On some generalized number theoretic functions and Ighachanea-Akkouchia Holder's inequalities

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Abstract

Recently, it has been shown by Ighachanea and Akkouchia [1] that using binomial coefficients, one can derive some new refinements of Holder's inequalities. This inequalities then can be applied to a wide class of special functions such as the Nielsen's beta function and some extended gamma functions. In this paper, we have derived some generalizations of previously known number theoretic functions. Furthermore, based on the results of Ighachanea and Akkouchia, Holder's inequalities for the derived generalized functions are established.

keywords: Number theoretic functions, Holder's inequalities, Nielsen's beta function, Extended gamma function.

Contents

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1 Introduction

Holder's inequalities have played an important role in wide application based areas of mathematics. In this paper, we have applied the generalizations of Holder's inequality given by Ighachanea and Akkouchia in [1] to a wider class of some special number theoretic functions which are also derived in this paper. The paper is arranged as follows. In this section, we have introduced some extended definitions of the gamma function which we will be subsequently using throughout the paper. In section 2, we present Holder's inequality and some preliminary results. In section 3, the results of Ighachanea and Akkouchia are presented that we will be applying throughout our paper to various special functions. In section 4, we have derived a analogue of Nielsen's beta function and derived some of its properties. In section 5, we have presented and extension of the Chaudhary-Zubair gamma function and derived some of its properties. The next sections are followed by applying inequalities from section 3 to the special functions derived in previous two sections.

The p-k gamma function or two parameter gamma functions is a parametric deformation of the classic gamma function given by:[1]

Definition 1.1. [[3], pg. 3](p-k Gamma Function) Given $x \in C/kZ^{-}; k, p \in R^{+} - \{0\}$ and $\Re(x) > 0$, then the integral representation of p-k Gamma Function is given by

$$ {}_p\Gamma_k(x) = \int_0^{\infty} e^{-\frac{t^k}{p}} t^{x-1} dt. \tag{1.1} $$

The above definition reduces to k-gamma function when $p = k$ [4]. $\Gamma_k(x)$ appears in a variety of contexts, such as, the combinatorics of creation and annihilation operators [5], [6] and the perturbative computation of Feynman integrals, see [7]. For more applications of k-gamma function refer to [8]-[21]

Definition 1.2. [[3], pg. 5] For $x \in C/kZ^{-}; k, p \in R^{+} - \{0\}$ and $\Re(x) > 0, n \in N$. The fundamental equations satisfied by p-k Gamma Function, ${}_p\Gamma_k(x)$ are,

$$ {}_p\Gamma_k(p) = \frac{p^{\frac{p}{k}}}{k} \Gamma(\frac{p}{k}). \tag{1.3} $$

[1] ${}_p\Gamma_k(x) \Rightarrow {}_k\Gamma_k(x) = \Gamma_k(x)$ as $p = k$ and ${}_p\Gamma_k(x) \Rightarrow {}_1\Gamma_1(x) = \Gamma(x)$ as $p, k \to 1$.

$$ {}_p\Gamma_k(x)\,{}_p\Gamma_k(-x) = \frac{\pi}{xk}\,\frac{1}{\sin(\frac{\pi x}{k})}. \tag{1.4}$$

$$ {}_p\Gamma_k(x)\,{}_p\Gamma_k(k-x) = \frac{p}{k^2}\,\frac{\pi}{\sin(\frac{\pi x}{k})}. \tag{1.5}$$

$$ \prod_{0\le r\le m-1} {}_p\Gamma_k(x+\frac{kr}{m}) = \frac{p^{\frac{m-1}{2}}}{k^{m-1}}(2\pi)^{\frac{(m-1)}{2}} m^{\frac{1}{2}-\frac{mx}{k}}\,{}_p\Gamma_k(mx); m = 2,3,4,\ldots. \tag{1.6}$$

Relation of p-k gamma function with k gamma function and the classic gamma function is given by

$$ {}_p\Gamma_k\left(x\right) = \left(\frac{p}{k}\right)^{\frac{x}{k}}\Gamma_k\left(x\right) = \frac{p^{\frac{x}{k}}}{k}\Gamma\left(\frac{x}{k}\right). \tag{1.7}$$

We kindly request readers to make themselves familiar with the k-gamma function introduced in [4]. Further generalizations of the k-gamma function and ordinary gamma function can be found in [27]-[29].

2 Holder's inequalities and some preliminary results

Theorem 2.1 (Holder's Inequality)**.** *Let $(\Omega,\mathcal{F},\mu)$ be a measure space where μ is a positive measure. Let $\xi,\tilde{\xi}:\Omega\to\boldsymbol{C}$ be two measurable functions. Then, for all $p,q\geq 1$ and $p^{-1}+q^{-1}=1$, we have*

$$ \int_{\Omega}\left|\xi\tilde{\xi}\right|d\mu\left(t\right)\leq\left(\int_{\Omega}\left|\xi\right|^{p}d\mu\left(t\right)\right)^{\frac{1}{p}}\left(\int_{\Omega}\left|\tilde{\xi}\right|^{p}d\mu\left(t\right)\right)^{\frac{1}{q}} \tag{2.1}$$

From [2], we have the following two theorems

Theorem 2.2. *Let n and m be two integers and let $a_i\in\mathbb{R}^+$. Set $i_0:=m$, $i_n:=0$ and $A:=\{(i_1,\ldots,i_{n-1}):0\leq i_k\leq i_{k-1},\ 1\leq k\leq n-1\}$. Then, we have*

$$ \left(\sum_{k=1}^{n}\nu_k a_k\right)^m = \sum_{(i_1,\ldots,i_{n-1})\in A} C_A\nu_1^{i_0-i_1}\nu_2^{i_1-i_2}\ldots\nu_n^{i_{n-1}-i_n}a_1^{i_0-i_1}a_2^{i_1-i_2}\ldots a_n^{i_{n-1}-i_n}, \tag{2.2}$$

where, $C_A=\binom{i_1}{i_2}\ldots\binom{i_{n-2}}{i_{n-1}}$, the $\binom{i_{k-1}}{i_k}$ is the binomial coefficient.

Theorem 2.3. *For $k=1,2,\ldots,n$, let $a_k\geq 0$ and let $\nu_k>0$ satisfy $\sum_{k=1}^{n}\nu_k=1$. Then for all integers $m\geq 1$, we have*

$$ \prod_{k=1}^{n}a_k^{\nu_k}+r_0^m\left(\sum_{k=1}^{n}a_k-n\sqrt[n]{\prod_{k=1}^{n}a_k}\right)\leq\left(\sum_{k=1}^{n}\nu_k a_k^{\frac{1}{m}}\right)^m\leq\sum_{k=1}^{n}\nu_k a_k, \tag{2.3}$$

where $r_0=\min\{\nu_k:k=1,\ldots,n\}$.

Moreover, if we set $U_m:=\left(\sum_{k=1}^{n}\nu_k a_k^{\frac{1}{m}}\right)^m$, then $\{U_m\}$ is a decreasing sequence and we have$\lim_{m\to\infty}U_m=\prod_{k=1}^{n}a_k^{\nu_k}$.

3 Ighachanea-Akkouchia Holder's inequalities

Using theorems 2.3 and 2.4, Ighachanea and Akkouchia [1] derived the following refinements of the Holder's inequality.

Theorem 3.1 (Ighachanea-Akkouchia inequality type-I). *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space where μ is a positive measure. Let n be a positive integer and let $\xi_1, \xi_2, \ldots, \xi_n$ be μ-measurable functions such that $\xi_k \in \mathcal{L}^{p_k}(\mu)$, for all $k = 1, \ldots, n$. Then for all integers $m \geq 2$, the inequalities*

$$\int_\Omega \prod_{k=1}^{n} |\xi_k(t)| d\mu(t) n r_0^m \prod_{k=1}^{n} ||\xi_k||_{p_k} \Big(1 - \prod_{k=1}^{n} ||\xi_k||_{p_k}^{\frac{-p_k}{n}} \int_\Omega \prod_{k=1}^{n} |\xi_k(t)|^{\frac{p_k}{n}} d\mu(t)\Big)$$
$$\leq \sum_{(i_1,\ldots,i_{n-1})\in A} C_A \frac{1}{p_1^{i_0-i_1} .. p_n^{i_{n-1}-i_n}} \prod_{k=1}^{n} ||\xi_k||_{p_k}^{1-\frac{p_k(i_k-i_{k-1})}{m}} \int_\Omega \prod_{k=1}^{n} |\xi_k(t)|^{\frac{p_k(i_k-i_{k-1})}{m}} d\mu(t) \leq \prod_{k=1}^{n} ||\xi_k||_{p_k}, \tag{3.1}$$

holds for $p_k > 1$, such that $\sum_{k=1}^{n} \frac{1}{p_k} = 1$, where $r_0 = \min\{\frac{1}{p_k} : k = 1, \ldots, n\}$.

Theorem 3.2 (Ighachanea-Akkouchia inequality type-II). *Let n, N be two integers and $\{Q_{j,k}\} \subset \mathbb{R}$, where $k = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, N$. Let $p_k > 1$, such that $\sum_{k=1}^{n} \frac{1}{p_k} = 1$, Then the inequalities*

$$\sum_{j=1}^{N} \Big|\prod_{k=1}^{n} Q_{j,k}\Big| n r_0^m \prod_{k=1}^{n} \Big(\sum_{j=1}^{N} |Q_{j,k}|^{p_k}\Big)^{\frac{1}{p_k}} \Big(1 - \prod_{k=1}^{n} \Big(\sum_{j=1}^{N} |Q_{j,k}|^{p_k}\Big)^{\frac{-1}{n}} \sum_{j=1}^{N} \prod_{k=1}^{n} |Q_{j,k}|^{\frac{p_k}{n}}\Big)$$
$$\leq \sum_{(i_1,\ldots,i_{n-1})\in A} C_A \frac{1}{p_1^{i_0-i_1} .. p_n^{i_{n-1}-i_n}} \prod_{k=1}^{n} \Big(\sum_{j=1}^{N} |Q_{j,k}|^{p_k}\Big)^{\frac{1}{p_k} - \frac{(i_k-i_{k-1})}{m}} \sum_{j=1}^{N} \prod_{k=1}^{n} |Q_{j,i}|^{\frac{p_k(i_k-i_{k-1})}{m}}$$
$$\leq \prod_{k=1}^{n} \Big(\sum_{j=1}^{N} |Q_{j,k}|^{p_k}\Big)^{\frac{1}{p_k}}, \tag{3.2}$$

is valid, where $r_0 = \min\{\frac{1}{p_k} : k = 1, \ldots, n\}$.

The detailed proof of above two theorems can be found in [1].

4 On some analogues of Nielsen's beta function

4.1 Basic properties

Definition 4.1. For $x > 0$, we define Nielsen's β function as follows[2]

$$\beta(x) = \int_0^1 \frac{t^{x-1}}{1+t} dt = \int_0^\infty \frac{e^{-xt}}{1+e^{-t}} dt = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+x} = \frac{1}{2}\left\{\psi\left(\frac{x+1}{2}\right) - \psi\left(\frac{x}{2}\right)\right\}. \tag{4.1}$$

where $\psi(x) = \frac{d}{dx} \log \Gamma(x)$.

[2] [16]-[19]

Nielsen's β function satisfies the following properties

$$\beta(x+1) = \frac{1}{x} - \beta(x), \tag{4.2}$$

$$\beta(x) + \beta(1-x) = \frac{\pi}{\sin \pi x}. \tag{4.3}$$

Further additional properties can be found in [20]. From [3], we have

$$\frac{1}{{}_p\Gamma_k(x)} = \frac{x}{kp^{\frac{x}{k}}} e^{\frac{x}{k}\gamma} \prod_{n=1}^{\infty} \left[\left(1 + \frac{x}{nk}\right) e^{-\frac{x}{nk}}\right]. \tag{4.4}$$

From, Eqn. (4.4), we get the following value for p-k digamma function

$$ {}_p\psi_k(x) = \frac{d}{dx} \ln\left({}_p\Gamma_k(x)\right) = \frac{\ln p - \gamma}{k} - \frac{1}{x} + \sum_{n=1}^{\infty} \frac{x}{nk(nk+x)} \tag{4.5}$$

$$= \frac{\ln p - \gamma}{k} + \sum_{n=0}^{\infty} \left(\frac{1}{nk+k} - \frac{1}{nk+x}\right). \tag{4.6}$$

Theorem 4.1. *For $x, y > 0$ and $p, k \in \mathbb{R}^+$, we have*

$$ {}_p\psi_k(x) - {}_p\psi_k(y) = \psi_k(x) - \psi_k(y) \tag{4.7}$$

where ${}_p\psi_k(x)$ is the p-k digamma function from Eqn. (4.5) and $\psi_k(x)$ is the k-digamma function[3] [21].

Proof. Proof follows from the definition of p-k digamma function and k-digamma function. ∎

Definition 4.2. The p-k extension of the Nielsen's β function ${}_p\beta_k(x)$ for $x > 0$ is defined as

$$ {}_p\beta_k(x) = \frac{p}{2}\left\{{}_p\psi_k\left(\frac{x+k}{2}\right) - {}_p\psi_k\left(\frac{x}{2}\right)\right\} \tag{4.8}$$

$$= \frac{p}{k}\sum_{n=0}^{\infty}\left(\frac{k}{2nk+x} - \frac{k}{2nk+x+k}\right) \tag{4.9}$$

$$= \frac{p}{k}\int_0^{\infty} \frac{e^{-\frac{xt}{k}}}{1+e^{-t}} dt \tag{4.10}$$

$$= \frac{p}{k}\int_0^{1} \frac{t^{\frac{x}{k}-1}}{1+t} dt, \tag{4.11}$$

where ${}_p\beta_k(x) = \beta_k(x)$ when $p = k$ and ${}_p\beta_k(x) = \beta(x)$ when $p = k = 1$.

[3] k-digamma function can be obtained from Eqn. (4.5) by letting $p = k$.

Theorem 4.2. ${}_p\beta_k\left(x\right)$ *satisfies the functional equation*

$$ {}_p\beta_k\left(x+k\right)=\frac{p}{x}-{}_p\beta_k\left(x\right) \tag{4.12}$$

and the reflection formula

$$ {}_p\beta_k\left(x\right)+{}_p\beta_k\left(k-x\right)=\frac{p^2}{k^2}\frac{\pi}{\sin\frac{\pi x}{k}}. \tag{4.13}$$

Proof. From Eqn. (4.11), we have

$$ {}_p\beta_k\left(x+k\right)+{}_p\beta_k\left(x\right)=\frac{p}{k}\int_0^1\frac{t^{\frac{x}{k}}+t^{\frac{x}{k}-1}}{1+t}dt=\frac{p}{k}\int_0^1 t^{\frac{x}{k}-1}dt=\frac{p}{x}. \tag{4.14}$$

Now,

$$ {}_p\beta_k\left(x\right)+{}_p\beta_k\left(k-x\right)=\frac{p}{2}\left\{{}_p\psi_k\left(\frac{x}{2}+\frac{k}{2}\right)-{}_p\psi_k\left(\frac{x}{2}\right)+{}_p\psi_k\left(k-\frac{x}{2}\right)-{}_p\psi_k\left(\frac{k}{2}-\frac{x}{2}\right)\right\}, \tag{4.15}$$

$$ =\frac{p}{2}\left\{{}_p\psi_k\left(k-\left(\frac{k}{2}-\frac{x}{2}\right)\right)-{}_p\psi_k\left(\frac{k}{2}-\frac{x}{2}\right)+{}_p\psi_k\left(k-\frac{x}{2}\right)-{}_p\psi_k\left(\frac{x}{2}\right)\right\}. \tag{4.16}$$

Now, logarithmically differentiating Eqn. (1.5), we get

$$ {}_p\psi_k\left(x\right)+{}_p\psi_k\left(k-x\right)=\frac{p}{k^2}\pi\cot\frac{\pi x}{k}. \tag{4.17}$$

Using the above relation in Eqn. (4.16), we get

$$ {}_p\beta_k\left(x\right)+{}_p\beta_k\left(k-x\right)=\frac{p}{2}\left\{\frac{p}{k^2}\pi\cot\frac{\pi}{k}\left(\frac{k}{2}-\frac{x}{2}\right)+\frac{p}{k^2}\pi\cot\frac{\pi x}{2k}\right\} \tag{4.18}$$

$$ =\frac{p^2\pi}{2k^2}\left\{\cot\left(\frac{\pi}{2}-\frac{\pi x}{2k}\right)+\cot\frac{\pi x}{2k}\right\} \tag{4.19}$$

$$ =\frac{p^2\pi}{2k^2}\left\{\tan\left(\frac{\pi x}{2k}\right)+\cot\frac{\pi x}{2k}\right\} \tag{4.20}$$

$$ =\frac{p^2}{k^2}\frac{\pi}{2\cos\left(\frac{\pi x}{2k}\right)\sin\left(\frac{\pi x}{2k}\right)} \tag{4.21}$$

$$ =\frac{p^2}{k^2}\frac{\pi}{\sin\left(\frac{\pi x}{k}\right)}. \tag{4.22}$$

This completes our proof. ∎

Definition 4.3. For $x > 0$, $p, k \in \mathbb{R}^+$ and $n \in \mathbb{N}$, we have

$$ {}_p\beta_k^{(n)}(x) = \frac{p}{2^{n+1}} \left\{ {}_p\psi_k^{(n)}\left(\frac{x+k}{2}\right) - {}_p\psi_k^{(n)}\left(\frac{x}{2}\right) \right\} \tag{4.23} $$

$$ = \frac{(-1)^n p}{k^{n+1}} \int_0^\infty \frac{t^n e^{-\frac{xt}{k}}}{1+e^{-t}} dt \tag{4.24} $$

$$ = \frac{p}{k^{n+1}} \int_0^1 \frac{(\ln t)^n t^{\frac{x}{k}-1}}{1+t} dt, \tag{4.25} $$

$$ {}_p\beta_k^{(n)}(x+k) = (-1)^n \frac{n!p}{x^{n+1}} - {}_p\beta_k^{(n)}(x). \tag{4.26} $$

Theorem 4.3. *i)* ${}_p\beta_k(x)$ *is positive and decreasing.*
ii) ${}_p\beta_k^{(n)}(x)$ *is positive and decreasing when* n *is an even integer.*
iii)${}_p\beta_k^{(n)}(x)$ *is negative and increasing if* n *is an odd integer.*

Proof. Proof trivially follows from Eqn. (4.24). ∎

Theorem 4.4. *i)* ${}_p\beta_k(x)$ *is logarihmically convex on* $(0, \infty)$.
ii) ${}_p\beta_k(x)$ *is completely monotonic on* $(0, \infty)$.

Proof. i) Let $r, s > 1$, $\frac{1}{r} + \frac{1}{s} = 1$ and $x, y \in (0, \infty)$, then, using Eqn. (4.11) and Hölder's inequality, we have

$$ \left[\frac{k}{p} {}_p\beta_k\left(\frac{x}{r} + \frac{y}{s}\right)\right] = \int_0^1 \frac{t^{\frac{x}{kr}+\frac{y}{ks}-1}}{1+t} dt \tag{4.27} $$

$$ = \int_0^1 \frac{t^{\frac{x-k}{kr}}}{(1+t)^{\frac{1}{r}}} \frac{t^{\frac{y-k}{ks}}}{(1+t)^{\frac{1}{s}}} dt \tag{4.28} $$

$$ \leq \left(\int_0^1 \frac{t^{\frac{x}{k}-1}}{1+t} dt\right)^{\frac{1}{r}} \left(\int_0^1 \frac{t^{\frac{y}{k}-1}}{1+t} dt\right)^{\frac{1}{s}} \tag{4.29} $$

$$ = \left[\frac{k}{p} {}_p\beta_k(x)\right]^{\frac{1}{r}} \left[\frac{k}{p} {}_p\beta_k(y)\right]^{\frac{1}{s}}. \tag{4.30} $$

ii) Using Eqn. (4.24), we have

$$ (-1)^n {}_p\beta_k^{(n)} = \frac{(-1)^{2n} p}{k^{n+1}} \int_0^\infty \frac{t^n e^{-\frac{xt}{k}}}{1+e^{-t}} dt \geq 0. \tag{4.31} $$

Therefore, ${}_p\beta_k(x)$ is completely monotonic on $(0, \infty)$. This completes our proof. ∎

4.2 On the equivalent conditions for log-convexity

If f is any function differentiable over an interval and is logarithmically convex, then the function satisfies the following two inequalities:
i) For $x, y > 0$, we have

$$\log f(x) \geq \log f(y) + \frac{f'(y)}{f(y)}(x-y) \tag{4.32}$$

which equivalently can be written as

$$\left(\frac{f(x)}{f(y)}\right)^{\frac{1}{x-y}} \geq \exp\left(\frac{f'(y)}{f(y)}\right). \tag{4.33}$$

ii) For $x > 0$, we have

$$f''(x) f(x) \geq f'(x)^2. \tag{4.34}$$

Therefore, we obtain the following theorem.

Theorem 4.5. *For $x, y > 0$ and $p, k \in \mathbb{R}^+$, we have the following inequalities*
i)

$$\left(\frac{{}_p\beta_k(x)}{{}_p\beta_k(y)}\right)^{\frac{1}{x-y}} \geq \exp\left(\frac{{}_p\beta'_k(y)}{{}_p\beta_k(y)}\right) \tag{4.35}$$

ii)

$${}_p\beta_k{}''(x)_p\beta_k(x) \geq_p \beta_k{}'(y)^2. \tag{4.36}$$

Proof. Substitute $f(x)$ with ${}_p\beta_k(x)$ is Eqn. (4.33) and (4.34) and the desire result readily follows. ■

Proposition 4.1. The following relation holds true

$${}_p\beta_k(x) = \frac{p}{k}\beta_k(x) = p\beta\left(\frac{x}{k}\right) \tag{4.37}$$

where $\beta_k(x)$ is the k-extension of Nielsen's beta function as introduced in [22].

Proof. The first equality follows from Eqn. (4.8), (4.7) and Definition 2.2 from [22]. Similarly, the second inequality follows from Definition 2.2 from [22] and the 4th equality of Eqn. (4.1).
Another way of proving the above proposition is using taking the counter examples. Authors in [22] have proved the following inequality for $\beta_k(x)$ for $x, y \in [0, \infty)$

$$\beta_k(x+k)\beta_k(y+k) \leq \ln 2\beta_k(x+y+k). \tag{4.38}$$

Therefore, if the second equality in Eqn. (4.37) is true, then

$$k\beta\left(\frac{x}{k}+1\right)\beta\left(\frac{y}{k}+1\right) \leq \ln 2\beta\left(\frac{x}{k}+\frac{y}{k}+1\right) \tag{4.39}$$

must also be true. By taking various counter examples, it turns that the above inequality holds true, therefore, we can conclude that the second equality in Eqn. (4.37) is true. Similarly, one can prove the first equality by substituting $f(x)$ with ${}_p\beta_k(x)$ in Eqn. (4.32) or by using Eqn. (4.35) and further taking various counter examples. ■

Theorem 4.6. *For $x, y \in [0, \infty)$ and $p, k \in \mathbb{R}^+$, we have*

$$ {}_p\beta_k(x+k)_p\beta_k\,(y+k) \leq \frac{p\ln 2}{k}{}_p\beta_k\,(x+y+k) \tag{4.40}$$

Proof. Multiply Eqn. (4.38) with p/k twice and use relation 4.37 to arrive at the desire result. ■

Authors in [22] have established the following two results for the k-extension of the Nielsen's beta function valid for $k > 0$:
i) For $x, y, z \in \mathbb{R}^+$, we have

$$\beta_k\,(x)\,\beta_k\,(x+y+z) - \beta_k\,(x+y)\,\beta_k\,(x+z) > 0 \tag{4.41}$$

ii) For $a \geq 1$ and $x \in [0, 1]$, we have the following inequality which reverses if $0 < a \leq 1$

$$\frac{[\beta_k\,(1+k)]^a}{\beta_k\,(a+k)} \leq \frac{[\beta_k\,(x+k)]^a}{\beta_k\,(ax+k)} \leq (\ln 2)^{a-1}. \tag{4.42}$$

From the above two results, we can deduce the following theorems respectively.

Theorem 4.7. *For $x, y, z \in \mathbb{R}^+$ and $p, k \in \mathbb{R}^+$, we have*

$$ {}_p\beta_k(x)_p\beta_k\,(x+y+z) - {}_p\beta_k(x+y)_p\beta_k\,(x+z) > 0. \tag{4.43}$$

Theorem 4.8. *For $a \geq 1$, $x \in [0, 1]$ and $p, k \in \mathbb{R}^+$, we have the following inequality which reverses for $0 < a \leq 1$*

$$\frac{[{}_p\beta_k\,(1+k)]^a}{{}_p\beta_k\,(a+k)} \leq \frac{[{}_p\beta_k\,(x+k)]^a}{{}_p\beta_k\,(ax+k)} \leq \frac{p^{a-1}}{k^{a-1}}(\ln 2)^{a-1}. \tag{4.44}$$

4.3 Some additional results for the n-th order

For $n \in \mathbb{N}_0$, we define

$$\left|{}_p\beta_k^{(n)}\,(x)\right| = (-1)^n{}_p\beta_k^{(n)}\,(x) \tag{4.45}$$

which is decreasing for all $n \in \mathbb{N}$. From Eqn. (4.26), we get the following relation

$$\left|{}_p\beta_k^{(n)}\,(x+k)\right| = \frac{n!p}{x^{n+1}} - \left|{}_p\beta_k^{(n)}\,(x)\right|. \tag{4.46}$$

Proposition 4.2. For $x > 0$, $n \in \mathbb{N}$ and $p, k \in \mathbb{R}^+$, define

$$\Delta_n\,(x) = \frac{x^{n+1}}{n!}\left|{}_p\beta_k^{(n)}\,(x)\right|. \tag{4.47}$$

Then we have

$$\lim_{x\to 0}\Delta_n\,(x) = p \tag{4.48}$$

and

$$\lim_{x\to 0}\Delta'_n\,(x) = 0. \tag{4.49}$$

Proof. From Eqn. (4.46), we have

$$\lim_{x\to 0}\Delta_n(x) = \lim_{x\to 0}\frac{x^{n+1}}{n!}\left|{}_p\beta_k^{(n)}(x)\right| \tag{4.50}$$

$$= \lim_{x\to 0}\frac{x^{n+1}}{n!}\left(\frac{n!p}{x^{n+1}} - \left|{}_p\beta_k^{(n)}(x+k)\right|\right) \tag{4.51}$$

$$= \lim_{x\to 0}\left(p - \frac{x^{n+1}}{n!}\left|{}_p\beta_k^{(n)}(x+k)\right|\right) \tag{4.52}$$

$$= p. \tag{4.53}$$

And,

$$\lim_{x\to 0}\Delta'_n(x) = \lim_{x\to 0}\frac{d}{dx}\left(\frac{x^{n+1}}{n!}\left|{}_p\beta_k^{(n)}(x)\right|\right) \tag{4.54}$$

$$= \lim_{x\to 0}\left(\frac{x^{n+1}}{n!}\left|{}_p\beta_k^{(n+1)}(x+k)\right| - \frac{(n+1)\,x^{n+1}}{n!}\left|{}_p\beta_k^{(n)}(x+k)\right|\right) \tag{4.55}$$

$$= 0. \tag{4.56}$$

This completes our proof. ■

Theorem 4.9. *For $n\in\mathbb{N}_0$, $r>0$, $s>0$, $\frac{1}{r}+\frac{1}{s}=1$ and $p,k\in\mathbb{R}^+$, we have*

$$\left[\left|{}_p\beta_k^{(n)}\left(\frac{x}{r}+\frac{y}{s}\right)\right|\right] \le \left[\left|{}_p\beta_k^{(n)}(x)\right|\right]^{\frac{1}{r}}\left[\left|{}_p\beta_k^{(n)}(y)\right|\right]^{\frac{1}{s}}. \tag{4.57}$$

Proof. Using Eqn. (4.24) and Hölder's inequality, we have

$$\left[\left|{}_p\beta_k^{(n)}\left(\frac{x}{r}+\frac{y}{s}\right)\right|\right] = \frac{p}{k^{n+1}}\int_0^\infty \frac{t^n e^{-\left(\frac{x}{kr}+\frac{y}{ks}\right)t}}{1+e^{-t}}dt \tag{4.58}$$

$$= \frac{p}{k^{n+1}}\int_0^\infty \frac{t^{\frac{n}{r}}e^{-\frac{xt}{kr}}}{(1+e^{-t})^{\frac{1}{r}}}\frac{t^{\frac{n}{s}}e^{-\frac{yt}{ks}}}{(1+e^{-t})^{\frac{1}{s}}}dt \tag{4.59}$$

$$\le \left(\frac{p}{k^{n+1}}\int_0^\infty \frac{t^n e^{-\frac{xt}{k}}}{(1+e^{-t})}dt\right)^{\frac{1}{r}}\left(\frac{p}{k^{n+1}}\int_0^\infty \frac{t^n e^{-\frac{yt}{k}}}{(1+e^{-t})}dt\right)^{\frac{1}{s}} \tag{4.60}$$

$$= \left[\left|{}_p\beta_k^{(n)}(x)\right|\right]^{\frac{1}{r}}\left[\left|{}_p\beta_k^{(n)}(y)\right|\right]^{\frac{1}{s}}. \tag{4.61}$$

This completes our proof. From this, follows the following theorem. ■

Theorem 4.10. $\left|{}_p\beta_k^{(n)}(x)\right|$ *is logarithmically convex for all $n\in\mathbb{N}$ on $(0,\infty)$.*

Theorem 4.11. *For $x, y > 0$ and $p, k \in \mathbb{R}^+$, we have the following inequalities*
i)

$$\left(\frac{\left|{}_p\beta_k^{(n)}(x)\right|}{\left|{}_p\beta_k^{(n)}(y)\right|}\right)^{\frac{1}{x-y}} \geq \exp\left(\frac{\left|{}_p\beta_k^{(n+1)}(y)\right|}{\left|{}_p\beta_k^{(n)}(y)\right|}\right) \tag{4.62}$$

ii)

$$\left|{}_p\beta_k^{(n+2)}(x)\right|\left|{}_p\beta_k^{(n)}(x)\right| - \left|{}_p\beta_k^{(n+1)}(x)\right|^2 \geq 0. \tag{4.63}$$

Proof. The result follows from Eqn. (4.33) and (4.34). ■

Proposition 4.3. For $n \in \mathbb{N}_0$, we have

$$ {}_p\beta_k^{(n)}(x) = \frac{p}{k}\beta_k^{(n)}(x). \tag{4.64}$$

Proof. Using Eqn. (4.24) and (A.1.6), we have

$$ {}_p\beta_k^{(n)}(x) = \frac{(-1)^n p}{k^{n+1}}\int_0^\infty \frac{t^n e^{-\frac{xt}{k}}}{1+e^{-t}}dt = \frac{p}{k}\beta_k^{(n)}(x). \tag{4.65}$$

Thus

$$ {}_p\beta_k^{(n)}(x) = \frac{p}{k}\beta_k^{(n)}(x) \tag{4.66}$$

This completes our proof. ■

It follows from the above proposition that

$$\left|{}_p\beta_k^{(n)}(x)\right| = \frac{p}{k}\left|\beta_k^{(n)}(x)\right| \tag{4.68}$$

where

$$\left|\beta_k^{(n)}(x)\right| = (-1)^n\beta_k^{(n)}(x). \tag{4.69}$$

Theorem 4.12. *For $n \in \mathbb{N}_0$, $x, y > 0$ and $p, k \in \mathbb{R}^+$, we have*

$$\left|{}_p\beta_k^{(n)}(x+y)\right| < \left|{}_p\beta_k^{(n)}(x)\right| + \left|{}_p\beta_k^{(n)}(y)\right| \tag{4.70}$$

Proof. Multiply Eqn. (A.1.8) with $\frac{p}{k}$ and use Eqn. (4.68) to get the desire result. ■

Theorem 4.13. *Let $n \in \mathbb{N}_0$, $a > 0$, and $x > 0$, then the inequalities*

$$\left|{}_p\beta_k^{(n)}(ax)\right| \leq a\left|{}_p\beta_k^{(n)}(x)\right| \tag{4.71}$$

if $a \geq 1$, and

$$\left|{}_p\beta_k^{(n)}(ax)\right| \geq a\left|{}_p\beta_k^{(n)}(x)\right| \tag{4.72}$$

if $a \leq 1$ are satisfied.

Proof. Multiply Eqn. (A.1.9) and (A.1.10) with $\frac{p}{k}$ and use Eqn. (4.68) to get the desire result. ∎

Theorem 4.14. *Let $k > 0$ and $n \in \mathbb{N}_0$, then the inequality*

$$\left|{}_p\beta_k^{(n)}(xy)\right| < \left|{}_p\beta_k^{(n)}(x)\right| + \left|{}_p\beta_k^{(n)}(y)\right| \tag{4.73}$$

holds for $x > 0$ and $y \geq 1$.

Proof. Multiply Eqn. (A.1.11) with $\frac{p}{k}$ and use Eqn. (4.68) to get the desire result. ∎

4.4 On some multiplicative convex properties

Theorem 4.15. *For $x > 0$, $n \in \mathbb{N}_{odd}$ and $p, k \in \mathbb{R}^+$, ${}_p\beta_k^{(n)}(x)$ is multiplicatively Convex on the interval $(0, \infty)$.*

Proof. Using Lemma 2.3.4 (i) from [23], we can say that a function is strictly multiplicatively convex when it is logarithmically convex and increasing. Therefore using Eqn. (4.24) and Hölder's inequality, we have

$$\begin{aligned}
{}_p\beta_k^{(n)}\left(\frac{x}{r}+\frac{y}{s}\right) &= \frac{(-1)^n p}{k^{n+1}} \int_0^\infty \frac{t^n e^{-\left(\frac{x}{kr}+\frac{x}{ks}\right)t}}{1+e^{-t}} dt && (4.74)\\
&= \frac{(-1)^n p}{k^{n+1}} \int_0^\infty \frac{t^{\frac{n}{r}} e^{-\frac{xt}{kr}}}{(1+e^{-t})^{\frac{1}{r}}} \frac{t^{\frac{n}{s}} e^{-\frac{yt}{ks}}}{(1+e^{-t})^{\frac{1}{s}}} dt && (4.75)\\
&\leq \left(\frac{(-1)^n p}{k^{n+1}} \int_0^\infty \frac{t^n e^{-\frac{xt}{k}}}{1+e^{-t}} dt\right)^{\frac{1}{r}} \left(\frac{(-1)^n p}{k^{n+1}} \int_0^\infty \frac{t^n e^{-\frac{yt}{k}}}{1+e^{-t}} dt\right)^{\frac{1}{s}} && (4.76)\\
&= \left[{}_p\beta_k^{(n)}(x)\right]^{\frac{1}{r}} \left[{}_p\beta_k^{(n)}(y)\right]^{\frac{1}{s}}. && (4.77)
\end{aligned}$$

Therefore, we can say that ${}_p\beta_k^{(n)}(x)$ is logarithmically convex. And, using Theorem 4.3, we can say that ${}_p\beta_k^{(n)}(x)$ is increasing when n is odd. Therefore, we can conclude that ${}_p\beta_k^{(n)}(x)$ is multiplicatively convex for $n \in \mathbb{N}_{odd}$ ∎

Theorem 4.16. *Let I be the interval $(0, \infty)$, $n \in \mathbb{N}_{odd}$ and $p, k \in \mathbb{R}^+ - \{0\}$, then for all $x_1 \leq x_2 \leq x_3$ in I, we have*

$$\begin{vmatrix} 1 & \log x_1 & \log\left({}_p\beta_k^{(n)}(x_1)\right) \\ 1 & \log x_2 & \log\left({}_p\beta_k^{(n)}(x_2)\right) \\ 1 & \log x_3 & \log\left({}_p\beta_k^{(n)}(x_3)\right) \end{vmatrix} \geq 0 \tag{4.78}$$

or equivalently

$$\begin{aligned}
{}_p\beta_k^{(n)}(x_1)^{\log x_3}\, {}_p\beta_k^{(n)}(x_2)^{\log x_1}\, {}_p\beta_k^{(n)}(x_3)^{\log x_2} \geq {}_p\beta_k^{(n)}(x_1)^{\log x_2}\, {}_p\beta_k^{(n)}(x_2)^{\log x_3} \\
\times\, {}_p\beta_k^{(n)}(x_3)^{\log x_1}. && (4.79)
\end{aligned}$$

Proof. Using Theorem 4.15 and Lemma 2.3.1 from [[23], pg. 77], the desire result readily follows. ■

Theorem 4.17. *Let I be the interval $(0,\infty)$, $n \in \mathbb{N}_{odd}$ and $p,k \in \mathbb{R}^+$. $x_1 \geq x_2 \geq .... \geq x_n$ and $y_1 \geq y_2 \geq .... \geq y_n$ are two families of numbers in a subinterval I of $(0,\infty)$ such that*

$$x_1 \geq y_1$$

$$x_1 x_2 \geq y_1 y_2$$

$$\vdots$$

$$x_1 x_2 .... x_{n-1} \geq y_1 y_2 .... y_{n-1}$$

$$x_1 x_2 .... x_n \geq y_1 y_2 .... y_n.$$

Then

$$_p\beta_k^{(n)}(x_1)\, {}_p\beta_k^{(n)}(x_2) ....{}_p\beta_k^{(n)}(x_n) \geq {}_p\beta_k^{(n)}(y_1)\, {}_p\beta_k^{(n)}(y_2) ....{}_p\beta_k^{(n)}(y_n)\,. \tag{4.80}$$

Proof. Using Theorem 4.15 and Proposition 2.3.5 from from [[23], pg. 80], the desire result readily follows. ■

Theorem 4.18. *Let I be the interval $(0,\infty)$, $m \in \mathbb{N}_{odd}$ and $p,k \in \mathbb{R}^+$. Let $A \in M_n(\mathbb{C})$ be any matrix having the eigenvalues $\lambda_1, ..., \lambda_n$ and the singular numbers $s_1, ..., s_n$, listed such that $|\lambda_1| \geq .... \geq |\lambda_n|$ and $s_1, ..., s_n$. Then*

$$\prod_{1\leq k\leq n} {}_p\beta_k^{(m)}(s_k) \geq \prod_{1\leq k\leq n} {}_p\beta_k^{(m)}(|\lambda_k|). \tag{4.81}$$

Proof. Using Theorem 4.15 and Proposition 2.3.6 from from [[23], pg. 80], the desire result readily follows. ■

Remark 4.1. *In a similar manner presented above, we can prove using the Hölder's inequality that both $\beta_k^{(n)}(x)$ and $\beta^{(n)}(x)$ are multiplicatively convex on the interval $(0,\infty)$. And therefore, both will satisfy the above theorems.*

4.5 On some monotonicity and convexity properties

Theorem 4.19. *If $F(x)$ is defined as*

$$F(x) = x^a \left| {}_p\beta_k^{(n)}(x) \right| \tag{4.82}$$

then, for $a \in \mathbb{R}$, $k \in \mathbb{R}^+$, $n \in \mathbb{N}_0$ and $x > 0$, $F(x)$ is decreasing if $\frac{a}{k} \leq n+1$ and increasing if $\frac{a}{k} \geq n+1+e^{-1}$.

Proof. Using Eqn. (4.24) and convolution theorem for Laplace transform, we have

$$F'(x) = ax^{a-1}\left|{}_p\beta_k^{(n)}(x)\right| - x^a\left|{}_p\beta_k^{(n+1)}(x)\right| \tag{4.83}$$

$$\frac{F'(x)}{x^a} = \frac{a}{x}\left|{}_p\beta_k^{(n)}(x)\right| - \left|{}_p\beta_k^{(n+1)}(x)\right| \tag{4.84}$$

$$= \frac{a}{k}\frac{p}{k^{n+1}}\int_0^\infty e^{-\frac{x}{k}t}dt\int_0^\infty \frac{t^n e^{-\frac{x}{k}t}}{1+e^{-t}}dt - \frac{p}{k^{n+1}}\int_0^\infty \frac{t^{n+1}e^{-\frac{x}{k}t}}{1+e^{-t}}dt \tag{4.85}$$

$$= \frac{a}{k}\frac{p}{k^{n+1}}\int_0^\infty\left[\int_0^t \frac{s^n}{1+e^{-s}}ds\right]e^{-\frac{x}{k}t}dt - \frac{p}{k^{n+1}}\int_0^\infty \frac{t^{n+1}e^{-\frac{x}{k}t}}{1+e^{-t}}dt \tag{4.86}$$

$$= \frac{p}{k^{n+1}}\int_0^\infty \xi_n(t)\,e^{-\frac{x}{k}t}dt, \tag{4.87}$$

where

$$\xi_n(t) = \frac{a}{k}\int_0^t \frac{s^n}{1+e^{-s}}ds - \frac{t^{n+1}}{1+e^{-t}} \tag{4.88}$$

Therefore, $\xi_n(0) = \lim_{t\to 0^+}\xi_n(t) = 0$ and

$$\xi'_n(t) = \frac{a}{k}\frac{t^n}{1+e^{-t}} - \frac{(n+1)\,t^n}{1+e^{-t}} - \frac{t^{n+1}e^{-t}}{(1+e^{-t})^2} \tag{4.89}$$

$$= \frac{t^n}{1+e^{-t}}\left[\frac{a}{k} - (n+1) - \frac{te^{-t}}{1+e^{-t}}\right]. \tag{4.90}$$

If $\frac{a}{k} \leq n+1$, then $\xi'_n(t) < 0$ which implies that $F'(x) < 0$, thus it gives the desire result. Similarly, we can prove that if $\frac{a}{k} \geq n+1+e^{-1}$ then $\xi'_n(t) > 0$ which implies that $F'(x) > 0$. This completes our proof. ■

Theorem 4.20. *Let $m \in \mathbb{N}$, the the inequality*

$$\left|{}_p\beta_k^{(m)}(xy)\right| \leq \left|{}_p\beta_k^{(m)}(x)\right| + \left|{}_p\beta_k^{(m)}(y)\right| \tag{4.91}$$

holds true for $x > 0$ and $y \geq 1$.

Proof. Let

$$G(x,y) = \left|{}_p\beta_k^{(m)}(xy)\right| - \left|{}_p\beta_k^{(m)}(x)\right| - \left|{}_p\beta_k^{(m)}(y)\right| \tag{4.92}$$

Fix y and differentiate with respect to x to get

$$\frac{\partial}{\partial x}G(x,y) = -y\left|{}_p\beta_k^{(m)}(xy)\right| + \left|{}_p\beta_k^{(m+1)}(x)\right| \tag{4.93}$$

$$= \frac{1}{x}\left[x\left|{}_p\beta_k^{(m+1)}(x)\right| - xy\left|{}_p\beta_k^{(m)}(xy)\right|\right]. \tag{4.94}$$

From theorem 4.19, we know that $x\left|{}_p\beta_k^{(m)}(x)\right|$ is decreasing. Since $y \geq 1$, this implies that $xy \geq x$, which therefore states that $G'(x,y) \geq 0$ and thus is increasing. Then for $0 < x < \infty$, we have

$$G(x,y) \leq \lim_{x\to\infty} G(x,y) = -\left|{}_p\beta_k^{(m)}(y)\right| < 0. \tag{4.95}$$

Putting the value of $G(x,y)$ in the above inequality yields the desire results. ∎

Theorem 4.21. *Let $n \in \mathbb{N}$, then the function*

$$H(x) = x\left|{}_p\beta_k^{(n)}(x)\right| \tag{4.96}$$

is strictly completely monotonic on $(0,\infty)$.

Proof. Differentiate Eqn. (4.96) to get

$$H'(x) = \left|{}_p\beta_k^{(n)}(x)\right| - x\left|{}_p\beta_k^{(n+1)}(x)\right| \tag{4.97}$$

$$H''(x) = -2\left|{}_p\beta_k^{(n+1)}(x)\right| + x\left|{}_p\beta_k^{(n+2)}(x)\right| \tag{4.98}$$

$$H'''(x) = 3\left|{}_p\beta_k^{(n+2)}(x)\right| - x\left|{}_p\beta_k^{(n+3)}(x)\right| \tag{4.99}$$

and therefore

$$H^{(m)}(x) = (-1)^{m-1}m\left|{}_p\beta_k^{(m+n-1)}(x)\right| + (-1)^m x\left|{}_p\beta_k^{(m+n)}(x)\right|. \tag{4.100}$$

Furthermore, we have

$$\frac{(-1)^m H^{(m)}(x)}{x} = \frac{-m}{x}\left|{}_p\beta_k^{(n+m-1)}(x)\right| + \left|{}_p\beta_k^{(n+m)}(x)\right|. \tag{4.101}$$

Using the convolution theorem for Laplace transform, we have

$$= \frac{-m}{k}\frac{p}{k^{n+1}}\int_0^\infty e^{-\frac{x}{k}t}dt\int_0^\infty \frac{t^{n+m-1}e^{-\frac{x}{k}t}}{1+e^{-t}}dt + \frac{p}{k^{n+1}}\int_0^\infty \frac{t^{n+m}e^{-\frac{x}{k}t}}{1+e^{-t}}dt \tag{4.102}$$

$$= \frac{-m}{k}\frac{p}{k^{n+1}}\int_0^\infty \left[\int_0^t \frac{s^{n+m-1}}{1+e^{-s}}ds\right]e^{-\frac{x}{k}t}dt + \frac{p}{k^{n+1}}\int_0^\infty \frac{t^{n+m}e^{-\frac{x}{k}t}}{1+e^{-t}}dt \tag{4.103}$$

$$= \frac{p}{k^{n+1}}\int_0^\infty \Omega_n(t)\,e^{-\frac{x}{k}t}dt, \tag{4.104}$$

where

$$\Omega_n(t) = \frac{-m}{k}\int_0^t \frac{s^{n+m-1}}{1+e^{-s}}ds + \frac{t^{n+m}}{1+e^{-t}} \tag{4.105}$$

Therefore, $\Omega_n(0) = \lim_{t\to 0^+}\Omega_n(t) = 0$ and

$$\Omega'_n(t) = \frac{-m}{k}\frac{t^{n+m-1}}{1+e^{-t}} + \frac{(n+m)\,t^{n+m-1}}{1+e^{-t}} + \frac{t^{n+m}e^{-t}}{(1+e^{-t})^2} \tag{4.106}$$

$$= \frac{t^{n+m-1}}{1+e^{-t}}\left[\frac{-m}{k} + (n+m) + \frac{te^{-t}}{1+e^{-t}}\right] > 0. \tag{4.107}$$

Hence $\Omega_n(t)$ is increasing. Therefore, for $t > 0$, we have $\Omega_n(t) > \Omega_n(0) = 0$ and thus $(-1)^m H^{(m)} > 0$. This competes our proof. ■

5 On some extensions of Chaudhary-Zubair gamma function

5.1 *p*-*k*-Chaudhary-Zubair gamma function

The aim of this section is to provide a p-k extesion of the Chaudhary-Zubair gamma function [24] defined as follows for $p, x > 0$

$$\Gamma_p(x) = \int_0^\infty t^{x-1}e^{\left(-t-\frac{p}{t}\right)}dt. \tag{5.1}$$

When $p =$, $\Gamma_p(x)$ reduces to $\Gamma(x)$. It satisfies the following properties

$$\Gamma_p(x+1) = x\Gamma_p(x) + p\Gamma_p(x-1), \tag{5.2}$$

$$\Gamma_p(-x) = p^{-x}\Gamma_p(x). \tag{5.3}$$

Differentiating Eqn. (5.1) n times yields

$$\Gamma_p^{(n)}(x) = \int_0^\infty (\ln t)^n t^{x-1}e^{\left(-t-\frac{p}{t}\right)}dt \tag{5.4}$$

We now establish the following extension of 5.1 for $x > 0$ and $c, p, k \in \mathbb{R}^+ - \{0\}$.

$$\Gamma_{CZ:(c,p,k)}(x) = \int_0^\infty t^{x-1}e^{\left(-\frac{t^k}{p}-\frac{c}{\frac{t^k}{p}}\right)}dt. \tag{5.5}$$

Note that we have slightly changed the notation by replacing p with c in Eqn. (5.1) to avoid getting confused it with the p that will appear in our above extended definition. Differentiating Eqn. (2.2) n times yields

$$\Gamma_{CZ:(c,p,k)}^{(n)}(x) = \int_0^\infty (\ln t)^n t^{x-1}e^{\left(-\frac{t^k}{p}-\frac{c}{\frac{t^k}{p}}\right)}dt. \tag{5.6}$$

In this section, we are going to explore some properties of $\Gamma_{CZ:(c,p,k)}^{(n)}(x)$ and $\Gamma_{CZ:(c,p,k)}(x)$.

5.2 Holder's inequalities for *p*-*k*-Chaudhary-Zubair gamma function

Theorem 5.1. *For $x, y > 0$, $\alpha, \beta \in (0,1)$, $\alpha+\beta=1$, $m, n \in \{2s : s \in \mathbb{N}_0\}$ and $p, k \in \mathbb{R}^+ - \{0\}$, $\Gamma^{(n)}_{CZ:(c,p,k)}(x)$ satisfies the following inequality*

$$\Gamma^{(\alpha m+\beta n)}_{CZ:(c,p,k)}(\alpha x+\beta y) \leq \left[\Gamma^{(m)}_{CZ:(c,p,k)}(x)\right]^{\alpha}\left[\Gamma^{(n)}_{CZ:(c,p,k)}(y)\right]^{\beta}. \tag{5.7}$$

Proof. Using Eqn. (5.6), we have

$$\Gamma^{(\alpha m+\beta n)}_{CZ:(c,p,k)}(\alpha x+\beta y) = \int_0^\infty (\ln t)^{\alpha m+\beta n} t^{\alpha x+\beta y-(\alpha+\beta)} e^{\left(-\frac{t^k}{p}-\frac{c}{\frac{t^k}{p}}\right)(\alpha+\beta)} dt \tag{5.8}$$

$$= \int_0^\infty (\ln t)^{\alpha m}(\ln t)^{\beta n} t^{\alpha(x-1)} t^{\beta(x-1)} e^{\alpha\left(-\frac{t^k}{p}-\frac{c}{\frac{t^k}{p}}\right)} e^{\beta\left(-\frac{t^k}{p}-\frac{c}{\frac{t^k}{p}}\right)} dt \tag{5.9}$$

$$= \int_0^\infty (\ln t)^{\alpha m} t^{\alpha(x-1)} e^{\alpha\left(-\frac{t^k}{p}-\frac{c}{\frac{t^k}{p}}\right)} (\ln t)^{\beta n} t^{\beta(x-1)} e^{\beta\left(-\frac{t^k}{p}-\frac{c}{\frac{t^k}{p}}\right)} dt. \tag{5.10}$$

Now, using Hölder's inequality. we have

$$\int_0^\infty (\ln t)^{\alpha m} t^{\alpha(x-1)} e^{\alpha\left(-\frac{t^k}{p}-\frac{c}{\frac{t^k}{p}}\right)} (\ln t)^{\beta n} t^{\beta(x-1)} e^{\beta\left(-\frac{t^k}{p}-\frac{c}{\frac{t^k}{p}}\right)} dt \tag{5.11}$$

$$\leq \left[\int_0^\infty \left[(\ln t)^{\alpha m} t^{\alpha(x-1)} e^{\alpha\left(-\frac{t^k}{p}-\frac{c}{\frac{t^k}{p}}\right)}\right]^{\frac{1}{\alpha}} dt\right]^{\alpha} \left[\int_0^\infty \left[(\ln t)^{\beta n} t^{\beta(y-1)} e^{\beta\left(-\frac{t^k}{p}-\frac{c}{\frac{t^k}{p}}\right)}\right]^{\frac{1}{\beta}} dt\right]^{\beta} \tag{5.12}$$

$$= \left[\int_0^\infty (\ln t)^{m} t^{(x-1)} e^{\left(-\frac{t^k}{p}-\frac{c}{\frac{t^k}{p}}\right)} dt\right]^{\alpha} \left[\int_0^\infty (\ln t)^{n} t^{(y-1)} e^{\left(-\frac{t^k}{p}-\frac{c}{\frac{t^k}{p}}\right)} dt\right]^{\beta} \tag{5.13}$$

$$= \left[\Gamma^{(m)}_{CZ:(c,p,k)}(x)\right]^{\alpha}\left[\Gamma^{(n)}_{CZ:(c,p,k)}(y)\right]^{\beta}. \tag{5.14}$$

This completes our proof. ■

Corollary 5.1.1. *We have*

$$\Gamma^{(n)}_{CZ:(c,p,k)}(\alpha x+\beta y) \leq \left[\Gamma^{(n)}_{CZ:(c,p,k)}(x)\right]^{\alpha}\left[\Gamma^{(n)}_{CZ:(c,p,k)}(y)\right]^{\beta}. \tag{5.15}$$

Proof. Let $m = n$ in Eqn. (5.7) and the desire result readily follows. ■

Corollary 5.1.2. *We have*

$$\Gamma^{\left(\frac{m+n}{2}\right)}_{CZ:(c,p,k)}\left(\frac{x+y}{2}\right) \leq \sqrt{\left[\Gamma^{(m)}_{CZ:(c,p,k)}\left(x\right)\right]\left[\Gamma^{(n)}_{CZ:(c,p,k)}\left(y\right)\right]}. \tag{5.16}$$

Proof. Let $\alpha = \beta = \frac{1}{2}$ in Eqn. (5.7) and the desire result readily follows. ■

Corollary 5.1.3. *We have*

$$\Gamma_{CZ:(c,p,k)}\left(\alpha x + \beta y\right) \leq \left[\Gamma_{CZ:(c,p,k)}\left(x\right)\right]^{\alpha}\left[\Gamma_{CZ:(c,p,k)}\left(y\right)\right]^{\beta}. \tag{5.17}$$

Proof. Let $m = n = 0$ in Eqn. (5.7) and the desire result readily follows. ■

Corollary 5.1.4. $\Gamma^{(n)}_{CZ:(c,p,k)}\left(x\right)$ *is logarithmically convex on the interval* $(0,\infty)$ *and increasing therefore, it is multiplicatively convex.*

Theorem 5.2. *For* $x, y > 0$*,* $\alpha, \beta \in (0,1)$*,* $\alpha + \beta = 1$ *and* $p, k \in \mathbb{R}^+ - \{0\}$*,* $\Gamma_{CZ:(c,p,k)}\left(x\right)$ *satisfies the following inequality*

$$\Gamma_{CZ:(c,p,k)}\left(x+y\right) \leq \left[\Gamma_{CZ:(c,p,k)}\left(\frac{x}{\alpha}\right)\right]^{\alpha}\left[\Gamma_{CZ:(c,p,k)}\left(\frac{y}{\beta}\right)\right]^{\beta}. \tag{5.18}$$

Proof. Using Eqn. (5.5), we have

$$\Gamma_{CZ:(c,p,k)}\left(x+y\right) = \int_0^\infty t^{(x+y)-1} e^{\left(-\frac{t^k}{p} - \frac{c}{\frac{t^k}{p}}\right)} dt \tag{5.19}$$

$$= \int_0^\infty t^{(x+y)-(\alpha+\beta)} e^{\left(-\frac{t^k}{p} - \frac{c}{\frac{t^k}{p}}\right)(\alpha+\beta)} dt \tag{5.20}$$

$$= \int_0^\infty t^{x-\alpha} e^{\left(-\frac{t^k}{p} - \frac{c}{\frac{t^k}{p}}\right)\alpha} t^{y-\beta} e^{\left(-\frac{t^k}{p} - \frac{c}{\frac{t^k}{p}}\right)\beta} dt \tag{5.21}$$

Using Hölder's inequality, we have

$$\int_0^\infty t^{x-\alpha} e^{\left(-\frac{t^k}{p}-\frac{c}{\frac{t^k}{p}}\right)\alpha} t^{y-\beta} e^{\left(-\frac{t^k}{p}-\frac{c}{\frac{t^k}{p}}\right)\beta} dt$$

$$\leq \left[\int_0^\infty \left[t^{x-\alpha} e^{\left(-\frac{t^k}{p}-\frac{c}{\frac{t^k}{p}}\right)\alpha}\right]^{\frac{1}{\alpha}} dt\right]^\alpha \left[\int_0^\infty \left[t^{y-\beta} e^{\left(-\frac{t^k}{p}-\frac{c}{\frac{t^k}{p}}\right)\beta}\right]^{\frac{1}{\beta}} dt\right]^\beta \tag{5.22}$$

$$= \left[\int_0^\infty t^{\frac{x}{\alpha}-1} e^{\left(-\frac{t^k}{p}-\frac{c}{\frac{t^k}{p}}\right)} dt\right]^\alpha \left[\int_0^\infty t^{\frac{y}{\beta}-1} e^{\left(-\frac{t^k}{p}-\frac{c}{\frac{t^k}{p}}\right)} dt\right]^\beta \tag{5.23}$$

$$= \left[\Gamma_{CZ:(c,p,k)}\left(\frac{x}{\alpha}\right)\right]^\alpha \left[\Gamma_{CZ:(c,p,k)}\left(\frac{y}{\beta}\right)\right]^\beta. \tag{5.24}$$

This completes our proof. ■

Corollary 5.2.1. *We have*

$$\Gamma_{CZ:(c,p,k)}(x+y) \leq \alpha\Gamma_{CZ:(c,p,k)}\left(\frac{x}{\alpha}\right) + \beta\Gamma_{CZ:(c,p,k)}\left(\frac{x}{\beta}\right) \tag{5.25}$$

Proof. Using Young's inequality (A.2.1), the desire inequality readily follows. ■

Theorem 5.3. *For $x, y > 0$, $q \geq 1$, $m, n \in \{2s : s \in \mathbb{N}_0\}$ and $p, k \in \mathbb{R}^+ - \{0\}$, $\Gamma^{(n)}_{CZ:(c,p,k)}(x)$ satisfies the following inequality*

$$\left[\Gamma^{(m)}_{CZ:(c,p,k)}(x) + \Gamma^{(n)}_{CZ:(c,p,k)}(y)\right]^{\frac{1}{q}} = \left[\Gamma^{(m)}_{CZ:(c,p,k)}(x)\right]^{\frac{1}{q}} + \left[\Gamma^{(n)}_{CZ:(c,p,k)}(y)\right]^{\frac{1}{q}}. \tag{5.26}$$

Proof. Using Eqn. (5.6), we have

$$\left[\Gamma^{(m)}_{CZ:(c,p,k)}(x) + \Gamma^{(n)}_{CZ:(c,p,k)}(y)\right]^{\frac{1}{q}}$$

$$= \left[\int_0^\infty (\ln t)^m t^{x-1} e^{\left(-\frac{t^k}{k}-\frac{c}{\frac{t^k}{k}}\right)} dt + \int_0^\infty (\ln t)^n t^{y-1} e^{\left(-\frac{t^k}{k}-\frac{c}{\frac{t^k}{k}}\right)} dt\right]^{\frac{1}{q}} \tag{5.27}$$

$$= \left[\int_0^\infty \left[(\ln t)^{\frac{m}{q}} t^{\frac{x-1}{q}} e^{\frac{1}{q}\left(-\frac{t^k}{k}-\frac{c}{\frac{t^k}{k}}\right)}\right]^q + \left[(\ln t)^{\frac{n}{q}} t^{\frac{y-1}{q}} e^{\frac{1}{q}\left(-\frac{t^k}{k}-\frac{c}{\frac{t^k}{k}}\right)}\right]^q dt\right]^{\frac{1}{q}}. \tag{5.28}$$

Using Minkowski's inequality (A.2.2), we have

$$\left[\int_0^\infty \left[(\ln t)^{\frac{m}{q}} t^{\frac{x-1}{q}} e^{\frac{1}{q}\left(-\frac{t^k}{k}-\frac{c}{\frac{t^k}{k}}\right)}\right]^q + \left[(\ln t)^{\frac{n}{q}} t^{\frac{y-1}{q}} e^{\frac{1}{q}\left(-\frac{t^k}{k}-\frac{c}{\frac{t^k}{q}}\right)}\right]^q dt\right]^{\frac{1}{q}}$$

$$\leq \left[\int_0^\infty \left[(\ln t)^{\frac{m}{q}} t^{\frac{x-1}{q}} e^{\frac{1}{q}\left(-\frac{t^k}{k}-\frac{c}{\frac{t^k}{k}}\right)}\right]^q + \left[(\ln t)^{\frac{n}{q}} t^{\frac{y-1}{q}} e^{\frac{1}{q}\left(-\frac{t^k}{k}-\frac{c}{\frac{t^k}{k}}\right)}\right]^q dt\right]^{\frac{1}{q}} \tag{5.29}$$

$$\leq \left[\int_0^\infty (\ln t)^m t^{x-1} e^{\left(-\frac{t^k}{k}-\frac{c}{\frac{t^k}{k}}\right)} dt\right]^{\frac{1}{q}} + \left[\int_0^\infty (\ln t)^n t^{y-1} e^{\left(-\frac{t^k}{k}-\frac{c}{\frac{t^k}{k}}\right)} dt\right]^{\frac{1}{q}} \tag{5.30}$$

$$= \left[\Gamma^{(m)}_{CZ:(c,p,k)}(x)\right]^{\frac{1}{q}} + \left[\Gamma^{(n)}_{CZ:(c,p,k)}(y)\right]^{\frac{1}{q}}. \tag{5.31}$$

This completes our proof. ■

Theorem 5.4. *For $x > 0$, $m, r \in \{2s : s \in \mathbb{N}_0\}$ such that $m \geq r$ and $p, k \in \mathbb{R}^+ - \{0\}$, $\Gamma^{(n)}_{CZ:(c,p,k)}(x)$ satisfies the following inequality*

$$\exp\left(\Gamma^{(m-r)}_{CZ:(c,p,k)}(x)\right)\exp\left(\Gamma^{(m+r)}_{CZ:(c,p,k)}(x)\right) \geq \left(\exp\left(\Gamma^{(m)}_{CZ:(c,p,k)}(x)\right)\right)^2. \tag{5.32}$$

Proof. Using Eqn. (5.6), we have

$$\frac{1}{2}\left(\Gamma^{(m-r)}_{CZ:(c,p,k)}(x) + \Gamma^{(m+r)}_{CZ:(c,p,k)}(x)\right) - \Gamma^{(m)}_{CZ:(c,p,k)}(x)$$

$$= \frac{1}{2}\left(\int_0^\infty (\ln t)^{m-r} t^{x-1} e^{\left(-\frac{t^k}{k}-\frac{c}{\frac{t^k}{k}}\right)} dt + \int_0^\infty (\ln t)^{m+r} t^{x-1} e^{\left(-\frac{t^k}{k}-\frac{c}{\frac{t^k}{k}}\right)} dt\right)$$

$$- \int_0^\infty (\ln t)^m t^{x-1} e^{\left(-\frac{t^k}{k}-\frac{c}{\frac{t^k}{k}}\right)} dt \tag{5.33}$$

$$= \frac{1}{2}\int_0^\infty \left[\frac{1}{(\ln t)^r} + (\ln t)^r - 2\right](\ln t)^m t^{x-1} e^{\left(-\frac{t^k}{k}-\frac{c}{\frac{t^k}{k}}\right)} dt \tag{5.34}$$

$$= \frac{1}{2}\int_0^\infty [1-(\ln t)^r]^2 (\ln t)^{m-r} t^{x-1} e^{\left(-\frac{t^k}{k}-\frac{c}{\frac{t^k}{k}}\right)} \tag{5.35}$$

$$\geq 0. \tag{5.36}$$

Therefore,

$$\frac{1}{2}\left(\Gamma^{(m-r)}_{CZ:(c,p,k)}(x) + \Gamma^{(m+r)}_{CZ:(c,p,k)}(x)\right) - \Gamma^{(m)}_{CZ:(c,p,k)}(x) \geq 0 \tag{5.37}$$

$$\Gamma^{(m-r)}_{CZ:(c,p,k)}(x) + \Gamma^{(m+r)}_{CZ:(c,p,k)}(x) \geq 2\Gamma^{(m)}_{CZ:(c,p,k)}(x). \tag{5.38}$$

Now, take the exponent of the above equation and the desire result readily follows. ■

Similar inequalities for the original version of chaudhry-Zubair gamma function can be found in [25].

5.3 On some multiplicative convex properties

Using corollary 5.1.4, we now know that $\Gamma^{(n)}_{CZ:(c,p,k)}(x)$ is multiplicatively convex. Therefore, it satisfies following theorems.

Theorem 5.5. *Let I be the interval $(0,\infty)$, $n \in \{2s : s \in \mathbb{N}_0\}$ and $c,p,k \in \mathbb{R}^+ - \{0\}$, then for all $x_1 \leq x_2 \leq x_3$ in I, we have*

$$\begin{vmatrix} 1 & \log x_1 & \log\left(\Gamma^{(n)}_{CZ:(c,p,k)}(x_1)\right) \\ 1 & \log x_2 & \log\left(\Gamma^{(n)}_{CZ:(c,p,k)}(x_2)\right) \\ 1 & \log x_3 & \log\left(\Gamma^{(n)}_{CZ:(c,p,k)}(x_3)\right) \end{vmatrix} \geq 0 \tag{5.39}$$

for all $x_1 \leq x_2 \leq x_3$ in I; equivalently, if and only if

$$\begin{aligned} &\Gamma^{(n)}_{CZ:(c,p,k)}(x_1)^{\log x_3}\Gamma^{(n)}_{CZ:(c,p,k)}(x_2)^{\log x_1}\Gamma^{(n)}_{CZ:(c,p,k)}(x_3)^{\log x_2} \\ &\qquad \geq \Gamma^{(n)}_{CZ:(c,p,k)}(x_1)^{\log x_2}\Gamma^{(n)}_{CZ:(c,p,k)}(x_2)^{\log x_3}\Gamma^{(n)}_{CZ:(c,p,k)}(x_3)^{\log x_1}. \end{aligned} \tag{5.40}$$

Proof. Using Corollary 5.1.4 and from from [[23], pg. 77], the desire result readily follows. ■

Theorem 5.6. *Let I be the interval $(0,\infty)$, $n \in \{2s : s \in \mathbb{N}_0\}$ and $c,p,k \in \mathbb{R}^+ - \{0\}$. $x_1 \geq x_2 \geq .... \geq x_n$ and $y_1 \geq y_2 \geq .... \geq y_n$ are two families of numbers in a subinterval I of $(0,\infty)$ such that*

$$x_1 \geq y_1$$

$$x_1x_2 \geq y_1y_2$$

$$.$$

$$.$$

$$.$$

$$x_1x_2....x_{n-1} \geq y_1y_2....y_{n-1}$$

$$x_1x_2....x_n \geq y_1y_2....y_n.$$

Then

$$\begin{aligned} &\Gamma^{(n)}_{CZ:(c,p,k)}(x_1)\,\Gamma^{(n)}_{CZ:(c,p,k)}(x_2)\,....\Gamma^{(n)}_{CZ:(c,p,k)}(x_3) \\ &\qquad \geq \Gamma^{(n)}_{CZ:(c,p,k)}(y_1)\,\Gamma^{(n)}_{CZ:(c,p,k)}(y_2)\,....\Gamma^{(n)}_{CZ:(c,p,k)}(y_n). \end{aligned} \tag{5.41}$$

Proof. Using Corollary 5.1.4 and from from [[23], pg. 80] the desire result readily follows. ■

Theorem 5.7. *Let I be the interval $(0,\infty)$, $m\in\{2s:s\in\mathbb{N}_0\}$ and $c,p,k\in\mathbb{R}^+-\{0\}$. Let $A\in M_n(\mathbb{C})$ be any matrix having the eigenvalues $\lambda_1,....,\lambda_n$ and the singular numbers $s_1,....,s_n$, listed such that $|\lambda_1|\geq....\geq|\lambda_n|$ and $s_1,....,s_n$. Then*

$$\prod_{1\leq k\leq n}\Gamma^{(m)}_{CZ:(c,p,k)}(s_k)\geq\prod_{1\leq k\leq n}\Gamma^{(m)}_{CZ:(c,p,k)}(|\lambda_k|). \tag{5.42}$$

Proof. Using Corollary 5.1.4 and from [[23], pg. 80], the desire result readily follows. ∎

6 Ighachanea-Akkouchia Holder's inequalities for *p*-*k*- analogue of Nielsen's beta function

In section 3, we defined the p-k-analogue of the Nielsen's beta function as

$$ {}_p\beta_k(x)=\frac{p}{k}\int_0^1\frac{t^{\frac{x}{k}-1}}{1+t}dt. \tag{6.1}$$

For the sake of this section, we change the subscripts of ${}_p\beta_k(x)$ from p and k to u and v:

$$ {}_u\beta_v(x)=\frac{u}{v}\int_0^1\frac{t^{\frac{x}{v}-1}}{1+t}dt. \tag{6.2}$$

Theorem 6.1. *Let $p_k>1$ for $k=1,2,\ldots,n$ with $\sum_{k=1}^n\frac{1}{p_k}=1$ and $x_k\geq 0$. Then for all integers $m\geq 2$, we have*

$$\begin{aligned} {}_u\beta_v\Big(\sum_{k=1}^n\frac{1}{p_k}x_k\Big) \quad &+ \quad nr_0^m\prod_{k=1}^n{}_u\beta_v^{\frac{1}{p_k}}(x_k)\Big(1-\prod_{k=1}^n{}_u\beta_v^{\frac{-1}{n}}(x_k){}_u\beta_v\Big(\sum_{k=1}^n\frac{1}{n}x_k\Big)\Big)\\ &\leq\sum_{(i_1,\ldots,i_{n-1})\in A}C_A\frac{1}{p_1^{i_0-i_1}..p_n^{i_{n-1}-i_n}}\prod_{k=1}^n{}_u\beta_v(x_k)^{1-\frac{i_k-i_{k-1}}{m}}{}_u\beta_v\Big(\sum_{k=1}^n\frac{(i_k-i_{k-1})x_k}{m}\Big)\leq\prod_{k=1}^n{}_u\beta_v^{\frac{1}{p_k}}(x_k),\end{aligned}$$

where, $r_0=\min\{\frac{1}{p_k},\ k=1,\ldots,n\}$.

Proof. To apply Theorem 3.1, we set $\Omega:=(0,1)$ and take the measure $d\mu(t):=\frac{1}{t(t+1)}dt$. Then we choose $\xi_k(t)=\frac{u}{v}t^{\frac{x_k}{vp_k}}$, for $k=1,2,\ldots,n$. So we have the following equalities:

$$\int_\Omega\prod_{k=1}^n|\xi_k(t)|d\mu(t)={}_u\beta_v\Big(\sum_{k=1}^n\frac{1}{p_k}x_k\Big), \tag{6.3}$$

$$\int_\Omega\prod_{k=1}^n|\xi_k(t)|^{\frac{p_k}{n}}d\mu(t)={}_u\beta_v\Big(\sum_{k=1}^n\frac{1}{n}x_k\Big), \tag{6.4}$$

$$||\xi_k||_{p_k} = \Big[{}_u\beta_v(x_k)\Big]^{1/p_k}, \tag{6.5}$$

and

$$\int_\Omega \prod_{k=1}^n |\xi_k(t)|^{\frac{p_k(i_k-i_{k-1})}{m}} d\mu(t) = {}_u\beta_v\Big(\sum_{k=1}^n \frac{(i_k-i_{k-1})x_k}{m}\Big). \tag{6.6}$$

Now, using Theorem 3.1, we have

$$\begin{aligned} {}_u\beta_v\Big(\sum_{k=1}^n \frac{1}{p_k}x_k\Big) \quad &+ \quad nr_0^m \prod_{k=1}^n {}_u\beta_v^{\frac{1}{p_k}}(x_k)\Big(1-\prod_{k=1}^n {}_u\beta_v^{\frac{-1}{n}}(x_k){}_u\beta_v\Big(\sum_{k=1}^n \frac{1}{n}x_k\Big)\Big) \\ &\leq \sum_{(i_1,\ldots,i_{n-1})\in A} C_A \frac{1}{p_1^{i_0-i_1}..p_n^{i_{n-1}-i_n}} \prod_{k=1}^n {}_u\beta_v(x_k)^{1-\frac{i_k-i_{k-1}}{m}} {}_u\beta_v\Big(\sum_{k=1}^n \frac{(i_k-i_{k-1})x_k}{m}\Big) \leq \prod_{k=1}^n {}_u\beta_v^{\frac{1}{p_k}}(x_k). \end{aligned}$$

This completes our proof. ■

Recall the nth derivative of Eq. (6.2) in section 3, we have

$${}_u\beta_v^{(N)}(x) = \frac{(-1)^N u}{v^{N+1}} \int_0^\infty \frac{t^N e^{-\frac{xt}{v}}}{1+e^{-t}} dt \tag{6.7}$$

Notice that here again we had a slight change in the notation, we have denoted the order by N instead of n. Now, applying theorem 3.1 on ${}_u\beta_v^{(N)}(x)$ gives the following theorem.

Theorem 6.2. *Let $p_k > 1$ for $k = 1, 2, \ldots, n$ with $\sum_{k=1}^n \frac{1}{p_k} = 1$ and $x_k \geq 0$. Then for all integers $m \geq 2$ and $N \geq 1$, we have*

$$\begin{aligned} \Big|{}_u\beta_v^{(N)}\Big(\sum_{k=1}^n \frac{1}{p_k}x_k\Big)\Big| \quad &+ \quad nr_0^m \prod_{k=1}^n \Big|{}_u\beta_v^{(N)}(x_k)\Big|^{\frac{1}{p_k}}\Big(1-\prod_{k=1}^n \Big|{}_u\beta_v^{(N)}(x_k)\Big|^{\frac{-1}{n}}\Big|{}_u\beta_v^{(N)}\Big(\sum_{k=1}^n \frac{1}{n}x_k\Big)\Big|\Big) \\ \leq \sum_{(i_1,\ldots,i_{n-1})\in A} C_A \frac{1}{p_1^{i_0-i_1}..p_n^{i_{n-1}-i_n}} &\prod_{k=1}^n \Big|{}_u\beta_v^{(N)}(x_k)\Big|^{1-\frac{i_k-i_{k-1}}{m}} \Big|{}_u\beta_v^{(N)}\Big(\sum_{k=1}^n \frac{(i_k-i_{k-1})x_k}{m}\Big)\Big| \leq \prod_{k=1}^n \Big|{}_u\beta_v^{(N)}(x_k\Big|^{\frac{1}{p_k}}, \end{aligned} \tag{6.8}$$

where, $r_0 = \min\{\frac{1}{p_k},\ k = 1, \ldots, n\}$.

7 Ighachanea-Akkouchia Holder's inequalities for extended Chaudhary-Zubair gamma function

7.1 For *p-k-* extended Chaudhary-Zubair gamma function

Recall the definition of extended Chaudhary-Zubair gamma function that we presented in section 5:

$$\Gamma_{CZ:(c,u,v)}(x) = \int_0^\infty t^{x-1} e^{\left(-\frac{t^v}{u} - \frac{c}{\frac{t^v}{u}}\right)} dt. \tag{7.1}$$

Again here we have replaced p and k with u and v. The N^{th} derivative of 7.1 is given by

$$\Gamma^{(N)}_{CZ:(c,u,v)}(x) = \int_0^\infty (\ln t)^N t^{x-1} e^{\left(-\frac{t^v}{u} - \frac{c}{\frac{t^v}{u}}\right)} dt. \tag{7.2}$$

Theorem 7.1. *Let $p_k > 1$ for $k = 1, 2, \ldots, n$ with $\sum_{k=1}^n \frac{1}{p_k} = 1$ and $x_k \geq 0$. Let $u, v > 0$ Then for all integers $m \geq 2$, we have*

$$\begin{aligned}
&\Gamma_{CZ:(c,u,v)}\Big(\sum_{k=1}^n \frac{1}{p_k} x_k\Big) n r_0^m \prod_{k=1}^n \Gamma^{\frac{1}{p_k}}_{CZ:(c,u,v)}(x_k)\Big(1 - \prod_{k=1}^n \Gamma^{\frac{-1}{n}}_{CZ:(c,u,v)}(x_k)\Gamma_{CZ:(c,u,v)}\Big(\sum_{k=1}^n \frac{1}{n} x_k\Big)\Big) \\
&\quad \leq \sum_{(i_1,\ldots,i_{n-1})\in A} C_A \frac{1}{p_1^{i_0-i_1}..p_n^{i_{n-1}-i_n}} \prod_{k=1}^n \Gamma_{CZ:(c,u,v)}(x_k)^{1-\frac{i_k-i_{k-1}}{m}} \Gamma_{CZ:(c,u,v)}\Big(\sum_{k=1}^n \frac{(i_k - i_{k-1})x_k}{m}\Big) \\
&\quad \leq \prod_{k=1}^n \Gamma^{\frac{1}{p_k}}_{CZ:(c,u,v)}(x_k),
\end{aligned} \tag{7.3}$$

where, $r_0 = \min\{\frac{1}{p_k},\ k = 1, \ldots, n\}$.

For 7.2 we have the following theorem.

Theorem 7.2. *Let $p_k > 1$ for $k = 1, 2, \ldots, n$ with $\sum_{k=1}^n \frac{1}{p_k} = 1$ and $x_k \geq 0$. Let $u, v > 0$ Then for all integers $m \geq 2$, we have*

$$\begin{aligned}
&\Gamma^{(N)}_{CZ:(c,u,v)}\Big(\sum_{k=1}^n \frac{1}{p_k} x_k\Big) n r_0^m \prod_{k=1}^n \Gamma^{(N)\frac{1}{p_k}}_{CZ:(c,u,v)}(x_k)\Big(1 - \prod_{k=1}^n \Gamma^{(N)\frac{-1}{n}}_{CZ:(c,u,v)}(x_k)\Gamma^{(N)}_{CZ:(c,u,v)}\Big(\sum_{k=1}^n \frac{1}{n} x_k\Big)\Big) \\
&\quad \leq \sum_{(i_1,\ldots,i_{n-1})\in A} C_A \frac{1}{p_1^{i_0-i_1}..p_n^{i_{n-1}-i_n}} \prod_{k=1}^n \Gamma^{(N)}_{CZ:(c,u,v)}(x_k)^{1-\frac{i_k-i_{k-1}}{m}} \Gamma^{(N)}_{CZ:(c,u,v)}\Big(\sum_{k=1}^n \frac{(i_k - i_{k-1})x_k}{m}\Big) \\
&\quad \leq \prod_{k=1}^n \Gamma^{(N)\frac{1}{p_k}}_{CZ:(c,u,v)}(x_k),
\end{aligned} \tag{7.4}$$

where, $r_0 = \min\{\frac{1}{p_k},\ k = 1, \ldots, n\}$.

7.2 For ordinary Chaudhary-Zubair gamma function

If $u = v$ in theorems 7.1 and 7.2, then we get the corresponding inequalities for the ordinary Chaudhary-Zubair gamma function:

$$\Gamma_c(x) = \int_0^\infty t^{x-1} e^{\left(-t-\frac{c}{t}\right)} dt. \tag{7.5}$$

Theorem 7.3. *Let $p_k > 1$ for $k = 1, 2, \dots, n$ with $\sum_{k=1}^{n} \frac{1}{p_k} = 1$ and $x_k \geq 0$. Let $u, v > 0$ Then for all integers $m \geq 2$, we have*

$$\Gamma_c\Big(\sum_{k=1}^{n} \frac{1}{p_k} x_k\Big) n r_0^m \prod_{k=1}^{n} \Gamma_c^{\frac{1}{p_k}}(x_k)\Big(1 - \prod_{k=1}^{n} \Gamma_c^{\frac{-1}{n}}(x_k)\Gamma_c\Big(\sum_{k=1}^{n} \frac{1}{n} x_k\Big)\Big)$$
$$\leq \sum_{(i_1,\dots,i_{n-1})\in A} C_A \frac{1}{p_1^{i_0-i_1} .. p_n^{i_{n-1}-i_n}} \prod_{k=1}^{n} \Gamma_c(x_k)^{1-\frac{i_k-i_{k-1}}{m}} \Gamma_c\Big(\sum_{k=1}^{n} \frac{(i_k - i_{k-1})x_k}{m}\Big) \leq \prod_{k=1}^{n} \Gamma_c^{\frac{1}{p_k}}(x_k), \quad (7.6)$$

where, $r_0 = \min\{\frac{1}{p_k},\ k = 1, \dots, n\}$.

Theorem 7.4. *Let $p_k > 1$ for $k = 1, 2, \dots, n$ with $\sum_{k=1}^{n} \frac{1}{p_k} = 1$ and $x_k \geq 0$. Let $u, v > 0$ Then for all integers $m \geq 2$, we have*

$$\Gamma_c^{(N)}\Big(\sum_{k=1}^{n} \frac{1}{p_k} x_k\Big) n r_0^m \prod_{k=1}^{n} \Gamma_c^{(N)\frac{1}{p_k}}(x_k)\Big(1 - \prod_{k=1}^{n} \Gamma_c^{(N)\frac{-1}{n}}(x_k)\Gamma_c^{(N)}\Big(\sum_{k=1}^{n} \frac{1}{n} x_k\Big)\Big)$$
$$\leq \sum_{(i_1,\dots,i_{n-1})\in A} C_A \frac{1}{p_1^{i_0-i_1} .. p_n^{i_{n-1}-i_n}} \prod_{k=1}^{n} \Gamma_c^{(N)}(x_k)^{1-\frac{i_k-i_{k-1}}{m}} \Gamma_c^{(N)}\Big(\sum_{k=1}^{n} \frac{(i_k - i_{k-1})x_k}{m}\Big) \leq \prod_{k=1}^{n} \Gamma_c^{(N)\frac{1}{p_k}}(x_k), \quad (7.7)$$

where, $r_0 = \min\{\frac{1}{p_k},\ k = 1, \dots, n\}$.

7.3 For v-extended Chaudhary-Zubair gamma function

In [30], authors derieved the following extension of the Chaudhary-Zubair gamma function:

$$\Gamma_{b,v}(z) = \int_0^\infty t^{z-1} e^{-\frac{t^v}{v} - \frac{b^v t^{-v}}{v}} dt \quad (7.8)$$

for $\Re z > 0$ and $b \geq 0$ and $v > 0$. The N^{tk} derivative of $\Gamma_{b,v}(z)$ can be given by

$$\Gamma_{b,v}^{(N)}(z) = \int_0^\infty (\ln t)^N t^{z-1} e^{-\frac{t^v}{v} - \frac{b^v t^{-v}}{v}} dt \quad (7.9)$$

Applying theorem 3.1 on $\Gamma_{b,v}(z)$ and $\Gamma_{b,v}^{(N)}(z)$ gives the following theorems.

Theorem 7.5. *Let $p_k > 1$ for $k = 1, 2, \dots, n$ with $\sum_{k=1}^{n} \frac{1}{p_k} = 1$ and $x_k \geq 0$. Let $u, v > 0$ Then for all integers $n \geq 2$, we have*

$$\Gamma_{b,v}\Big(\sum_{k=1}^{n}\frac{1}{p_k}x_k\Big)nr_0^m\prod_{k=1}^{n}\Gamma_{b,v}^{\frac{1}{p_k}}(x_k)\Big(1-\prod_{k=1}^{n}\Gamma_{b,v}^{\frac{-1}{n}}(x_k)\Gamma_{b,v}\Big(\sum_{k=1}^{n}\frac{1}{n}x_k\Big)\Big)$$
$$\leq\sum_{(i_1,\ldots,i_{n-1})\in A}C_A\frac{1}{p_1^{i_0-i_1}..p_n^{i_{n-1}-i_n}}\prod_{k=1}^{n}\Gamma_{b,v}(x_k)^{1-\frac{i_k-i_{k-1}}{m}}\Gamma_{b,v}\Big(\sum_{k=1}^{n}\frac{(i_k-i_{k-1})x_k}{m}\Big)\leq\prod_{k=1}^{n}\Gamma_{b,v}^{\frac{1}{p_k}}(x_k), \tag{7.10}$$

where, $r_0=\min\{\frac{1}{p_k},\ k=1,\ldots,n\}$.

Theorem 7.6. *Let* $p_k>1$ *for* $k=1,2,\ldots,n$ *with* $\sum_{k=1}^{n}\frac{1}{p_k}=1$ *and* $x_k\geq 0$*. Let* $u,v>0$ *Then for all integers* $m\geq 2$*, we have*

$$\Gamma_{b,v}^{(N)}\Big(\sum_{k=1}^{n}\frac{1}{p_k}x_k\Big)nr_0^m\prod_{k=1}^{n}\Gamma_{b,v}^{(N)\frac{1}{p_k}}(x_k)\Big(1-\prod_{k=1}^{n}\Gamma_{b,v}^{(N)\frac{-1}{n}}(x_k)\Gamma_{b,v}^{(N)}\Big(\sum_{k=1}^{n}\frac{1}{n}x_k\Big)\Big)$$
$$\leq\sum_{(i_1,\ldots,i_{n-1})\in A}C_A\frac{1}{p_1^{i_0-i_1}..p_n^{i_{n-1}-i_n}}\prod_{k=1}^{n}\Gamma_{b,v}^{(N)}(x_k)^{1-\frac{i_k-i_{k-1}}{m}}\Gamma_{b,v}^{(N)}\Big(\sum_{k=1}^{n}\frac{(i_k-i_{k-1})x_k}{m}\Big)\leq\prod_{k=1}^{n}\Gamma_{b,v}^{(N)\frac{1}{p_k}}(x_k), \tag{7.11}$$

where, $r_0=\min\{\frac{1}{p_k},\ k=1,\ldots,n\}$.

8 Conclusion

In this paper, using the theory of $k-$special functions and some extended versions of the gamma function, we have derived some new number theoretic functions such as the Nielsen's beta function and the extended Chaudhary-Zubair gamma function. Some monotonicity properties of this functions are also proved and modified Holder's inequalities which were derived by Ighachanea and Akkouchia in their work are applied in deriving some inequalities for the functionsthat we have presented in this paper.

9 Appendix 1

1.1.[22] For $x, k > 0$

$$\beta_k(x) = \frac{k}{2}\left\{\psi_k\left(\frac{x+k}{2}\right) - \psi_k\left(\frac{x}{2}\right)\right\} \tag{A.1.1}$$

$$= \sum_{n=0}^{\infty}\left(\frac{k}{2nk+x} - \frac{k}{2nk+x+k}\right) \tag{A.1.2}$$

$$= \int_0^{\infty} \frac{e^{-\frac{xt}{k}}}{1+e^{-t}}dt \tag{A.1.3}$$

$$= \int_0^1 \frac{t^{\frac{x}{k}-1}}{1+t}dt. \tag{A.1.4}$$

1.2.[22] For $n \in \mathbb{N}_0$, we have

$$\beta_k^{(n)}(x) = \frac{k}{2^{n+1}}\left\{\psi_k^{(n)}\left(\frac{x+k}{2}\right) - \psi_k^{(n)}\left(\frac{x}{2}\right)\right\} \tag{A.1.5}$$

$$= \frac{(-1)^n}{k^n}\int_0^{\infty} \frac{t^n e^{-\frac{xt}{k}}}{1+e^{-t}}dt \tag{A.1.6}$$

$$= \int_0^1 \frac{(\ln t)^n t^{\frac{x}{k}-1}}{1+t}dt. \tag{A.1.7}$$

1.3.[22] For $n \in \mathbb{N}_0$ and $x, y > 0$, the following inequality holds true

$$\left|\beta_k^{(n)}(x+y)\right| < \left|\beta_k^{(n)}(x)\right| + \left|\beta_k^{(n)}(y)\right| \tag{A.1.8}$$

1.4.[22] Let $n \in \mathbb{N}_0$, $a > 0$, and $x > 0$, then the inequalities

$$\left|\beta_k^{(n)}(ax)\right| \leq a\left|\beta_k^{(n)}(x)\right| \tag{A.1.9}$$

if $a \geq 1$, and

$$\left|\beta_k^{(n)}(ax)\right| \geq a\left|\beta_k^{(n)}(x)\right| \tag{A.1.10}$$

if $a \leq 1$ are satisfied.

1.5.[22] Let $k > 0$ and $n \in \mathbb{N}_0$, then the inequality

$$\left|\beta_k^{(n)}(xy)\right| < \left|\beta_k^{(n)}(x)\right| + \left|\beta_k^{(n)}(y)\right| \tag{A.1.11}$$

holds for $x > 0$ and $y \geq 1$.

10 Appendix 2

2.1. (Young's Inequality): If $u, v \geq 0$ and $(\alpha, \beta) \in (0, 1)$ sucn that $\alpha + \beta = 1$, then the inequality

$$u^{\alpha} v^{\beta} \leq \alpha u + \beta v \tag{A.2.1}$$

holds.
2.2. (Minkowski's inequality): Let $u \geq 1$. If $f(t)$ and $g(t)$ are continuous real-valued function on $[a, b]$, then inequality

$$\left(\int_a^b |f(t) + g(t)|^u dt \right)^{\frac{1}{u}} \leq \left(\int_a^b |f(t)|^u dt \right)^{\frac{1}{u}} + \left(\int_a^b |g(t)|^u dt \right)^{\frac{1}{u}} \tag{A.2.2}$$

holds.

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