

A Generalized α -Laplace L_{evy} Process

Abstract

Random time changed L_{evy} Processes are getting increased attention of late as they can account for a variety of features in data. In this article we discuss α -Laplace L_{evy} Process and a generalization of it. Both are random time changed α -stable L_{evy} Processes. We obtained a characterization of α -Laplace L_{evy} Process and discuss the first passage time distribution of a generalized α -Laplace L_{evy} Process. Interestingly, this first passage time follows a discrete distribution.

KeyWords: α -Laplace, characterization, first passage time, Laplace transform, L_{evy} Processes, moment generating function

1 Introduction

Brownian motion (BM) are L_{evy} Processes (see, theorems 1.1 and 1.2 below) where $X(1)$ has a normal distribution. There are situations where a Laplace model is preferred to a Gaussian one. While Hsu (1979) used it to model the pooled position errors in a large navigation system, Sethia and Anderson (1984) used a stationary autoregressive model with Laplace marginals in communication engineering. Such possibilities motivated the introduction of Laplace process in Satheesh (1990) as a possible alternative to BM. Madan

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and Seneta (1990) proposed the variance gamma (VG) processes (same as the Laplace process) to model long tailedness inherent in data. Typically, Laplace process accounts for distributions of increments that are more peaked at the mode with thick tails. See, Satheesh (2021) for a review of data driven models starting with BM and resulting in a variety of L_{evy} Processes including fractional BM, fractional Laplace process and fractional α -stable process and also Samorodnitsky and Taqqu (2000) and Jondeau et al. (2007).

A larger class of L_{evy} processes can be derived from an α -stable L_{evy} Process $fX(t)g$ by randomizing its time parameter t using a positive continuous random variable T . The advantage is, while certain features of $fX(t)g$ are retained, those of T can be augmented to alter some others to get new useful processes. See, Schoutens (2003) and Gajda et al. (2018), for more on this.

In proposition 3.2 a generalized α -Laplace law is derived as an exponential mixture of α -stable laws. In terms of L_{evy} processes, this is equivalent to randomising the time parameter of α -stable L_{evy} process by the unit exponential.

A characterization using stochastic integrals and the first passage time distribution of Laplace process were obtained in Satheesh and Pillai (1988) and Satheesh (1990b), stated as corollaries 2.2 and 3.7 here. In this article we generalise these results to α -Laplace process and a generalized α -Laplace processes. We now brief the background needed.

Theorem 1.1. (Capasso and Bakstein, 2012, p.154) $fX(t); t \geq 0g$ is a L_{evy} process if (i) $X(0) = 0$ almost surely (ii) $X(t)$ has stationary and independent increments and (iii) $X(t)$ is continuous in probability, that is, X_s

$P \square \square!$

$s!t$

$X_t.$

Theorem 1.2. (Feller, 1971, p.303, Capasso and Bakstein, 2012, p.159) $fX(t); t \geq 0$ is a Lévy process if the distribution of $X(1)$ is infinitely divisible.

Theorem 1.3. (Capasso and Bakstein, 2012, p.160) Any Lévy process can be decomposed as $X(t) = \mu B(t) + S(t); \mu > 0$, where $B(t)$ is a Brownian motion (BM) with drift and $S(t)$ is a pure jump process.

Theorem 1.4. (Feller, 1971, p.588) A random variable X or its distribution is in class-L (or self-decomposable) if its characteristic function (CF) $\phi(s)$ has the property that $\phi(s) = \phi(cs)$ is a CF $\phi_c(s)$ for each $c \in (0; 1)$. Similar definition in

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Theorem 1.5. (Klebanov et al., 1984) A random variable X or its distribution is geometrically infinitely divisible (geometrically α -stable) if its CF $\phi(s)$ has the property that $\phi(s) = \phi_1$

$\phi_1(s)$ such that e^{-cs} is infinitely divisible (α -stable).

Similar definition in terms of MGFs and LTs holds.

α -Laplace laws are defined by their CF ϕ_1

$\phi_1(s) = \int_0^\infty e^{-cs} d\mu(c); 0 < \alpha \leq 2; c > 0$. They are

mixtures of symmetric α -stable laws, where the mixing distribution is exponential.

They are self-decomposable, geometrically infinitely divisible (George,

1990) and hence infinitely divisible (Sandhya and Pillai, 1999). Hence one can

define the corresponding α -Laplace Lévy processes (α -LLP). For $\alpha = 2$ the α -

Laplace law is Laplace and the corresponding Lévy process is Laplace process.

Laplace Process was introduced and discussed in Satheesh (1990a, 2021) as a

possible alternative to BM and was compared and contrasted with BM. It is

known that α

2 -stable law is the first passage time distribution (FPT/FPTD) of

BM with zero drift (Feller, 1971, p.174).

Eaton et al. (1971) introduced the MGF of α -stable laws. They call it

extreme stable since the parameter α in the stable model is set as $\alpha = 1$.

They have taken the location parameter also as zero. Here we refer to them as

α -stable laws. Satheesh (2022) used this to define and discuss α -stable Lévy

processes.

Theorem 1.6. Eaton et al. (1971) The function $\exp\{-b(1 - |s|)^\alpha\}; 0 < \alpha$

$\leq 2; 0 < b < \infty; \alpha \neq 1; b > 0$ are MGFs of α -stable laws.

Using this we define a generalised α -Laplace law and the corresponding

Lévy process, viz. generalised α -LLP (G-LLP) and derive its FPTD. FPTD

of processes are important as they give the distribution of the time taken for

the process to reach/ cross a barrier/ threshold. If $a > 0$ is the barrier, then

the random variable $T(a) = \inf\{t > 0 : X(t) \geq a\}$ denote the FPT of

$X(t)$. Here $t > 0$, since $X(0) = 0$ for a Lévy process.

Lukacs (1969) conceived a stochastic integral

R_B

A $g(t) dX(t)$ corresponding to a Lévy process $fX(t); t \geq 0$ in the sense of convergence in probability, where $g(t)$ is continuous in $[A;B] \subseteq \mathbb{R}$ and proved the following theorem.

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Theorem 1.7. Lukacs (1969) Let $fX(t); t \geq 0$ be a Lévy process and $g(t)$ a continuous function in $[A;B] \subseteq \mathbb{R}$. Let $f(u)$ be the CF of $X(1)$ and $h(u)$ that of the corresponding stochastic integral. Then $\ln[h(u)] =$

$$\int_A^B \ln[f(ug(t))] dt.$$

With this background we obtained a characterization of α -LLP using the above theorem in the next section. In section 3 we derive and discuss a generalization of α -Laplace law and its divisibility properties such as selfdecomposability, infinite divisibility etc.. Then we derive the FPTD of the G -LLP. These processes are obtained from α -stable Lévy processes by randomising the time parameter by the unit exponential law, see remark 3.1. Interestingly, the first passage time has a discrete distribution.

2 A characterization of α -LLP

Theorem 2.1. A Lévy process $fX(t); t \geq 0$ for which the distribution of $X(1)$ is symmetric, is α -LLP if and only if, the CF $h(u)$ of the stochastic integral $\int_0^1 g(t) dX(t)$ is given by $\ln[h(u)] = \int_0^1 (1 + |uj|^\alpha) \ln[1 + |uj|^\alpha]$:

Proof. Let $h(u)$ be the CF of the stochastic integral

$$\int_0^1 g(t) dX(t) \text{ where}$$

$$X(t) \text{ is } \alpha\text{-LLP with CF } f(u) = \int_0^1 (1 + |uj|^\alpha) \ln[1 + |uj|^\alpha]$$

$\int_0^1 \ln[f(ug(t))] dt$: Denoting $|uj|^\alpha$ by k in the following integration, we have;

$$\ln[h(u)] = \int_0^1 \ln(1 + |jut|^\alpha) dt = \int_0^1 \ln(1 + kt) dt$$

$$= \int_0^1 [t \ln(1 + kt)] dt$$

$$= \int_0^1 \ln(1 + k) + 1 - \int_0^1 \ln(1 + kt) dt;$$

$$= \ln(1 + k) + 1 - \int_0^1 \ln(1 + kt) dt;$$

$$= \ln(1 + k) + 1 - \int_0^1 \ln(1 + kt) dt;$$

—
since

$$\frac{1 + kt}{1 + kt} = 1$$

$$= \int_0^1 \ln(1 + kt) dt$$

$$\ln(1 + k) = \int_0^1 \ln(1 + kt) dt$$

Conversely, let $f(u)$ be the CF of $X(1)$, $\ln[h(u)] = \int_0^1 \ln[1 + juj] du$.

We need to find $f(u)$. Since $X(1)$ is symmetric, $f(u)$ is real and even and so

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we need to evaluate it for $u > 0$ only. Putting $g(u) = \ln[f(u)]$,

$$\int_0^1 \ln(1 + u_1) dt = \int_0^1 \ln[f(ut_1)] dt$$

$$= \int_0^u \ln(z) z^{-1} dz \quad (z = ut_1 \text{ \& } dz = u dt)$$

That is,

$$\int_0^u \ln(z) z^{-1} dz = \int_0^1 \ln(1 + u_1) du$$

$$\frac{d}{du} \int_0^1 \ln(1 + u_1) du =$$

$$\int_0^1 \frac{1}{1 + u_1} du$$

$$\frac{d}{du} \int_0^1 (1-u)^{\alpha-1} \ln(1+u) du$$

$$= \int_0^1 \frac{d}{du} (1-u)^{\alpha-1} \ln(1+u) du$$

$$= \int_0^1 (1-u)^{\alpha-1} \frac{d}{du} \ln(1+u) du$$

$$= \int_0^1 (1-u)^{\alpha-1} \frac{1}{1+u} du$$

$$= \int_0^1 (1-u)^{\alpha-1} \frac{1}{1+u} du$$

That is; $\int_0^1 (1-u)^{\alpha-1} \ln(1+u) du = \int_0^1 (1-u)^{\alpha-1} \frac{1}{1+u} du$

That completes the proof.

Corollary 2.2. With $\alpha = 2$, theorem 2.1 characterizes Laplace Process.

3 FPTD of G_{LLP}

Theorem 3.1. The function $M(s) = \int_0^1 (1-u)^{\alpha-1} e^{-su} du$

$0 \leq \text{Re}(s) < 1$; $0 < \alpha < 2$; $\alpha \neq 1$; $b > 0$ are MGFs of probability laws.

Proof. By Loève (1960, p.213), if $\phi(s)$ is analytic in the strip $0 < \text{Re}(s) < a$, continuous in $0 \leq \text{Re}(s) < a$ and $\phi(0)$ is the characteristic function (CF) of a probability law, then $\phi(s)$ is the MGF of that probability law. Now, $\int_0^1 (1-u)^{\alpha-1} e^{-su} du$

$\int_0^1 (1-u)^{\alpha-1} e^{-su} du$

analytic in the strip $0 < \text{Re}(s) < 1$ and continuous in $0 \leq \text{Re}(s) < 1$. Again, $\Gamma(b(1-\alpha)s)$ is analytic for $\text{Re}(s) > 0$ and continuous for $\text{Re}(s) \geq 0$. Hence $M(s) = \int_0^\infty e^{-st} f(t) dt$

$\int_0^\infty e^{-st} t^{b(1-\alpha)s-1} dt$ is analytic in the strip $0 < \text{Re}(s) < 1$ and continuous in $0 \leq \text{Re}(s) < 1$. Since $\exp\{-b(1-\alpha)t\}$ is the CF of α -stable laws (Eaton et al., 1971), $M(s)$ is the CF of geometrically α -stable laws (Klebanov et al.

1984), and hence $M(s)$ is the MGF of a probability law.

Note. For $\alpha = 2$ we get $M(s) = \int_0^\infty e^{-st} \frac{1}{2} e^{-bt} dt$

$\frac{1}{2} e^{-bs^2}$, the MGF of Laplace law. α -Laplace

laws are exponential mixtures of symmetric α -stable laws. By Eaton et al. (1971), the α -stable laws in theorem 1.6 are not symmetric. Hence we call the MGF $M(s)$ in the above theorem as that of a generalized α -Laplace (G_α) law. For $1 < \alpha \leq 2$ it has finite mean. One may prove theorem 3.1 with a more probabilistic flavour, as follows.

Proposition 3.2. The function $M(s) = \int_0^\infty e^{-st} t^{b(1-\alpha)s-1} dt$

$\int_0^\infty e^{-st} t^{b(1-\alpha)s-1} dt$; $0 \leq \text{Re}(s) < 1$; $0 < \alpha \leq 2$; $\int_0^\infty e^{-st} t^{b(1-\alpha)s-1} dt = 1$; $b > 0$ are MGFs of G_α laws.

Proof. Let the random variable X be α -stable with MGF $\exp\{-b(1-\alpha)t\}$. Then for $c > 0$, the MGF of $c_{1-\alpha}X$ is $\exp\{-c b(1-\alpha)t\}$. Let c be a random variable having the unit exponential law. Then the MGF of $c_{1-\alpha}X$ is

$$E_c \int_0^\infty e^{-st} t^{b(1-\alpha)s-1} dt = \int_0^\infty e^{-st} t^{b(1-\alpha)s-1} \int_0^\infty e^{-ct} dt = \int_0^\infty e^{-st} t^{b(1-\alpha)s-1} dt = 1$$

We are finding the MGF of the scale mixture of α -stable laws where the mixing distribution is unit exponential. If the MGF of the random variable Y is $M(s)$ and $E_c \sim \text{Exp}(1)$, then $Y = E_{c_{1-\alpha}X}$ is the stochastic representation of Y .

Proposition 3.3. G_α laws are geometric(p)-sum of its own type for every $p \in (0, 1)$. Hence they are geometrically infinitely divisible, infinitely divisible and also self-decomposable.

Proof. The probability generating function (PGF) of a geometric(p) law on $\{1, 2, 3, \dots\}$ is $P(s) = \frac{ps}{1-(1-p)s}$.

Hence the MGF of the geometric(p)-sum is; $P(M(s)) = \frac{pM(s)}{1-(1-p)M(s)}$. Taking $M(s)$ as the MGF of G_α we have,

$$P(M(p_{1-\alpha}s)) = \frac{p[1 + b(1-\alpha)(p_{1-\alpha}s)]}{1 - (1-p)[1 + b(1-\alpha)(p_{1-\alpha}s)]} = \frac{p}{p + b(1-\alpha)ps} = 1$$

$$1 + b(1 - \alpha)s_1 :$$

Since $0 < p_1 = \alpha < 1$, and this is true for any $p \in (0, 1)$, G_α laws are geometric(p)-sum of its own type for every $p \in (0, 1)$. Hence they are geometrically infinitely divisible and infinitely divisible, Sandhya and Pillai (1999).

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Now, rewriting the third and first lines we have,

$$\frac{1}{1 + b(1 - \alpha)s_1} = \frac{1}{[1 + b(1 - \alpha)(p_1 = s_1)]}$$

$$\frac{1}{1 - (1 - \alpha)p} = [1 + b(1 - \alpha)(p_1 = s_1)]$$

That is; $M(s) = M(p_1 = s) \cdot P_1(M(p_1 = s))$;

where P_1 is the PGF of the geometric law on $f_0; 1; 2; \dots; g$. Since $P_1(M(p_1 = s))$ is also an MGF, $0 < p_1 = \alpha < 1$ and the above equation is true for any $p \in (0, 1)$, G_α laws are self-decomposable.

Definition 3.4. Lévy processes $fX(t)$; $t \geq 0$ are generalized α -LLP (G_α -LLP), if the distribution of $X(1)$ has MGF $M(s) =$

$$\frac{1}{1 + b(1 - \alpha)s_1}; 0 \leq \text{Re}(s) < 1; 0 < \alpha < 1; \alpha \neq 1; b > 0.$$

Remark 3.1. Now, in terms of Lévy processes, proposition 3.2 means that the G_α -LLP are obtained by randomising the time parameter of α -stable Lévy process in Satheesh (2022) by the unit exponential law. Similarly, by randomising the time parameter of symmetric α -stable Lévy process by the unit exponential, α -LLP are obtained.

Since the location parameter is zero for the generalized α -Laplace laws considered here, the G_α -LLP has zero drift. We now derive the FPTD of G_α -LLP using standard arguments based on optional sampling theorem applied to the following martingale of $fX(t)$.

Proposition 3.5. For the G_α -LLP $fX(v)$; $v \geq 0$, $W(v) = \exp\{fsX(v) - \alpha v\}$, $s > 0$ a constant, is a martingale, where $\alpha = -\ln[1 + b(1 - \alpha)s_1]$.

Proof. Since, E

\square

$$E s X(v)$$

$$= e_{-\alpha v}, E(jW(v)) = E(W(v)) = e_{-\alpha v} E$$

\square

$$E s X(v)$$

\square

\square

$1 < 1$. Since Lévy processes have stationary and independent increments, for $u \leq v$, $X(v) - X(u)$ is independent of F_u , the filtration up to time u . Now, $E(W(v) | F_u) = E(\exp\{fsX(v) - \alpha v\} | F_u) =$

$$= e_{-\alpha v} E$$

\square

$$E[e^{sX(v) - X(u)}] = F_u$$

—
E

□

$$E[e^{sX(u)}] = F_u$$

$$= e^{-\rho u} E$$

□

$$E[e^{sX(v) - X(u)}]$$

$$E[e^{sX(u)}]$$

$$= e^{-\rho u} E[e^{sX(u)}]$$

$$= E[e^{sX(u) - \rho u}] = W(u):$$

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That completes the proof.

Theorem 3.6. The FPTD of G_{LLP} for $1 < \rho < 2$, is discrete ρ -stable.

Proof. Let the random variable $T(\rho) = T$ denote the FPT for the G_{LLP} $fX(t)$; $t \geq 0$ to reach or cross $\rho > 0$. We saw that for $fX(t)$, $W(t) = \exp\{sX(t) - \rho t\}$ is a martingale, where $\rho = -\ln[1 + b(1 - \rho)s]$. For a martingale $fW(t)$ and for the FPT T (which is a stopping time), $EfW(0) = EfW(T \wedge t)$. As $X(0) = 0$, $W(0) = 1$ and hence $EfW(T \wedge t) = 1$. That is,

$$E[\exp\{sX(T \wedge t) - \rho(T \wedge t)\}] = 1; \quad (1)$$

Note that for $\rho > 1$; $\rho = -\ln[1 + b(1 - \rho)s] > 0$, and so $0 \leq W(T \wedge t) \leq e^{-\rho t}$:

Now assuming $PfT < 1$ (we will justify this at the end of the proof) we may pass to the limit as $t \rightarrow \infty$ under the expectation in (1) by the optional sampling theorem, yielding;

$$1 = \lim_{t \rightarrow \infty}$$

t!1

$$E[\exp\{sX(T \wedge t) - \rho(T \wedge t)\}] = E[e^{-\rho T}]$$

—

$$E[e^{-\rho T}] = E$$

—

$$E[e^{-\rho T}] = E[e^{-\rho s}]:$$

$$\text{Now } \rho = -\ln[1 + b(1 - \rho)s] \Rightarrow s =$$

n

$$\frac{e^{-\rho} - 1}{b(1 - \rho)}$$

$$0 = 1 -$$

=

n

$$\frac{1 - e^{-\rho}}{b(\rho - 1)}$$

$$0 = 1 -$$

, and we get

the LT of the FPT as,

E

$$E[e^{-\lambda T}] = \exp$$

$$\int_0^\infty (1 - e^{-\lambda t})^{1-\alpha} [b(\lambda - 1)]^{1-\alpha} dt$$

$$= \exp$$

$$\int_0^\infty (1 - e^{-\lambda t})^{1-\alpha} dt$$

; which is that of discrete α -stable law, see Steutel and van Harn (1979). Finally, since $PfT < 1g = \lim_{\lambda \rightarrow 0} E$

$$E[e^{-\lambda T}]$$

$\lambda = 1$; T has a proper distribution, justifying our assumption $PfT < 1g = 1$. Remark 3.2. E

$$E[e^{-\lambda T}]$$

$= \int_0^\infty (1 - e^{-\lambda t})^{1-\alpha} dt$ is the LT of a probability distribution only when $0 < 1-\alpha < 1 \Rightarrow \alpha > 1$ (Satheesh and Nair, 2002, Feller, 1971, p.448) and by the one-to-one correspondence $P(e^{-\lambda t}) = L(\lambda)$; $\lambda \geq 0$, between the probability generating function P and the LT L of a discrete distribution. Also, in the proof here we need $\alpha > 0 \Rightarrow \alpha > 1$. These are the reasons for restricting the range of α to $1 < \alpha \leq 2$ in the above theorem. Corollary 3.7. When $\alpha = 2$, we have the Laplace process and the LT of T is E

$$E[e^{-\lambda T}]$$

$$= e$$

$$\int_0^\infty (1 - e^{-\lambda t})^{1-2} dt$$

which is that of discrete α -stable. Recall that the Satheesh 9 FPTD of BM is α -stable.

Remark 3.3. That the FPTD of Laplace process is discrete α -stable has intrigued the author for long, because it is the distribution of time, that is

continuous for the process. If one defines an exponential Lévy process on the same lines and find its FPTD as in theorem 3.6, it is Poisson. This is not entirely surprising, knowing the close relation between exponential and Poisson laws in the context of renewal processes. But here, we need an interpretation for this conclusion. Note that the increase in an exponential Lévy process is in jumps and hence T represents the number of jumps, which is discrete, to reach or cross the barrier u . Thus one possible reason is that the change (increase/ decrease as Laplace law is the difference of identical exponential laws) in Laplace process is in jumps. This intuition is substantiated by theorem 1.3 quoted in the introduction, which implies that among Lévy processes only BM has almost sure continuity of paths. Thus the changes in the u -LLP are also in jumps and so what T , the FPT, represents here is the number of jumps required to reach or cross the barrier. One may also note that the structure of the martingale $W(t)$ here is comparable with that of the corresponding Wald's martingale, see Karlin and Taylor (1975, p.243).

Remark 3.4. We saw that the FPTD of exponential Lévy process is Poisson. Along with the discussion in Karlin and Taylor (1975, p.321), this is a martingale proof of the inter-arrival time characterization of Poisson process.

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