A Henstock approach of the PUL-integral

Abstract
The PUL-integral is a McShane type of de_nition in which the notion of a partition of unity is of great importance. It was _rst introduced by Kurzweil and Jarnik. Recently, Boonpogkrong revisited this de_nition and presented its, relatively, simpli_ed approach. In this paper, a Henstock-Kurzweil approach of this integral including its fundamental properties will be presented.

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1 Introduction

The PU integral is de_ned is such a way that it utilizes the notion of a partition of unity which is known to be applicable in de_ning an integral de_ned on a smooth manifold. The concept of de_ning this integral in terms of its covering system, unlike the Henstock integral, equivalently the generalized Riemann integral, is that the partitions of the domain of the integrand allows overlapping of intervals in the collection. On the other hand, a McShane integral is an integration process which is a Henstock type of de_nition in which the tag is not of Perron type. Another variant of this de_nition, in some sense, is the PUL-integral. Boonpogkrong [1] revisited the PUL integral in its more simpli_ed approach. There, he showed the application of the PUL integral in a manifold setting. Flores and Benitez [3, 4] further de_ned a generalized version of this de_nition in a Banach setting in its Stieltjes form and presented some its theorem on convergence.

In this section, the PU-integral will be revisited in its simpli_ed approach and some of its simple properties will be given. In what follows, with no confusion arises, we denote

a closed and bounded interval in R_n by

$$[a; b] = Y_{i=1}^{k} [a_i; b_i];$$

and

$$\_([a; b]) = vol([a; b])$$

where a_i; b_i 2 R and _ is the Lebesgue Measure of [a; b].

Given a funtion f : X ! R, the support of f, written as supp f, is de_ned as the closure of the set fx 2 X : f(x) 6= 0g.

2 Main Results

Definition 2.1. [1] A _nite collection $f'_k g^m_{k=1}$ of smooth functions on E is a partial partition of unity if

(i) $'_k$ _ 0 on E for each k = 1; 2; _ _ _ ;m; and

(ii) $X_{k=1}^{m} '_k$ _ 1 a.e. on E.

If $X_{k=1}^{m} '_k$ = 1 a.e. in E, then we say that $f'_k g^m_{k=1}$ is called a partition of unity.

Definition 2.2. Let ' be a smooth function on E, I be a closed and bounded interval in R_n, _ be a gauge on E, and _ 2 E. Then a triple (_; I; ') is a _-_ne in a sense of PU if

_ 2 I and
supp ' _ I _ B(_; _(_)):
Definition 2.3. Let D = f(_k; Ik; 'k)gmk
=1 be _nite collection of triples. Then D is a
_-_ne PU-partial division of E if f'kgmk
=1 is a partial partition of unity. If f'kgmk
=1 is a
partition of unity, then we say that D is a _-_ne PU-division of E.
Remark 2.4. For a _-_ne division D = f_; I; 'g, I's may be overlapping.
Note that if ' : [a; b] ! R is continuous, then
Z
E
' exists for any closed and bounded
subinterval E of [a; b].
Definition 2.5. Let f : [a; b] ! X and g : [a; b] ! R. We de_ne the PU-sum by
S(f;D) =
Xm
k=1
f(_k) (R)
Z
Ik
'k

where D is a _-_ne PU-division of [a; b]. For convenience, we denote a _-_ne PU-division
of [a; b] by D = f(_; I;)g and a PU-sum of f with respect to D by
S(f;D) =
X
D
f(_)
Z
I
= (D)
X
f(_)
Z
I
:
Definition 2.6. Let f : E ! R, where E is a compact set in Rn. Then f is said to be
PU-integrable to a real number A on E if for every _ > 0, there exists a gauge _ de_ned
on E such that for any _-_ne division D = f_k; Ik; 'kgmk
=1 of E, we have
jS(f;D) □ Aj < _:
We denote A by (P)
Z
E
f.
Example 2.7. De_ne f : [0; 1] ! R by
f(x) =
(
1; if x 2 Q \ [0; 1]
0; otherwise
for all X 2 [0; 1]. Let _ > 0. Note that Q is a countable set; thus, we write Q = fqng1 n=1.
De_ne _ on [0; 1] by

$$\delta(x) = \begin{cases} \dfrac{\varepsilon}{2^{n+1}}; & \text{if } x \in \mathbb{Q} \cap [0;1] \\ 1; & \text{otherwise} \end{cases}$$

for all $x \in [0;1]$. Here, δ is a gauge on $[0;1]$. Let $D = \{(\xi; I; \varphi)\}$, a δ-fine PU-division of $[0;1]$. Observe that

$$\left| (D)\sum f(\xi)\int_I \varphi - 0 \right| = \left| (D)\sum f(\xi)\int_I \varphi \right| = \left| (D)\sum_{\xi\in[0;1]\cap\mathbb{Q}} f(\xi)\int_I \varphi + (D)\sum_{\xi\in[0;1]\setminus\mathbb{Q}} f(\xi)\int_I \varphi \right| = \left| (D)\sum_{\xi\in[0;1]\cap\mathbb{Q}} f(\xi)\int_I \varphi \right| =$$

X
D
_2[0;1]\Q
Z
I
'
______
=
X
D
_2[0;1]\Q
_____
Z
I
'
_____
_
X
D
_2[0;1]\Q
Z
I
j'j _
X
D
_2[0;1]\Q
Z
I
1 =
X
D
_2[0;1]\Q
_(I)
<
X
D
_2[0;1]\Q
_(_) <
1X
n=1
_(_) =
1X
n=1
_
2n+1
= _;
where _ is the Lebesgue measure. This means that f, the Dirichlet function, also called

as the Weierstrass function, is PU-integrable to 0 on [0; 1].
Recall that the Dirichlet function fails to be Riemann integrable; hence the latter example portays an important facet of the PU-integral.
Now, we will establish some of the elementary properties of the PU-integral.
Theorem 2.8. The PU-integral of f over [a; b] is unique.
Theorem 2.9. Assume that f : [a; b] ! R and g : [a; b] ! R are PU-integrable over [a; b]. If c 2 R, then cf and f + g are PU-integrable over [a; b]. Moreover,
(P)

Z
[a;b]
(cf) = c _ (P)
Z
[a;b]
f
and
(P)
Z
[a;b]
(f + g) = (P)
Z
[a;b]
f + (P)
Z
[a;b]
g:
Proof : Let a; b 2 R. Fix _ > 0. Choose a gauge _ on [a; b] such that if D is any _-_ne PU-division of [a; b], then

_____
S(f;D) □ (P)
Z
[a;b]
f

_____
<

_
1 + jcj
:
Let D be a _-_ne PU-division of [a; b]. Then

_____
S(cf;D) □ (P)
Z
[a;b]
cf

_____
= jcj _
_____S(f;D) □ (P)
Z
[a;b]
f

_____
< jcj _

_
1 + jcj
< _:
This means that cf1 is PU-integrable over [a; b] and
(P)
Z
[a;b]
cf = c _ (P)
Z
[a;b]

f:
Now, we will verify that f + g is PU-integrable over [a; b] and that
(P)
Z
[a;b]
(f + g) = (P)
Z
[a;b]
f + (P)
Z
[a;b]
g:
To this end, let _ > 0. Then choose a gauge $\_1$ on [a; b] such that if D is a $\_1$-_ne
PU-division of [a; b], then

_____
S(f;D) □ (P)
Z
[a;b]
f

_____
<

_
2
:

In similar fashion, we choose a gauge $\_2$ on [a; b] such that if D_0 is a $\_2$-_ne PU-division
of [a; b], then _____
S(g;D_0) □ (P)
Z
[a;b]
g

_____
<

_
2
:
Let _ = minf$_1$; $_2$g on [a; b]. Here, _ is a gauge on [a; b]. Let D be a _-_ne PU-division
of [a; b]. Then D is both $\_1$-_ne PU-division and $\_2$-_ne PU-division of [a; b]. Thus,

_____
S(f + g;D) □

_
(P)
Z
[a;b]
f + (P)
Z
[a;b]
g

______
=

_____

_
S(f;D) □ (P)

$$\int_{[a,b]} f + S(g;D) - (P)\int_{[a,b]} g\Bigg|$$

$$\le \left|S(f;D) - (P)\int_{[a,b]} f\right| + \left|S(g;D) - (P)\int_{[a,b]} g\right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Therefore, $f + g$ is PU-integrable over $[a, b]$ and

$$(P)\int_{[a,b]} (f + g) = (P)\int_{[a,b]} f + (P)\int_{[a,b]} g. \quad \square$$

Remark 2.10. Define $\mathcal{P} = \{f : [a, b] \to \mathbb{R} \mid f \text{ is PU-integrable on } [a, b]\}$. Then $\mathcal{P}$ is linear over $\mathbb{R}$.

3 Cauchy Criterion

We give a characterization of the PU-integral using Cauchy criterion.

Theorem 3.1. (Cauchy Criterion) A function $f : [a, b] \to \mathbb{R}$ is PU-integrable over $[a, b]$ if and only if for any $\epsilon > 0$, there exists a gauge δ on $[a, b]$ such that for any pair of δ-fine PU-divisions D_1 and D_2 of $[a, b]$, we have

$$|S(f;D_1) - S(f;D_2)| < \epsilon.$$

Proof : ($\Rightarrow$) Let $\epsilon > 0$. Then choose a gauge δ on $[a, b]$ such that if D is a δ-fine PU-division

of $[a;b]$, we have
$$\left|\overline{S}(f;D) - (P)\int_{[a;b]} f\right| < \frac{\epsilon}{2}.$$
Fix D_1 and D_2, δ-fine PU-divisions of $[a;b]$. Then
$$\left|\overline{S}(f;D_1) - \overline{S}(f;D_2)\right| \le \left|\overline{S}(f;D_1) - (P)\int_{[a;b]} f\right| + \left|(P)\int_{[a;b]} f - \overline{S}(f;D_2)\right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

($\Leftarrow$) By hypothesis, for each $n \in \mathbb{N}$, choose a gauge δ_n on $[a;b]$ such that if D_n and D'_n are any two δ_n-fine PU-divisions of $[a;b]$, we have
$$|S(f;D_n) - S(f;D'_n)| < \frac{1}{n}. \quad (3.1)$$
Without loss of generality, we assume that $\delta_n \ge \delta_{n+1}$ on $[a;b]$ for all $n \in \mathbb{N}$. For every $n \in \mathbb{N}$, let D_n be a fixed δ_n-fine division of $[a;b]$ and consider its corresponding PU-sum $s_n = S(f;D_n)$. Here, the sequence $\langle s_n\rangle_{n=1}^{\infty}$ is Cauchy, and so $\langle s_n\rangle_{n=1}^{\infty}$ converges in X, say,
$$\lim_{n\to\infty} s_n = A.$$
We now show that f is a PU-integrable over $[a;b]$ and

$$(P)\int_{[a,b]} f = A.$$

Let $\varepsilon > 0$. Since $\lim_{n\to\infty} s_n = A$, we may choose $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$

$$|S(f;D_n) - A| = |s_n - A| < \frac{\varepsilon}{2}. \quad (3.2)$$

Now, choose $N_2 \in \mathbb{N}$ such that $\frac{1}{N_2} < \frac{\varepsilon}{2}$. Take $N = N_1 \vee N_2$ and put $\delta = \delta_N$ on $[a; b]$. Let D be any δ-fine division of $[a; b]$. Then D is δ_N-fine division of $[a; b]$. Since $N \geq N_2$, by (3.1) we have

$$|S(f;D) - S(f;D_N)| < \frac{1}{N} \leq \frac{1}{N_2} < \frac{\varepsilon}{2}. \quad (3.3)$$

Also, since $N \geq N_1$, inequality (3.2) for $n = N$, i.e.

$$|S(f;D_N) - A| < \frac{\varepsilon}{2}. \quad (3.4)$$

Hence, by (3.3) and (3.4) we have

$$|S(f;D) - A| \leq |S(f;D) - S(f;D_N)| + |S(f;D_N) - A| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This means that f is a PU-integrable over $[a; b]$ and

$$(P)\int_{[a,b]} f = A. \quad \square$$

In what follows, let $\mathcal{I}_n([a; b])$ be the set of all compact subintervals of $[a; b] \subseteq \mathbb{R}^n$.

Corollary 3.2. If $f : [a; b] \to X$ is PU-integrable over $[a; b]$ and $I \in \mathcal{I}_n([a; b])$, then f is PU-integrable over I and

$$(P)\int$$

I
f = (P)
Z
[a;b]
f _ _I

Corollary 3.3. Let f be any real-valued funtion on [a; b]. Suppose that f is PU-integrable on the subintervals $F_1;F_2$ 2 $I_m([a; b])$ with F_1 and F_2 is a partition of [a; b]. Then f is PU-integrable over F_1
S
F_2 and
(P)
Z
F_1
S
F_2
f = (P)
Z
F_1
f + (P)
Z
F_2
f:

Theorem 3.4. Suppose that f and g are PU-integrable over [a:b]. If f _ g on [a; b], then
(P)
Z
[a;b]
f _ (P)
Z
[a;b]
g:

Proof : Fix _ > 0. Since f and g are PU-integrable over [a; b], we may choose the smallest possible gauge _ on [a; b] such that if D is a _-_ne PU-division of [a; b], then

S(f;D) □ (P)
Z
[a;b]
f
<
_
2
;
and
S(g;D) □ (P)
Z
[a;b]
g
<
_
2
:

Notice that

(P)
Z
[a;b]
f □ S(f;D) _
_____
S(f;D) □ (P)
Z
[a;b]
f
_____
<
_
2
and
S(f;D) □ (P)
Z
[a;b]
g _
_____
S(g;D) □ (P)
Z
[a;b]
g
_____
<
_
2
implies
(P)
Z
[a;b]
f □
_
2
< S(f;D)
and
S(g;D) < (P)
Z
[a;b]
g +
_
2
:
But S(f; g) _ S(g;D). Hence, we now have
(P)
Z
[a;b]
f □
_
2
< S(f;D) _ S(g;D) < (P)
Z
[a;b]

g +

_
2
;

that is,
(P)
Z
[a;b]
f < (P)
Z
[a;b]
g + _:
The arbitrariness of _ implies
(P)
Z
[a;b]
f _ (P)
Z
[a;b]
g: _

4 Some Existence and Convergence Theorem

The following theorem exhibits a real-valued function on R^n which is a PU- integrable function.

Theorem 4.1. Let f : [a; b] ! X be continuous on [a; b] . Then f is PU-integrable over [a; b].

Proof : Let _ > 0. Note that the Riemann integral
Z
[a;b]
' exists, whenever ' is a partition of unity on [a; b]. Since f is continuous on [a; b], then f is uniformly continuous on [a; b]. Hence, there exists a _ > 0 such that for any x; y 2 [a; b] with kx yk < _(x), we have
jf(x) f(y)j <

_
2[vol([a; b]) + 1]
:

Let D_1 = f(_; I; ')g and D_2 = f(_; J;)g be any two _-_ne divisions of [a; b]. Let D_3 = f;K; _g be a _-_ne division of [a; b], where K = I \ J with I 2 D_1 and J 2 D_2.
for any interval I. Observe that
S(f;D_1) =
X
I2D_1
f(_)
Z
I
' =
X
I2D_1
f(_)
_ X
J2D_2
Z
I\J

'

_
=
X
$K2D_3$
f(_)
Z
K

_
and
$S(f;D_2)$ =
X
$J2D_2$
f(_)
Z
J
=
X
$J2D_2$
f(_)
_ X
$I2D_1$
Z
J\I

_
=
X
$K2D_3$
f(_)
Z
K
_:
Then

___
$S(f;D_1)$ □ $S(f;D_2)$
___
_
___
$S(f;D_1)$ □ $S(f;D_3)$
___
+
___
$S(f;D_3)$ □ $S(f;D_2)$
___
=
______
X
$K2D_3$
f(_)
Z
K
_ □
X

K2D3
f()
Z
K
–
_____
+
_____
X
K2D3
f()
Z
K
_ □
X
K2D3
f(_)
Z
K
–
_____

_
X
K2D3
"

f(_) □ f()

Rn _
_____
Z
K
–
_____
#
+
X
K2D3
"

f() □ f(_)

–
_____
Z
K
–
_____
#
<
X
K2D3
"

2[vol([a; b]) + 1]
–
Z
K
–
#
+
X
K2D3
"
2[vol([a; b]) + 1]
–
Z
K
–
#
=
2[vol([a; b]) + 1]
_ X
K2D3
Z
K
–
+
X
K2D3
Z
K
–
=
2[vol([a; b]) + 1]
□
2 vol([a; b])
< _:
Therefore, f is PU-integrable over [a; b]. _
We now establish the Uniform convergence theorem for this integral.
Lemma 4.2. Let f : [a; b] ! R be a PU-integrable function over [a; b]. If f is bounded

by M on $[a; b]$, then

$$\left|(P)\int_{[a;b]} f\right| \le M \cdot \mathrm{vol}([a; b]).$$

Proof : By assumption, $-M \le f(x) \le M$. Thus,

$$-M \cdot \mathrm{vol} \le (P)\int_{[a;b]} f(x) \le M \cdot \mathrm{vol}([a; b]).$$

This means that

$$\left|(P)\int_{[a;b]} f\right| \le M \cdot \mathrm{vol}([a; b]).$$

Theorem 4.3. (The Uniform Convergence Theorem)
Assume that $\langle f_n \rangle_{n=1}^{\infty}$ is a sequence of bounded and integrable functions over $[a; b]$. If $f_n \to f$ uniformly on $[a; b]$, then f is PU-integrable on $[a; b]$ and

$$\lim_{n\to\infty} (P)\int_{[a;b]} f_n = (P)\int_{[a;b]} f.$$

Proof : Let $\epsilon > 0$. Since f_n converges uniformly on $[a; b]$, we choose $N_1 \in \mathbb{N}$ such that if $n \ge N_1$ and $x \in [a; b]$, we have

$$|f_n(x) - f(x)| < \frac{\epsilon}{3 \cdot [\mathrm{vol}([a; b]) + 1]}. \quad (4.1)$$

If $m; n \ge N_1$ and $x \in [a; b]$, then by Equation 4.1

$$\begin{aligned} |f_n(x) - f_m(x)| &= |[f_n(x) - f(x)] + [f(x) - f_m(x)]| \\ &\le |[f_n(x) - f(x)]| + |[f(x) - f_m(x)]| \\ &< \frac{\epsilon}{3 \cdot [\mathrm{vol}([a; b]) + 1]} + \frac{\epsilon}{3 \cdot [\mathrm{vol}([a; b]) + 1]} \\ &= \frac{2 \cdot \epsilon}{3[\mathrm{vol}([a; b]) + 1]}. \end{aligned}$$

By Lemma 4.2 and by linearity, for each $n, m \geq N_1$,

$$\left|(P)\int_{[a;b]} f_m - (P)\int_{[a;b]} f_n\right| = \left|\int_{[a;b]} (f_m - f_n)\right| \leq \frac{2\varepsilon}{3[\mathrm{vol}([a;b]) + 1]}\,\mathrm{vol}([a;b]) < \frac{2\varepsilon}{3[\mathrm{vol}([a;b]) + 1]}\,\mathrm{vol}([a;b]) + 1 < \frac{2}{3}\varepsilon.$$

for all $m; n \geq N_1$. This shows that $\left\{(P)\int_{[a;b]} f_n\right\}$ is cauchy, and so $(P)\int_{[a;b]} f_n$ converges to, say, A. So, choose an $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$,

$$\left|(P)\int_{[a;b]} f_n - A\right| < \frac{\varepsilon}{3}. \quad (4.2)$$

Take $N = \max\{N_1; N_2\}$. Since f_N is PU-integrable on $[a; b]$, we may choose a gauge δ on $[a; b]$ such that if D is a δ-fine PU-division of $[a; b]$, then

S(fN;D) (P)
Z
[a;b]
fN

<

3
: (4.3)
Observe that from Equation 4.1,
jS(f;D) S(fN;D)j =

(D)
X
f(_)
Z
I
' (D)
X
fN(_)
Z
I
'

=

(D)
X
(f(_) fN(_))
Z
I
'

_ (D)
X
jf(_) fN(_)j _

Z
I
'

<

3 _ [vol([a; b]) + 1]
_ (D)
XZ
I
'

_

3 _ [vol([a; b]) + 1]

$$\le \varepsilon \,\mathrm{vol}([a;b]) = \frac{\varepsilon}{3}.$$

Therefore,

$$\begin{aligned}|S(f;D) - A| &= \left| S(f;D) - S(f_N;D) + S(f_N;D) - (P)\int_{[a;b]} f_N + (P)\int_{[a;b]} f_N - A \right| \\ &\le |S(f;D) - S(f_N;D)| + \left| S(f_N;D) - (P)\int_{[a;b]} f_N \right| + \left| (P)\int_{[a;b]} f_N - A \right| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.\end{aligned}$$

This shows the integrablity of f over [a; b]. Hence,

$$\lim_{n\to\infty} (P)\int_{[a;b]} f_n = A = (P)\int_{[a;b]} f.$$

5 Conclusion and Recommendation

Results obtain in the literature are pretty much standard and, apparently, the fundamental concepts such as the Cauchy Criterion and the Uniform Convergence Theorem hold for this integral. As a recommendation, further convegence theorems and the Saks-Henstock Lemma and its corollary results are yet to be established.

References

[1] Boonpogkrong, V., Kursweil-Henstock Integration on Manifolds, Taiwanese Journal of Mathematics, Vol. 15, No. 2(2011), 559-571.

[2] Fleming, W., Functions of Several Variables, 2nd edition, Springer-Verlag, New York, 1977.

[3] Flores, G. C. and Benitez, J. V., Simple Properties of PUL-Stieltjes Integral in Banach Space, Journal of Ultra Scientist of Physical Sciences, Vol. 29, No. 4(2017), 126-134.

[4] Flores, G. C. and Benitez, J. V., Some Convergence Theorems of the PUL-Stieltjes Integral in Banach Space, Iranian Journal of Mathematical Sciences and Informatics, Vol. 16, No. 2(2021), 61-72.

[5] Jarnik, J. and Kurzweil, J., A nonabsolutely convergent integral which admits transformation and can be used for integration on manifolds, Czechoslovak Math. J., 35(1) (1985), 116-139.

[6] F. Jones, Lebesgue Integration on Euclidean Space, Jones and Bartlett Publishers, Inc., 2001.

[7] E. Kreyszig, Introductory Functional Analysis with Applications, John Wiley and Sons, Inc., 1978.

[8] J. R. Munkres, Topology, 2nd ed., Prentice Hall, Inc., 2000.

[9] Spivak, M. Calculus on Manifolds: A modern Approach to Classical Theorems of Advanced Calculus, Addison-Wesley Publishong Company, 1965.

[10] Tu, L. W., An Introduction to Manifolds, Springer Science + Business Media, LLC., 2008.

[11] Yeong., L. T., Henstock-Kurzweil Integration on Euclidean Spaces, World Scienti_c Publishing Co. Pte. Ltd., 2011, p. 21.