cases}$$

where,

$$E = (b^2 - 1) + b(d_1 - d_2),\ F = 2b + d_1 - d_2,$$
$$G = a(1-b) - (c_1 + bc_2) - a(d_1 - d_2),\ H = a + c_2$$

Through some simple algebraic computations, one can verify that the system (1.9) has four equilibrium points:

$$E_0(0,0),\ E_1(-\frac{G}{2E}, 0),\ E_2(0, \frac{H}{2}),\ E_3(\bar{q_1}, \bar{p_2}).$$

where

$$\bar{q_1} = \frac{(a-c_2)(d_2-d_1)+2(a-c_1)}{4+(d_1-d_2)^2},$$
$$\bar{p_2} = \frac{2c_2+c_1d_1-c_1d_2+b(2c_1-c_2d_1+c_2d_2)+a(2-d_1+d_1^2+b(-2+d_1-d_2)+d_2-2d_1d_2+d_2^2)}{4+(d_1-d_2)^2}.$$

The equilibrium point $E_1(-\frac{G}{2E},0)$ exists if and only if $\frac{G}{E}<0$.
By doing basic calculations, one may verify that in set Γ, $\bar{p_2}>0$ and

$$\bar{q_1} = \frac{(a-c_2)(d_2-d_1)+2(a-c_1)}{4+(d_1-d_2)^2} > 0$$

if and only if

$$d_1-d_2 < \frac{2(a-c_1)}{a-c_2}$$

.

In [12], the authors studied the model (1.9). They examined existence and stability of the fixed points. In their work, they pointed out some conditions for the existence of Nash equilibrium point. With the help of mathematica software, we found that for the existence of Nash equilibrium point we require the parameters domain Γ and satisfying the only condition $d_1-d_2<\frac{2(a-c_1)}{a-c_2}$. In [12], the authors used extra conditions for the existence of Nash equilibrium point. Moreover, we discussed rate of convergence, and we used some control techniques to control bifurcation and chaos in the model (1.9). In [12], the authors did not discussed rate of convergence and chaos control.

The point $E_3(\bar{q_1},\bar{p_2})$ is the unique equilibrium point of system (1.9) iff $d_1-d_2<\frac{2(a-c_1)}{a-c_2}$.
The Jacobian matrix of the model (1.9) at the point $(\bar{q_1},\bar{p_2})$ is defined as follows:

$$J = \begin{bmatrix} 1+\alpha_1(G+F\bar{p_2}+4E\bar{q_1}) & \alpha_1F\bar{q_1} \\ -\alpha_2F\bar{p_2} & 1+\alpha_2(h-4\bar{p_2}-F\bar{q_1}). \end{bmatrix} \tag{2.1}$$

The trace T and determinant D of the matrix J are

$$\begin{aligned} T &= 2+\alpha_1(G+Fp_2+4Eq_1)+\alpha_2(H-4p_2-Fq_1), \\ D &= 1+\alpha_2(H-4p_2-Fq_1)+\alpha_1(G+4E(1+\alpha_2H-4\alpha_2p_2)q_1+\alpha_2G(H-4p_2-Fq_1) \\ &\quad +F(p_2+\alpha_2Hp_2-4\alpha_2p_2^2-4\alpha_2Eq_1^2)) \end{aligned}$$

In accordance with the Jury condition, the equilibrium point $(\bar{q_1},\bar{p_2})$ is considered to be stable if and only if the following conditions are met:

$$\begin{cases} T+D+1>0, \\ -T+D+1>0, \\ D-1<0. \end{cases} \tag{2.2}$$

3 Rate of convergence

In this section, we are going to determine the rate of convergence of a solution that is going to converge to the Nash equilibrium point of the model (1.9).

The following result gives the rate of convergence of solutions of a system of difference equations

$$X_{n+1} = (A+B(n))X_n \tag{3.1}$$

Where X_n is an m-dimensional vector, $A\in C^{m\times m}$ is a constant matrix and $B:\mathbb{Z}^+\to C^{m\times m}$ is a matrix function satisfying

$$\lim_{n\to\infty} \|B_n\| = 0. \tag{3.2}$$

Here $\|.\|$ refers to any matrix norm that is connected to the vector norm

$$\| < x, y > \| = \sqrt{x^2 + y^2}.$$

Proposition 3.1. *[16] Assume that the condition* (3.2) *holds. If X_n is a solution of* (3.1)*, then either $X_n = 0$ for all large values of n or*

$$\rho = \lim_{n\to\infty} (\|X_n\|)^{\frac{1}{n}}$$

exists and is equal to the modulus of one of the eigenvalues of matrix A.

Proposition 3.2. *[16] Assume that the condition* (3.2) *holds. If X_n is a solution of* (3.1)*, then either $X_n = 0$ for all large values of n or*

$$\rho = \lim_{n\to\infty} \frac{\|X_{n+1}\|}{\|X_n\|}$$

exists and is equal to the modulus of one of the eigenvalues of matrix A.

Let$(q_1(t), p_2(t))$ be any solution of the model (1.9) such that $\lim_{t\to\infty} q_1(t) = \bar{q_1}(t)$ and $\lim_{n\to\infty} p_2(t) = \bar{p_2}(t)$. To find the error terms, one has from the model (1.9)

$$\begin{aligned} q_1(t+1) - \bar{q_1} &= q_1 + \alpha_1 q_1(t)\Big(G + 2Eq_1(t) + Fp_2(t)\Big) - \bar{q_1} - \alpha_1\bar{q_1}\Big(G + 2E\bar{q_1} + F\bar{p_2}\Big) \\ &= \Big(1 + \alpha_1(G + 2E(q_1(t) + \bar{q_1}) + F\bar{p_2})\Big)(q_1(t) - \bar{q_1}) + \alpha_1 F q_1(t)(p_2(t) - \bar{p_2}), \end{aligned}$$

and

$$\begin{aligned} p_2(t+1) - \bar{p_2} &= p_2(t) + \alpha_2 p_2(t)\Big(-Fq_1(t) - 2p_2(t) + H\Big) - \bar{p_2} - \alpha_2\bar{p_2}\Big(-F\bar{q_1} - 2\bar{p_2} + H)\Big) \\ &= -\alpha_2 F p_2(t)(q_1(t) - \bar{q_1}) + \Big(1 + \alpha_2(-F\bar{q_1} - 2(p_2(t) + \bar{p_2}) + H\Big)(p_2(t) - \bar{p_2}) \end{aligned}$$

Let, $e_t^1 = q_1(t) - \bar{q_1}$ and $e_t^2 = p_2(t) - \bar{p_2}$ then one has

$$e_{t+1}^1 = a_t e_t^1 + b_t e_t^2,$$

and

$$e_{t+1}^2 = c_t e_t^1 + d_t e_t^2,$$

where

$$\begin{aligned} a_t &= 1 + \alpha_1(G + 2E(q_1(t) + \bar{q_1}) + F\bar{p_2}), \\ b_t &= \alpha_1 F q_1(t), \\ c_t &= -\alpha_2 F p_2(t), \\ d_t &= 1 + \alpha_2(-F\bar{q_1} - 2(p_2(t) + \bar{p_2}) + H \end{aligned}$$

Moreover,

$$
\begin{aligned}
&\lim_{t\to\infty} a_t = 1+\alpha_1(G+4E\bar{q_1}+F\bar{p_2}),\\
&\lim_{t\to\infty} b_t = \alpha_1 F\bar{q_1},\\
&\lim_{t\to\infty} c_t = -\alpha_2 F\bar{p_2},\\
&\lim_{t\to\infty} d_t = 1+\alpha_2(-F\bar{q_1}-4\bar{p_2})+H
\end{aligned}
$$

Now the limiting system of error terms can be written as

$$
\begin{bmatrix} e^1_{t+1} \\ e^2_{t+1} \end{bmatrix} = \begin{bmatrix} 1+\alpha_1(G+4E\bar{q_1}+F\bar{p_2}) & \alpha_1 F\bar{q_1} \\ -\alpha_2 F p_2 & 1+\alpha_2(-F\bar{q_1}-4\bar{p_2})+H \end{bmatrix} \begin{bmatrix} e^1_n \\ e^2_n \end{bmatrix}
$$

which is similar to the linearized system of (1.9) about the Nash equilibrium point $(\bar{q_1}, \bar{p_2})$. Using the proposition (3.1), one has the following result.

Theorem 3.1. *Assume that* $\{(q_1(t), p_2(t))\}$ *be a positive solution of the model* (1.9) *such that* $\lim_{t\to\infty} q_1(t) = \bar{q_1}(t)$ *and* $\lim_{t\to\infty} p_2(t) = \bar{p_2}(t)$*, where* $(\bar{q_1}, \bar{p_2})$ *is the Nash equilibrium point of the model* (1.9)*. Then, the error vector* $e_t = \begin{pmatrix} e^1_t \\ e^2_t \end{pmatrix}$ *of every solution of* (1.9) *satisfies both of the following asymptotic relations*

$$
\lim_{t\to\infty} (\|e_t\|)^{1/t} = |\lambda_{1,2} J(\bar{q_1}, \bar{p_2})|,
$$
$$
\lim_{t\to\infty} \frac{\|e_{t+1}\|}{\|e_t\|} = |\lambda_{1,2} J(\bar{q_1}, \bar{p_2})|,
$$

where $\lambda_{1,2} J(\bar{q_1}, \bar{p_2})$ *are the characteristic roots of Jacobian matrix* $J(\bar{q_1}, \bar{p_2})$.

4 Hybrid Control Method

The study of methods for controlling chaos in discrete systems has surged in popularity in recent years. In dynamical systems, it is preferable to maximize the system in terms of certain success criterion while suppressing chaos.

The state feedback control approach (OGY method) [9, 11], the pole-placement technique [17], and the hybrid control method [21] are all viable options for achieving chaos control in discrete models. Only the hybrid control strategy will be discussed in this section. This method was originally developed to regulate the period-doubling bifurcation, but it also has the capability of managing the Neimark-Sacker bifurcation.

We take into consideration the controlled system associated with (1.9) as shown below.

$$
\begin{cases} q_1(t+1) = \beta(q_1(t)+\alpha_1 q_1(t)(G+2Eq_1(t)+Fp_2(t))) + (1-\beta)q_1(t), \\ p_2(t+1) = \beta(p_2(t)+\alpha_2 p_2(t)(H-2p_2(t)-Fq_1(t))) + (1-\beta)p_2(t), \end{cases} \tag{4.1}
$$

where $0<\beta<1$. Both the controlled system (4.1), and the original system (1.9), share the same fixed points. The Jacobian matrix of (4.1) evaluated at $E_3(\bar{q_1}, \bar{p_2})$ is

$$
J(E_3) = \begin{bmatrix} 1+\alpha_1\beta(G+F\bar{p_2}+4E\bar{q_1}) & \alpha_1\beta F\bar{q_1} \\ \alpha_2\beta F\bar{p_2} & 1+\alpha_2\beta(H-4p_2-F\bar{q_1}) \end{bmatrix}.
$$

The trace T and determinant D of the matrix J are

$$T = 2 + \alpha_1\beta(G + F\bar{p_2} + 4E\bar{q_1}) + \alpha_2\beta(H - 4\bar{p_2} - F\bar{q_1}),$$
$$D = 1 + \alpha_2\beta(H - 4\bar{p_2} - F\bar{q_1}) + \alpha_1\beta(G + 4E(1 + \alpha_2\beta(H - 4\bar{p_2}))\bar{q_1} + \alpha_2\beta G(H - 4\bar{p_2} - F\bar{q_1})$$
$$+ F(\bar{p_2} + \alpha_2\beta H\bar{p_2} - 4\alpha_2\beta\bar{p_2}^2 - 4\alpha_2\beta E\bar{q_1}^2))$$

In accordance with the Jury condition, the Nash equilibrium point $(\bar{q1}, \bar{p2})$ is considered to be stable if and only if the following requirements are met:

$$\begin{cases} T + D + 1 > 0, \\ -T + D + 1 > 0, \\ D - 1 < 0. \end{cases} \tag{4.2}$$

5 Numerical Simulations

In this part, we provide numerical examples to validate our earlier theoretical results about the model's many qualitative features.

5.1 Stability analysis and Bifurcation of the system (1.9) at E_3 by using a as bifurcation parameter:

For model (1.9), we set the parameters and beginning conditions as follows:
$b = 1,\ c_1 = 2,\ c_2 = 2,\ d_1 = 1,\ d_2 = 3,\ \alpha_1 = 0.1, \alpha_2 = 0.15,\ q_1(0) = 0.1,\ p_2(0) = 0.1.$

The bifurcation diagrams (1a,1b) of figure (1) depict that system experiences transcritical bifurcation for $a = 2$ and period doubling bifurcation for $a \approx 11.3333$. The Nash equilibrium point is stable for $2 < a < 11.3333$. All three inequalities of (2.2) are satisfied iff $2 < a < 11.3333$. The phase portrait depicted in figure (1e) is showing that Nash equilibrium point is stable for $a = 11.3$. The phase portrait depicted in figure (1f) is showing that Nash equilibrium point is unstable due to period doubling bifurcation for $a = 11.4$.

For model (4.1), we set the parameters and beginning conditions as follows:
$\beta = 0.95,\ b = 1,\ c_1 = 2,\ c_2 = 2,\ d_1 = 1,\ d_2 = 3,\ \alpha_1 = 0.1, \alpha_2 = 0.15,\ q_1(0) = 0.1,\ p_2(0) = 0.1.$

The bifurcation diagrams (1c,1d) of figure (1) depict that system experiences transcritical bifurcation for $a = 2$ and period doubling bifurcation for $a \approx 12.0351$. It confirms that period doubling bifurcation is delayed in controlled model (4.1).

5.2 Stability analysis and Bifurcation of the system (1.9) at E_3 by using b as bifurcation parameter:

For model (1.9), we set the parameters and beginning conditions as follows:
$a = 8,\ c_1 = 2,\ c_2 = 2,\ d_1 = 1,\ d_2 = 0.5,\ \alpha_1 = 0.1, \alpha_2 = 0.1,\ q_1(0) = 0.1,\ p_2(0) = 0.1.$

The bifurcation diagrams (2a,2b) of figure (2) depict that system experiences Neimark-Sacker bifurcation for $b \approx 1.036$. The Nash equilibrium point is stable for $0 < b < 1.036$. All three inequalities of (2.2) are satisfied iff $0 < b < 1.036$. The phase portrait depicted in figure (2e) is showing that Nash equilibrium point is stable for $b = 1.03$. The phase portrait depicted in figure (2f) is showing that Nash equilibrium point is unstable due to Neimark-Sacker bifurcation for $b = 1.04$.

For model (4.1), we set the parameters and beginning conditions as follows:
$\beta = 0.25,\ a = 8,\ c_1 = 2,\ c_2 = 2,\ d_1 = 1,\ d_2 = 0.5,\ \alpha_1 = 0.1, \alpha_2 = 0.1,\ q_1(0) = 0.1,\ p_2(0) = 0.1.$

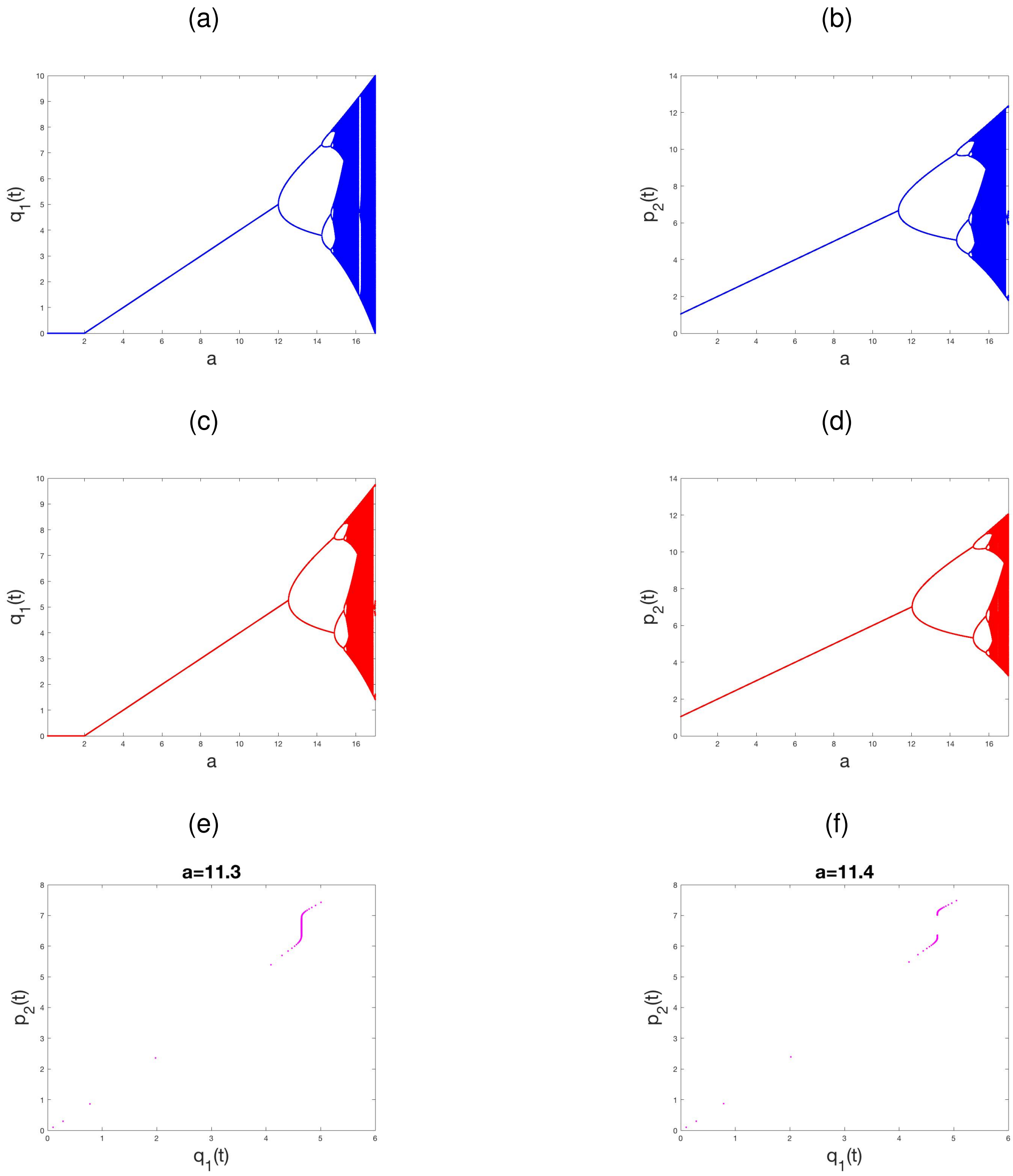

Figure 1: Bifurcation diagrams of (1.9) and (4.1) for $\beta = 0.95,\ b = 1,\ c_1 = 2,\ c_2 = 2,\ d_1 = 1,\ d_2 = 3,\ \alpha_1 = 0.1, \alpha_2 = 0.15,\ q_1(0) = 0.1,\ p_2(0) = 0.1, a \in (0, 17)$, and phase portraits of model (1.9) for $a = 11.3$ and $a = 11.4$.

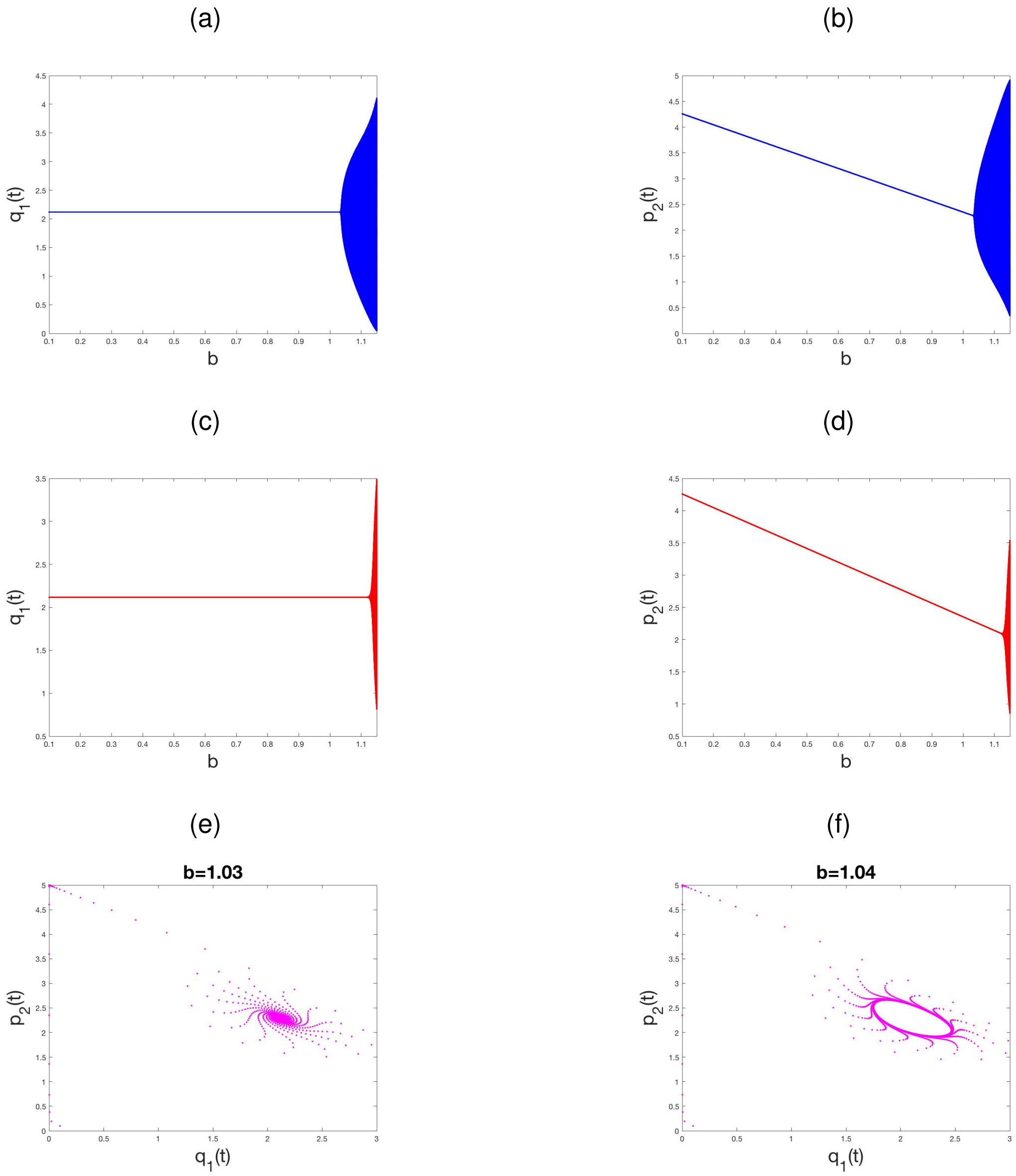

Figure 2: Bifurcation diagrams of (1.9) and (4.1) for $\beta = 0.25,\ a = 8,\ c_1 = 2,\ c_2 = 2,\ d_1 = 1,\ d_2 = 0.5,\ \alpha_1 = 0.1, \alpha_2 = 0.1,\ q_1(0) = 0.1,\ p_2(0) = 0.1, a \in (0.1, 1.15)$,and phase portraits of model (1.9) for $b = 1.03$ and $b = 1.04$.

The bifurcation diagrams (2c,2d) of figure (2) depict that system experiences Neimark-Sacker bifurcation for $b \approx 1.13789$. It confirms that Neimark-Sacker bifurcation is delayed in controlled model (4.1).

The similar investigation for other parameters $c_1, c_2, d_1, d_2, \alpha_1, \alpha_2$ can be done by taking one of the parameters as bifurcation parameter and fixing all other parameters. All parameters are affecting the stability of Nash equilibrium point of model (1.9).

6 Conclusion

We studied a dynamic version of the Cournot-Bertrand competition in this paper. In particular, we looked at how a duopoly game changes when one company competes on price and the other on quantity. Conditions for the existence and local stability of the Nash equilibrium are established. The impacts of parameters on Nash equilibrium point stability are investigated. Using numerical investigations, we have examined the study of bifurcation, which leads to qualitative changes in the behavior of games and leads to loss of stability of Nash equilibrium. When the parameters of the system exceed the boundary curve of the stable zone, three distinct types of bifurcations are discovered: period-doubling, Neimark-Sacker, and transcritical bifurcations. This indicates that the dynamic behavior of the outputprice game becomes more complicated. In addition, a hybrid control mechanism is used in order to bring bifurcation and chaos under control. It has been shown via numerical examples that the bifurcation in the modified controlled model occurs later than in original model as a result of the hybrid control strategy.

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