Original Research Article

A Henstock approach of the PUL-integral

Abstract

The PUL-integral is a McShane type of definition in which the notion of a partition of unity is of great importance. It was first introduced by Kurzweil and Jarnik. Recently, Boonpogkrong revisited this definition and presented its, relatively, simplified approach. In this paper, a Henstock-Kurzweil approach of this integral including its fundamental properties will be presented.

Keywords: Partition of unity, Perron-type, Convergence Theorems

1 Introduction

The PU integral is defined is such a way that it utilizes the notion of a partition of unity which is known to be applicable in defining an integral defined on a smooth manifold. The concept of defining this integral in terms of its covering system, unlike the Henstock integral, is that the partitions of the domain of the integrand allows overlapping of intervals in the collection. Another variant of this definition is the PUL-integral, a PU integral of McShane type in some sense. Boonpogkrong [1] revisited the PUL integral in its more simplified approach. There, he showed the application of the PUL integral in a manifold setting. Flores and Benitez [3, 4] further defined a generalized version of this definition in a Banach setting in its Stieltjes form and presented some its theorem on convergence.

In this section, the PU-integral will be revisited in its simplified approach and some of its simple properties will be given. In what follows, with no confusion arises, we denote a closed and bounded interval in $\mathbb{R}^n$ by

$$[\boldsymbol{a}, \boldsymbol{b}] = \prod_{i=1}^{k} [a_i, b_i],$$

and

$$\mu([\boldsymbol{a},\boldsymbol{b}]) = vol([\boldsymbol{a},\boldsymbol{b}])$$

where $a_i, b_i \in \mathbb{R}$ and μ is the *Lebesgue Measure* of $[\boldsymbol{a},\boldsymbol{b}]$.

Definition 1.1. [1] Let $f : X \to \mathbb{R}$ be any function. The *support of* f, written as supp f, is defined as the closure of the set $\{x \in X : f(x) \neq 0\}$.

Definition 1.2. [1] A finite collection $\{\varphi_k\}_{k=1}^m$ of smooth functions on $\boldsymbol{E}$ is a *partial partition of unity* if

(*i*) $\varphi_k \geq 0$ on $\boldsymbol{E}$ for each $k = 1, 2, \cdots, m$; and

(*ii*) $\displaystyle\sum_{k=1}^{m} \varphi_k \leq 1$ a.e. on $\boldsymbol{E}$.

If $\displaystyle\sum_{k=1}^{m} \varphi_k = 1$ a.e. in $\boldsymbol{E}$, then we say that $\{\varphi_k\}_{k=1}^m$ is called a *partition of unity.*

2 Main Results

Definition 2.1. Let φ be a smooth function on $\boldsymbol{E}$, $\boldsymbol{I}$ be a closed and bounded interval in $\mathbb{R}^n$, δ be a gauge on $\boldsymbol{E}$, and $\boldsymbol{\xi} \in \boldsymbol{E}$. Then a triple $(\boldsymbol{\xi}, \boldsymbol{I}, \varphi)$ is a δ*-fine* in a sense of *PU* if $\boldsymbol{\xi} \in \boldsymbol{I}$ and

$$\text{supp } \varphi \subseteq \boldsymbol{I} \subseteq B(\boldsymbol{\xi}, \delta(\boldsymbol{\xi})).$$

Definition 2.2. Let $D = \{(\boldsymbol{\xi}_k, \boldsymbol{I}_k, \varphi_k)\}_{k=1}^m$ be finite collection of triples. Then D is a δ*-fine PU-partial division* of $\boldsymbol{E}$ if $\{\varphi_k\}_{k=1}^m$ is a partial partition of unity. If $\{\varphi_k\}_{k=1}^m$ is a partition of unity, then we say that D is a δ*-fine PU-division* of $\boldsymbol{E}$.

Remark 2.3. For a δ-fine division $D = \{\boldsymbol{\xi}, \boldsymbol{I}, \varphi\}$, $\boldsymbol{I}$'s may be overlapping.

Note that if $\varphi : [\boldsymbol{a},\boldsymbol{b}] \to \mathbb{R}$ is continuous, then $\displaystyle\int_{\boldsymbol{E}} \varphi$ exists for any closed and bounded subinterval $\boldsymbol{E}$ of $[\boldsymbol{a},\boldsymbol{b}]$.

Definition 2.4. Let $f : [\boldsymbol{a},\boldsymbol{b}] \to X$ and $g : [\boldsymbol{a},\boldsymbol{b}] \to \mathbb{R}$. We define the *PU-sum* by

$$S(f, D) = \sum_{k=1}^{m} f(\boldsymbol{\xi}_k)\ (\mathcal{R}) \int_{I_k} \varphi_k$$

where D is a δ-fine PU-division of $[\boldsymbol{a},\boldsymbol{b}]$. For convenience, we denote a δ-fine PU-division of $[\boldsymbol{a},\boldsymbol{b}]$ by $D = \{(\boldsymbol{\xi}, \boldsymbol{I}, \psi)\}$ and a PU-sum of f with respect to D by

$$S(f, D) = \sum_{D} f(\boldsymbol{\xi}) \int_{\boldsymbol{I}} \psi = (D) \sum f(\boldsymbol{\xi}) \int_{\boldsymbol{I}} \psi.$$

Definition 2.5. Let $f : \boldsymbol{E} \to \mathbb{R}$, where $\boldsymbol{E}$ is a compact set in $\mathbb{R}^n$. Then f is said to be *PU-integrable* to a real number A on $\boldsymbol{E}$ if for every $\epsilon > 0$, there exists a gauge δ defined on $\boldsymbol{E}$ such that for any δ-fine division $D = \{\boldsymbol{\xi}_k, \boldsymbol{I_k}, \varphi_k\}_{k=1}^m$ of E, we have

$$|S(f, D) - A| < \epsilon.$$

We denote A by $(\mathcal{P})\int_{\boldsymbol{E}} f$.

Example 2.6. Define $f : [0, 1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \cap [0, 1] \\ 0, & \text{otherwise} \end{cases}$$

for all $X \in [0, 1]$. Let $\epsilon > 0$. Note that $\mathbb{Q}$ is a countable set; thus, we write $\mathbb{Q} = \{q_n\}_{n=1}^{\infty}$. Define δ on $[0, 1]$ by

$$\delta(x) = \begin{cases} \dfrac{\epsilon}{2^{n+1}}, & \text{if } x \in \mathbb{Q} \cap [0, 1] \\ 1, & \text{otherwise} \end{cases}$$

for all $x \in [0, 1]$. Here, δ is a gauge on $[0, 1]$. Let $D = \{(\xi, I, \varphi)\}$, a δ-fine *PU*-division of $[0, 1]$. Observe that

$$\begin{aligned}
\left|\sum_D f(\xi)\int_I \varphi - 0\right| &= \left|\sum_D f(\xi)\int_I \varphi\right| = \left|\sum_{\substack{D\\ \xi\in[0,1]\cap\mathbb{Q}}} f(\xi)\int_I \varphi + \sum_{\substack{D\\ \xi\in[0,1]\setminus\mathbb{Q}}} f(\xi)\int_I \varphi\right| \\
&= \left|\sum_{\substack{D\\ \xi\in[0,1]\cap\mathbb{Q}}} f(\xi)\int_I \varphi\right| = \left|\sum_{\substack{D\\ \xi\in[0,1]\cap\mathbb{Q}}} \int_I \varphi\right| = \sum_{\substack{D\\ \xi\in[0,1]\cap\mathbb{Q}}} \left|\int_I \varphi\right| \\
&\leq \sum_{\substack{D\\ \xi\in[0,1]\cap\mathbb{Q}}} \int_I |\varphi| \leq \sum_{\substack{D\\ \xi\in[0,1]\cap\mathbb{Q}}} \int_I 1 = \sum_{\substack{D\\ \xi\in[0,1]\cap\mathbb{Q}}} \mu(I) \\
&< \sum_{\substack{D\\ \xi\in[0,1]\cap\mathbb{Q}}} \delta(\xi) < \sum_{n=1}^{\infty} \delta(\xi) = \sum_{n=1}^{\infty} \frac{\epsilon}{2^{n+1}} \\
&= \epsilon,
\end{aligned}$$

where μ is the *Lebesgue* measure. This means that f, the *Dirichlet* function, also called as the *Weierstrass* function, is *PU*-integrable to 0 on $[0, 1]$.

Recall that the Dirichlet function fails to be Riemann integrable; hence the latter example portays an important facet of the *PU*-integral.

Now, we will establish some of the elementary properties of the *PU*-integral.

Theorem 2.7. *The PU-integral of f over $[\boldsymbol{a}, \boldsymbol{b}]$ is unique.*

Theorem 2.8. *Assume that* $f : [\boldsymbol{a}, \boldsymbol{b}] \to \mathbb{R}$ *and* $g : [\boldsymbol{a}, \boldsymbol{b}] \to \mathbb{R}$ *are PU-integrable over* $[\boldsymbol{a}, \boldsymbol{b}]$. *If* $c \in \mathbb{R}$, *then* cf *and* $f + g$ *are PU-integrable over* $[\boldsymbol{a}, \boldsymbol{b}]$. *Moreover,*

$$(\mathcal{P}) \int_{[\boldsymbol{a},\boldsymbol{b}]} (cf) = c \cdot (\mathcal{P}) \int_{[\boldsymbol{a},\boldsymbol{b}]} f$$

and

$$(\mathcal{P}) \int_{[\boldsymbol{a},\boldsymbol{b}]} (f+g) = (\mathcal{P}) \int_{[\boldsymbol{a},\boldsymbol{b}]} f + (\mathcal{P}) \int_{[\boldsymbol{a},\boldsymbol{b}]} g.$$

Proof: Let $a, b \in \mathbb{R}$. Fix $\epsilon > 0$. Choose a gauge δ on $[\boldsymbol{a}, \boldsymbol{b}]$ such that if D is any δ-fine PU-division of $[\boldsymbol{a}, \boldsymbol{b}]$, then

$$\left| S(f, D) - (\mathcal{P}) \int_{[\boldsymbol{a},\boldsymbol{b}]} f \right| < \frac{\epsilon}{1 + |c|}.$$

Let D be a δ-fine PU-division of $[\boldsymbol{a}, \boldsymbol{b}]$. Then

$$\begin{aligned} \left| S(cf, D) - (\mathcal{P}) \int_{[\boldsymbol{a},\boldsymbol{b}]} cf \right| &= |c| \cdot \left| S(f, D) - (\mathcal{P}) \int_{[\boldsymbol{a},\boldsymbol{b}]} f \right| \\ &< |c| \cdot \frac{\epsilon}{1 + |c|} \\ &< \epsilon. \end{aligned}$$

This means that cf_1 is PU-integrable over $[\boldsymbol{a}, \boldsymbol{b}]$ and

$$(\mathcal{P}) \int_{[\boldsymbol{a},\boldsymbol{b}]} cf = c \cdot (\mathcal{P}) \int_{[\boldsymbol{a},\boldsymbol{b}]} f.$$

Now, we will verify that $f + g$ is PU-integrable over $[\boldsymbol{a}, \boldsymbol{b}]$ and that

$$(\mathcal{P}) \int_{[\boldsymbol{a},\boldsymbol{b}]} (f+g) = (\mathcal{P}) \int_{[\boldsymbol{a},\boldsymbol{b}]} f + (\mathcal{P}) \int_{[\boldsymbol{a},\boldsymbol{b}]} g.$$

To this end, let $\epsilon > 0$. Then choose a gauge δ_1 on $[\boldsymbol{a}, \boldsymbol{b}]$ such that if D is a δ_1-fine PU-division of $[\boldsymbol{a}, \boldsymbol{b}]$, then

$$\left| S(f, D) - (\mathcal{P}) \int_{[\boldsymbol{a},\boldsymbol{b}]} f \right| < \frac{\epsilon}{2}.$$

In similar fashion, we choose a gauge δ_2 on $[\boldsymbol{a}, \boldsymbol{b}]$ such that if D' is a δ_2-fine PU-division of $[\boldsymbol{a}, \boldsymbol{b}]$, then

$$\left| S(g, D') - (\mathcal{P}) \int_{[\boldsymbol{a},\boldsymbol{b}]} g \right| < \frac{\epsilon}{2}.$$

Let $\delta = \min\{\delta_1, \delta_2\}$ on $[\boldsymbol{a}, \boldsymbol{b}]$. Here, δ is a gauge on $[\boldsymbol{a}, \boldsymbol{b}]$. Let D be a δ-fine PU-division of $[\boldsymbol{a},\boldsymbol{b}]$. Then D is both δ_1-fine PU-division and δ_2-fine PU-division of $[\boldsymbol{a},\boldsymbol{b}]$. Thus,

$$\begin{aligned}
&\left|S(f+g,D)-\left[(\mathcal{P})\int_{[\boldsymbol{a},\boldsymbol{b}]} f+(\mathcal{P})\int_{[\boldsymbol{a},\boldsymbol{b}]} g\right]\right| \\
&=\left|\left[S(f,D)-(\mathcal{P})\int_{[\boldsymbol{a},\boldsymbol{b}]} f\right]+\left[S(g,D)-(\mathcal{P})\int_{[\boldsymbol{a},\boldsymbol{b}]} g\right]\right| \\
&\leq\left|S(f,D)-(\mathcal{P})\int_{[\boldsymbol{a},\boldsymbol{b}]} f\right|+\left|S(g,D)-(\mathcal{P})\int_{[\boldsymbol{a},\boldsymbol{b}]} g\right| \\
&<\frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon.
\end{aligned}$$

Therefore, $f+g$ is PU-integrable over $[\boldsymbol{a},\boldsymbol{b}]$ and

$$(\mathcal{P})\int_{[\boldsymbol{a},\boldsymbol{b}]}(f+g)=(\mathcal{P})\int_{[\boldsymbol{a},\boldsymbol{b}]} f+(\mathcal{P})\int_{[\boldsymbol{a},\boldsymbol{b}]} g. \qquad \square$$

Remark 2.9. Define $\mathscr{P}=\{f:[\boldsymbol{a},\boldsymbol{b}]\to\mathbb{R}|\ f$ is PU-integrable on $[\boldsymbol{a},\boldsymbol{b}]\}$. Then $\mathscr{P}$ is linear over $\mathbb{R}$.

3 Cauchy Criterion

We give a characterization of the PU-integral using Cauchy criterion.

Theorem 3.1. (Cauchy Criterion) *A function $f:[\boldsymbol{a},\boldsymbol{b}]\to\mathbb{R}$ is PU-integrable over $[\boldsymbol{a},\boldsymbol{b}]$ if and only if for any $\epsilon>0$, there exists a gauge δ on $[\boldsymbol{a},\boldsymbol{b}]$ such that for any pair of δ-fine PU-divisions D_1 and D_2 of $[\boldsymbol{a},\boldsymbol{b}]$, we have*

$$|S\,(f,D_1)-S\,(f,D_2)\,|<\epsilon.$$

Proof: $(\Rightarrow)$ Let $\epsilon>0$. Then choose a gauge δ on $[\boldsymbol{a},\boldsymbol{b}]$ such that if D is a δ-fine PU-division of $[\boldsymbol{a},\boldsymbol{b}]$, we have

$$\left|S(f,D)-(\mathcal{P})\int_{[\boldsymbol{a},\boldsymbol{b}]} f\ \right|<\frac{\epsilon}{2}.$$

Fix D_1 and D_2, δ-fine PU-divisions of $[\boldsymbol{a},\boldsymbol{b}]$. Then

$$\begin{aligned}
&\left|S(f,D_1)-S(f,D_2)\right| \\
&\quad\leq\left|S\,(f,D_1)-(\mathcal{P})\int_{[\boldsymbol{a},\boldsymbol{b}]} f\right|+\left|(\mathcal{P})\int_{[\boldsymbol{a},\boldsymbol{b}]} f-S\,(f,D_2)\right| \\
&\quad<\frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon.
\end{aligned}$$

$(\Leftarrow)$ By hypothesis, for each $n\in\mathbb{N}$, choose a gauge δ_n on $[\boldsymbol{a},\boldsymbol{b}]$ such that if D_n and D'_n

are any two δ_n-fine PU-divisions of $[\boldsymbol{a}, \boldsymbol{b}]$, we have

$$|S(f, D_n) - S(f, D'_n)| < \frac{1}{n}. \tag{3.1}$$

Without loss of generality, we assume that $\delta_n \geq \delta_{n+1}$ on $[\boldsymbol{a}, \boldsymbol{b}]$ for all $n \in \mathbb{N}$. For every $n \in \mathbb{N}$, let D_n be a fixed δ_n-fine division of $[\boldsymbol{a}, \boldsymbol{b}]$ and consider its corresponding PU-sum $s_n = S(f, D_n)$. Here, the sequence $\langle s_n \rangle_{n=1}^{+\infty}$ is Cauchy, and so $\langle s_n \rangle_{n=1}^{+\infty}$ converges in X, say, $\lim_{n\to\infty} s_n = A$.

We now show that f is a PU-integrable over $[\boldsymbol{a}, \boldsymbol{b}]$ and

$$(\mathcal{P}) \int_{[\boldsymbol{a},\boldsymbol{b}]} f = A.$$

Let $\epsilon > 0$. Since $\lim_{n\to\infty} s_n = A$, we may choose $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$

$$|S(f, D_n) - A| = |s_n - A| < \frac{\epsilon}{2}. \tag{3.2}$$

Now, choose $N_2 \in \mathbb{N}$ such that $\frac{1}{N_2} < \frac{\epsilon}{2}$. Take $N = N_1 \vee N_2$ and put $\delta = \delta_N$ on $[\boldsymbol{a}, \boldsymbol{b}]$. Let D be any δ-fine division of $[\boldsymbol{a}, \boldsymbol{b}]$. Then D is δ_N-fine division of $[\boldsymbol{a}, \boldsymbol{b}]$. Since $N \geq N_2$, by (3.1) we have

$$|S(f, D) - S(f, D_N)| < \frac{1}{N} \leq \frac{1}{N_2} < \frac{\epsilon}{2}. \tag{3.3}$$

Also, since $N \geq N_1$, inequality (3.2) for $n = N$; i.e.

$$|S(f, D_N) - A| < \frac{\epsilon}{2}. \tag{3.4}$$

Hence, by (3.3) and (3.4) we have

$$\begin{aligned} |S(f, D) - A| &\leq |S(f, D) - S(f, D_N) \\ &\quad + |S(f, D_N) - A| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

This means that f is a PU-integrable over $[\boldsymbol{a}, \boldsymbol{b}]$ and

$$(\mathcal{P}) \int_{[\boldsymbol{a},\boldsymbol{b}]} f = A. \qquad \square$$

In what follows, let $\mathcal{I}_n([\boldsymbol{a}, \boldsymbol{b}])$ be the set of all compact subintervals of $[\boldsymbol{a}, \boldsymbol{b}] \subseteq \mathbb{R}^n$.

Corollary 3.2. If $f : [\boldsymbol{a}, \boldsymbol{b}] \to X$ is PU-integrable over $[\boldsymbol{a}, \boldsymbol{b}]$ and $\boldsymbol{I} \in \mathcal{I}_n([\boldsymbol{a}, \boldsymbol{b}])$, then f

is PU-integrable over $\boldsymbol{I}$ and

$$(\mathcal{P})\int_{\boldsymbol{I}} f = (\mathcal{P})\int_{[\boldsymbol{a},\boldsymbol{b}]} f\cdot\chi_{\boldsymbol{I}}$$

Corollary 3.3. *Let f be any real-valued funtion on $[\boldsymbol{a},\boldsymbol{b}]$. Suppose that f is PU-integrable on the subintervals $\boldsymbol{F_1},\boldsymbol{F_2}\in\mathcal{I}_m([\boldsymbol{a},\boldsymbol{b}])$ with $\boldsymbol{F_1}$ and $\boldsymbol{F_2}$ is a partition of $[\boldsymbol{a},\boldsymbol{b}]$. Then f is PU-integrable over $\boldsymbol{F_1}\bigcup\boldsymbol{F_2}$ and*

$$(\mathcal{P})\int_{\boldsymbol{F_1}\bigcup\boldsymbol{F_2}} f = (\mathcal{P})\int_{\boldsymbol{F_1}} f + (\mathcal{P})\int_{\boldsymbol{F_2}} f.$$

Theorem 3.4. *Suppose that f and g are PU-integrable over $[\boldsymbol{a}.\boldsymbol{b}]$. If $f\le g$ on $[\boldsymbol{a},\boldsymbol{b}]$, then*

$$(\mathcal{P})\int_{[\boldsymbol{a},\boldsymbol{b}]} f \le (\mathcal{P})\int_{[\boldsymbol{a},\boldsymbol{b}]} g.$$

Proof: Fix $\epsilon>0$. Since f and g are PU-integrable over $[\boldsymbol{a},\boldsymbol{b}]$, we may choose the smallest possible gauge δ on $[\boldsymbol{a},\boldsymbol{b}]$ such that if D is a δ-fine PU-division of $[\boldsymbol{a},\boldsymbol{b}]$, then

$$\left|S(f,D)-(\mathcal{P})\int_{[\boldsymbol{a},\boldsymbol{b}]} f\right| < \frac{\epsilon}{2};$$

and

$$\left|S(g,D)-(\mathcal{P})\int_{[\boldsymbol{a},\boldsymbol{b}]} g\right| < \frac{\epsilon}{2}.$$

Notice that

$$(\mathcal{P})\int_{[\boldsymbol{a},\boldsymbol{b}]} f - S(f,D) \le \left|S(f,D)-(\mathcal{P})\int_{[\boldsymbol{a},\boldsymbol{b}]} f\right| < \frac{\epsilon}{2}$$

and

$$S(f,D)-(\mathcal{P})\int_{[\boldsymbol{a},\boldsymbol{b}]} g \le \left|S(g,D)-(\mathcal{P})\int_{[\boldsymbol{a},\boldsymbol{b}]} g\right| < \frac{\epsilon}{2}$$

implies

$$(\mathcal{P})\int_{[\boldsymbol{a},\boldsymbol{b}]} f - \frac{\epsilon}{2} < S(f,D)$$

and

$$S(g,D) < (\mathcal{P})\int_{[\boldsymbol{a},\boldsymbol{b}]} g + \frac{\epsilon}{2}.$$

But $S(f,g)\le S(g,D)$. Hence, we now have

$$(\mathcal{P})\int_{[\boldsymbol{a},\boldsymbol{b}]} f - \frac{\epsilon}{2} < S(f,D) \le S(g,D) < (\mathcal{P})\int_{[\boldsymbol{a},\boldsymbol{b}]} g + \frac{\epsilon}{2},$$

that is,

$$(\mathcal{P})\int_{[\boldsymbol{a},\boldsymbol{b}]} f < (\mathcal{P})\int_{[\boldsymbol{a},\boldsymbol{b}]} g + \epsilon.$$

The arbitrariness of ϵ implies

$$(\mathcal{P})\int_{[\boldsymbol{a},\boldsymbol{b}]} f \leq (\mathcal{P})\int_{[\boldsymbol{a},\boldsymbol{b}]} g. \qquad \square$$

4 Some Existence and Convergence Theorem

The following theorem exhibits a real-valued function on $\mathbb{R}^n$ which is a PU- integrable function.

Theorem 4.1. *Let $f : [\boldsymbol{a},\boldsymbol{b}] \to X$ be continuous on $[\boldsymbol{a},\boldsymbol{b}]$. Then f is PU-integrable over $[\boldsymbol{a},\boldsymbol{b}]$.*

Proof: Let $\epsilon > 0$. Note that the Riemann integral $\displaystyle\int_{[\boldsymbol{a},\boldsymbol{b}]} \varphi$ exists, whenever φ is a partition of unity on $[\boldsymbol{a},\boldsymbol{b}]$. Since f is continuous on $[\boldsymbol{a},\boldsymbol{b}]$, then f is uniformly continuous on $[\boldsymbol{a},\boldsymbol{b}]$. Hence, there exists a $\delta > 0$ such that for any $\boldsymbol{x},\boldsymbol{y} \in [\boldsymbol{a},\boldsymbol{b}]$ with $\|\boldsymbol{x}-\boldsymbol{y}\| < \delta(\boldsymbol{x})$, we have

$$|f(\boldsymbol{x}) - f(\boldsymbol{y})| < \frac{\epsilon}{2[vol([\boldsymbol{a},\boldsymbol{b}]) + 1]}.$$

Let $D_1 = \{(\boldsymbol{\xi}, \boldsymbol{I}, \varphi)\}$ and $D_2 = \{(\boldsymbol{\zeta}, \boldsymbol{J}, \psi)\}$ be any two δ-fine divisions of $[\boldsymbol{a},\boldsymbol{b}]$. Let $D_3 = \{\boldsymbol{\gamma}, \boldsymbol{K}, \sigma\}$ be a δ-fine division of $[\boldsymbol{a},\boldsymbol{b}]$, where $\boldsymbol{K} = \boldsymbol{I} \cap \boldsymbol{J}$ with $\boldsymbol{I} \in D_1$ and $\boldsymbol{J} \in D_2$. for any interval $\boldsymbol{I}$. Observe that

$$S(f, D_1) = \sum_{\boldsymbol{I}\in D_1} f(\boldsymbol{\xi})\int_{\boldsymbol{I}} \varphi = \sum_{\boldsymbol{I}\in D_1} f(\boldsymbol{\xi})\left[\sum_{\boldsymbol{J}\in D_2}\int_{\boldsymbol{I}\cap\boldsymbol{J}} \varphi\right] = \sum_{\boldsymbol{K}\in D_3} f(\boldsymbol{\xi})\int_{\boldsymbol{K}} \sigma$$

and

$$S(f, D_2) = \sum_{\boldsymbol{J}\in D_2} f(\boldsymbol{\zeta})\int_{\boldsymbol{J}} \psi = \sum_{\boldsymbol{J}\in D_2} f(\boldsymbol{\zeta})\left[\sum_{\boldsymbol{I}\in D_1}\int_{\boldsymbol{J}\cap\boldsymbol{I}} \psi\right] = \sum_{\boldsymbol{K}\in D_3} f(\boldsymbol{\zeta})\int_{\boldsymbol{K}} \sigma.$$

Then

$$\begin{aligned}
&\big|S(f, D_1) - S(f, D_2)\big| \\
&\quad\leq \big|S(f, D_1) - S(f, D_3)\big| + \big|S(f, D_3) - S(f, D_2)\big| \\
&\quad= \left|\sum_{\boldsymbol{K}\in D_3} f(\boldsymbol{\xi})\int_{\boldsymbol{K}} \sigma - \sum_{\boldsymbol{K}\in D_3} f(\boldsymbol{\gamma})\int_{\boldsymbol{K}} \sigma\right| \\
&\qquad + \left|\sum_{\boldsymbol{K}\in D_3} f(\boldsymbol{\gamma})\int_{\boldsymbol{K}} \sigma - \sum_{\boldsymbol{K}\in D_3} f(\boldsymbol{\zeta})\int_{\boldsymbol{K}} \sigma\right|
\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{\boldsymbol{K}\in D_3}\left[\|f(\boldsymbol{\xi})-f(\gamma)\|_{\mathbb{R}^n}\cdot\left|\int_{\boldsymbol{K}}\sigma\right|\right]+\sum_{\boldsymbol{K}\in D_3}\left[\|f(\gamma)-f(\zeta)\|\cdot\left|\int_{\boldsymbol{K}}\sigma\right|\right]\\
&< \sum_{\boldsymbol{K}\in D_3}\left[\frac{\epsilon}{2[vol([\boldsymbol{a},\boldsymbol{b}])+1]}\cdot\left|\int_{\boldsymbol{K}}\sigma\right|\right]+\sum_{\boldsymbol{K}\in D_3}\left[\frac{\epsilon}{2[vol([\boldsymbol{a},\boldsymbol{b}])+1]}\cdot\left|\int_{\boldsymbol{K}}\sigma\right|\right]\\
&= \frac{\epsilon}{2[vol([\boldsymbol{a},\boldsymbol{b}])+1]}\cdot\left(\sum_{\boldsymbol{K}\in D_3}\left|\int_{\boldsymbol{K}}\sigma\right|+\sum_{\boldsymbol{K}\in D_3}\left|\int_{\boldsymbol{K}}\sigma\right|\right)\\
&= \frac{\epsilon}{2[vol([\boldsymbol{a},\boldsymbol{b}])+1]}\cdot\left(2\ vol([\boldsymbol{a},\boldsymbol{b}])\right)\\
&< \epsilon.
\end{aligned}
$$

Therefore, f is PU-integrable over $[\boldsymbol{a},\boldsymbol{b}]$. □

We now establish the Uniform convergence theorem for this integral.

Lemma 4.2. *Let $f:[\boldsymbol{a},\boldsymbol{b}]\to\mathbb{R}$ be a PU-integrable function over $[\boldsymbol{a},\boldsymbol{b}]$. If f is bounded by M on $[\boldsymbol{a},\boldsymbol{b}]$, then*

$$\left|(\mathcal{P})\int_{[\boldsymbol{a},\boldsymbol{b}]} f\right| \leq M\cdot vol([\boldsymbol{a},\boldsymbol{b}]).$$

Proof: By assumption, $-M\leq f(\boldsymbol{x})\leq M$. Thus,

$$-M\cdot vol \leq (\mathcal{P})\int_{[\boldsymbol{a},\boldsymbol{b}]} f(\boldsymbol{x}) \leq M\cdot vol([\boldsymbol{a},\boldsymbol{b}]).$$

This means that

$$\left|(\mathcal{P})\int_{[\boldsymbol{a},\boldsymbol{b}]} f\right| \leq M\cdot vol([\boldsymbol{a},\boldsymbol{b}]).$$

Theorem 4.3. (The Uniform Convergence Theorem)
Assume that $\langle f_n\rangle_{n=1}^{\infty}$ is a sequence of bounded and integrable functions over $[\boldsymbol{a},\boldsymbol{b}]$. If $f_n\to f$ uniformly on $[\boldsymbol{a},\boldsymbol{b}]$, then f is PU-integrable on $[\boldsymbol{a},\boldsymbol{b}]$ and

$$\lim_{n\to\infty}(\mathcal{P})\int_{[\boldsymbol{a},\boldsymbol{b}]} f_n = (\mathcal{P})\int_{[\boldsymbol{a},\boldsymbol{b}]} f.$$

Proof: Let $\epsilon>0$. Since f_n converges uniformly on $[\boldsymbol{a},\boldsymbol{b}]$, we choose $N_1\in\mathbb{N}$ such that if $n\geq N_1$ and $\boldsymbol{x}\in[\boldsymbol{a},\boldsymbol{b}]$, we have

$$|f_n(\boldsymbol{x})-f(\boldsymbol{x})| < \frac{\epsilon}{3\cdot[vol([\boldsymbol{a},\boldsymbol{b}])+1]}. \tag{4.1}$$

If $m,n\geq N_1$ and $\boldsymbol{x}\in[\boldsymbol{a},\boldsymbol{b}]$, then by Equation 4.1

$$
\begin{aligned}
|f_n(\boldsymbol{x})-f_m(\boldsymbol{x})| &= |[f_n(\boldsymbol{x})-f(\boldsymbol{x})]+[f(\boldsymbol{x})-f_m(\boldsymbol{x})]|\\
&\leq |[f_n(\boldsymbol{x})-f(\boldsymbol{x})]|+|[f(\boldsymbol{x})-f_m(\boldsymbol{x})]|
\end{aligned}
$$

$$
\begin{aligned}
&< \frac{\epsilon}{3 \cdot [vol([\boldsymbol{a},\boldsymbol{b}])+1]} + \frac{\epsilon}{3 \cdot [vol([\boldsymbol{a},\boldsymbol{b}])+1]} \\
&= \frac{2 \cdot \epsilon}{3[vol([\boldsymbol{a},\boldsymbol{b}])+1]}.
\end{aligned}
$$

By Lemma 4.2 and by linearity, for each $n, m \geq N_1$,

$$
\begin{aligned}
\left|(\mathcal{P})\int_{[\boldsymbol{a},\boldsymbol{b}]} f_m - (\mathcal{P})\int_{[\boldsymbol{a},\boldsymbol{b}]} f_n\right| = \left|\int_{[\boldsymbol{a},\boldsymbol{b}]} (f_m - f_n)\right| &\leq \frac{2 \cdot \epsilon}{3[vol([\boldsymbol{a},\boldsymbol{b}])+1]} \cdot vol([\boldsymbol{a},\boldsymbol{b}]) \\
&< \frac{2 \cdot \epsilon}{3[vol([\boldsymbol{a},\boldsymbol{b}])+1]} \cdot vol([\boldsymbol{a},\boldsymbol{b}]) + 1 \\
&< \frac{2}{3} \cdot \epsilon.
\end{aligned}
$$

for all $m, n \geq N_1$. This shows that $\left\langle (\mathcal{P})\int_{[\boldsymbol{a},\boldsymbol{b}]} f_n \right\rangle$ is cauchy, and so $(\mathcal{P})\int_{[\boldsymbol{a},\boldsymbol{b}]} f_n$ converges to, say, A. So, choose an $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$,

$$
\left|(\mathcal{P})\int_{[\boldsymbol{a},\boldsymbol{b}]} f_n - A\right| < \frac{\epsilon}{3}. \tag{4.2}
$$

Take $N = \max\{N_1, N_2\}$. Since f_N is PU-integrable on $[\boldsymbol{a},\boldsymbol{b}]$, we may choose a gauge δ on $[\boldsymbol{a},\boldsymbol{b}]$ such that if D is a δ-fine PU-division of $[\boldsymbol{a},\boldsymbol{b}]$, then

$$
\left|S(f_N, D) - (\mathcal{P})\int_{[\boldsymbol{a},\boldsymbol{b}]} f_N\right| < \frac{\epsilon}{3}. \tag{4.3}
$$

Observe that from Equation 4.1,

$$
\begin{aligned}
|S(f,D) - S(f_N,D)| &= \left|(D)\sum f(\boldsymbol{\xi})\int_{\boldsymbol{I}} \varphi - (D)\sum f_N(\boldsymbol{\xi})\int_{\boldsymbol{I}} \varphi\right| \\
&= \left|(D)\sum (f(\boldsymbol{\xi}) - f_N(\boldsymbol{\xi}))\int_{\boldsymbol{I}} \varphi\right| \\
&\leq (D)\sum |f(\boldsymbol{\xi}) - f_N(\boldsymbol{\xi})| \cdot \left|\int_{\boldsymbol{I}} \varphi\right| \\
&< \frac{\epsilon}{3 \cdot [vol([\boldsymbol{a},\boldsymbol{b}])+1]} \cdot (D)\sum \int_{\boldsymbol{I}} \varphi \\
&\leq \frac{\epsilon}{3 \cdot [vol([\boldsymbol{a},\boldsymbol{b}])+1]} \cdot vol([\boldsymbol{a},\boldsymbol{b}]) \\
&= \frac{\epsilon}{3}.
\end{aligned}
$$

Therefore,

$$\begin{aligned}
&|S(f,D) - A| \\
&= \Bigg| S(f,D) - S(f_N,D) + S(f_N,D) \\
&\qquad\qquad - (\mathcal{P}) \int_{[\boldsymbol{a},\boldsymbol{b}]} f_N + (\mathcal{P}) \int_{[\boldsymbol{a},\boldsymbol{b}]} f_N - A \Bigg| \\
&\leq |S(f,D) - S(f_N,D)| + \left| S(f_N,D) - (\mathcal{P}) \int_{[\boldsymbol{a},\boldsymbol{b}]} f_N \right| \\
&\quad + \left| (\mathcal{P}) \int_{[\boldsymbol{a},\boldsymbol{b}]} f_N - A \right| \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\
&= \epsilon.
\end{aligned}$$

This shows the integrablity of f over $[\boldsymbol{a},\boldsymbol{b}]$. Hence,

$$\lim_{n\to\infty} (\mathcal{P}) \int_{[\boldsymbol{a},\boldsymbol{b}]} f_n = A = (\mathcal{P}) \int_{[\boldsymbol{a},\boldsymbol{b}]} f.$$

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