A Generalised α-Laplace Lévy Process

Abstract

In this article we obtain a characterization of α-Laplace Lévy Process and discuss the first passage time distribution of a generalized α-Laplace Lévy Process.

Key Words: α-Laplace, characterization, first passage time, Laplace transform, moment generating function

1 Introduction

A characterization using stochastic integrals and the first passage time distribution of Laplace process were obtained in Satheesh and Pillai (1988) and Satheesh (1990b), stated as corollaries 2.2 and 3.7 here. In this article we generalise these results to α-Laplace process and a generalized α-Laplace processes. We now brief the background needed.

Theorem 1.1. *(Capasso and Bakstein, 2012, p.154)* $\{X(t), t \geq 0\}$ *is a Lévy process if (i)* $X(0) = 0$ *almost surely (ii)* $X(t)$ *has stationary and independent increments and (iii)* $X(t)$ *is continuous in probability.*

Theorem 1.2. *(Feller, 1971, p.303, Capasso and Bakstein, 2012, p.159)* $\{X(t), t \geq 0\}$ *is a Lévy process iff the distribution of* $X(1)$ *is infinitely divisible.*

Theorem 1.3. *(Capasso and Bakstein, 2012, p.159) Any Lévy process can be decomposed as* $X(t) = \sigma B(t) + S(t);\ \sigma > 0$, *where* $B(t)$ *is a Brownian motion (BM) with drift and* $S(t)$ *is a pure jump process.*

Definition 1.4. *(Feller,1971, p.588) A r.v.* X *or its distribution is in class-L (or self-decomposable) if its characteristic function (CF)* $\omega(s)$ *has the property that* $\omega(s)/\omega(cs)$ *is a CF* $\omega_c(s)$ *for each* $c \in (0, 1)$. *Similar definition in terms of moment generating functions (MGF) and Laplace Transforms (LT) holds.*

Definition 1.5. *(Klebanov et al.,1984) A r.v.* X *or its distribution is geometrically infinitely divisible (geometrically α-stable) iff its CF* $\omega(s)$ *has the property that* $\omega(s) = \frac{1}{1+\psi(s)}$ *such that* $e^{-\psi(s)}$ *is infinitely divisible (α-stable). Similar definition in terms of MGFs and LTs holds.*

α-Laplace laws are defined by their CF $\frac{1}{1+c|s|^{\alpha}};\ 0 < \alpha \le 2,\ c > 0$. They are mixtures of symmetric α-stable laws, where the mixing distribution is exponential. They are self-decomposable, geometrically infinitely divisible (George, 1990) and hence infinitely divisible (Sandhya and Pillai, 1999). Hence one can define the corresponding α-Laplace Lévy processes (αLLP). For $\alpha = 2$ the α-Laplace law is Laplace and the corresponding Lévy process is Laplace process. Laplace Process was introduced and discussed in Satheesh (1990a, 2021) as a possible alternative to BM and was compared and contrasted with BM. It is known that $\frac{1}{2}$-stable law is the first passage time distribution (FPT/FPTD) of BM with zero drift (Feller, 1971, p.174).

Eaton *et al.* (1971) introduced the MGF of α-stable laws. They call it extreme stable since the parameter β in the stable model is set as $\beta = 1$. They have taken the location parameter also as zero. Here we refer to them as α-stable laws. Satheesh (2022) used this to define and discuss α-stable Lévy processes.

Theorem 1.6. *Eaton et al. (1971) The function* $\exp\{-b(1-\alpha)s^{\alpha}\};\ 0 \le Re(s) < \infty;\ 0 < \alpha \le 2,\ \alpha \ne 1, b > 0$ *are MGFs of α-stable laws.*

Using this we define a generalised α-Laplace law and the corresponding Lévy process, *viz.* generalized αLLP (GαLLP) and derive its FPTD. FPTD of processes are important as they give the distribution of the time taken for the process to reach/ cross a barrier/ threshold. If $\lambda > 0$ is the barrier, then the random variable (*r.v.*) $T(\lambda) = T = \inf\{t > 0 : X(t) \geq \lambda\}$ denote the FPT of $X(t)$. Here $t > 0$, since $X(0) = 0$ for a Lévy process.

Lukacs (1969) conceived a stochastic integral $\int_A^B g(t)\, dX(t)$ corresponding to a Lévy process $\{X(t), t \in T\}$ in the sense of convergence in probability, where $g(t)$ is continuous in $[A, B] \subset T$ and proved the following theorem.

Theorem 1.7. *Lukacs (1969) Let* $\{X(t), t \in T\}$ *be a Lévy process and* $g(t)$ *a continuous function in* $[A, B] \subset T$*. Let* $f(u)$ *be the CF of* $X(1)$ *and* $h(u)$ *that of the corresponding stochastic integral. Then* $\ln[h(u)] = \int_A^B \ln[f(ug(t))]\, dt$*.*

With this background we obtain a characterization of αLLP using the above theorem in the next section and in section 3 we derive and discuss the FPTD of the GαLLP.

2 A characterization of αLLP

Theorem 2.1. *A Lévy process* $\{X(t), t \geq 0\}$ *for which the distribution of* $X(1)$ *is symmetric, is* α*LLP if and only if, the CF* $h(u)$ *of the stochastic integral* $\int_0^1 t^{1/\alpha}\, dX(t)$ *is given by* $\ln[h(u)] = 1 - (1 + |u|^{-\alpha})\ \ln[1 + |u|^{\alpha}]$*.*

Proof. Let $h(u)$ be the CF of the stochastic integral $\int_0^1 t^{1/\alpha}\, dX(t)$ where $X(t)$ is αLLP with CF $f(u) = \frac{1}{1+|u|^{\alpha}}$. Then by theorem 1.7, $\ln[h(u)] = \int_0^1 \ln[f(ut^{1/\alpha})]\, dt$. Denoting $|u|^{\alpha}$ by k in the following integration, we have;

$$
\begin{aligned}
\ln[h(u)] &= -\int_0^1 \ln(1 + |ut^{1/\alpha}|^{\alpha})\, dt = -\int_0^1 \ln(1 + kt)\, dt \\
&= -\left[w \ln(1 + kw)\right]_0^1 + \int_0^1 \frac{kw}{1 + kw}\, dw \\
&= -\ln(1 + k) + 1 - \int_0^1 \frac{1}{1 + kw} dw, \ \left(\textit{since}\ \frac{kw}{1 + kw} = 1 - \frac{1}{1 + kw}\right) \\
&= -\ln(1 + k) + 1 - \frac{1}{k}\ln(1 + k) = 1 - (1 + k^{-1})\ln(1 + k) \\
&= 1 - (1 + |u|^{-\alpha})\ \ln[1 + |u|^{\alpha}].
\end{aligned}
$$

Conversely, let $f(u)$ be the CF of $X(1)$, $\ln[h(u)] = 1 - (1 + |u|^{-\alpha})\ \ln[1 + |u|^{\alpha}]$. We need to find $f(u)$. Since $X(1)$ is symmetric, $f(u)$ is real and even and so we need to evaluate it for $u > 0$ only. Putting $\psi(u) = \ln[f(u)]$,

$$1 - \left(1 + u^{-\alpha}\right) \ln(1 + u^{\alpha}) = \int_0^1 \ln[f(ut^{1/\alpha})]\ dt = \int_0^1 \psi(ut^{1/\alpha})\ dt$$
$$= \frac{\alpha}{u^{\alpha}} \int_0^u \psi(z) z^{\alpha-1}\ dz\ (z = ut^{1/\alpha}\ \&\ dz = \frac{z\ u^{\alpha}}{\alpha\ z^{\alpha}} dt).$$

That is, $\int_0^u \psi(z) z^{\alpha-1}\ dz = \frac{u^{\alpha}}{\alpha}\{1 - (1 + u^{-\alpha}) \ln(1 + u^{\alpha})\}$. Hence,

$$\begin{aligned}
\psi(u)u^{\alpha-1} &= \frac{d}{du}\left[\frac{u^{\alpha}}{\alpha}\left\{1 - (1 + \frac{1}{u^{\alpha}}) \ln(1 + u^{\alpha})\right\}\right] \\
&= \frac{\alpha u^{\alpha-1}}{\alpha} - \frac{d}{du}\left[\frac{u^{\alpha}}{\alpha}\left\{\left(\frac{u^{\alpha}+1}{u^{\alpha}}\right) \ln(1 + u^{\alpha})\right\}\right] \\
&= u^{\alpha-1} - \frac{d}{du}\left[\left(\frac{1+u^{\alpha}}{\alpha}\right) \ln(1 + u^{\alpha})\right] \\
&= u^{\alpha-1} - u^{\alpha-1} \ln(1 + u^{\alpha}) - \frac{1+u^{\alpha}}{\alpha} \frac{1}{1+u^{\alpha}} \alpha u^{\alpha-1} \\
&= -u^{\alpha-1} \ln(1 + u^{\alpha})
\end{aligned}$$

$$\textit{That is, } \psi(u) = \ln[f(u)] = -\ln(1 + u^{\alpha}) \implies f(u) = \frac{1}{1 + |u|^{\alpha}},$$

completing the proof. □

Corollary 2.2. *With $\alpha = 2$, theorem 2.1 characterizes Laplace Process.*

3 FPTD of GαLLP

Theorem 3.1. *The function $M(s) = \frac{1}{1+b(1-\alpha)s^{\alpha}}$; $0 \le Re(s) < 1$; $0 < \alpha \le 2, \alpha \ne 1, b > 0$ are MGFs of probability laws.*

Proof. By Loève (1960, p.213), if $\Psi(s)$ is analytic in the strip $0 < Re(s) < a$, continuous in $0 \le Re(s) < a$ and $\Psi(is)$ is the characteristic function (CF) of a probability law, then $\Psi(s)$ is the MGF of that probability law. Now, $\frac{1}{1-s}$ is analytic in the strip $0 < Re(s) < 1$ and continuous in $0 \le Re(s) < 1$. Again, $-b(1 - \alpha)s^{\alpha}$ is analytic for $Re(s) > 0$ and continuous for $Re(s) \ge 0$. Hence $M(s) = \frac{1}{1+b(1-\alpha)s^{\alpha}}$ is analytic in the strip $0 < Re(s) < 1$ and continuous in $0 \le Re(s) < 1$. Since $\exp\{-b(1 - \alpha)(is)^{\alpha}\}$ is the CF of α-stable laws (Eaton *et al.*, 1971), $M(is)$ is the CF of geometrically α-stable laws (Klebanov *et al.*

1984), and hence $M(s)$ is the MGF of a probability law. □

Note. For $\alpha = 2$ we get $M(s) = \frac{1}{1-bs^2}$, the MGF of Laplace law. α-Laplace laws are exponential mixtures of symmetric α-stable laws. By Eaton *et al.* (1971), the α-stable laws in theorem 1.6 are not symmetric. Hence we call the MGF $M(s)$ in the above theorem as that of a generalized α-Laplace (GαL) law. For $1 < \alpha \leq 2$ it has finite mean. One may prove theorem 3.1 with a more probabilistic flavour, as follows.

Proposition 3.2. *The function* $M(s) = \frac{1}{1+b(1-\alpha)s^\alpha}$; $0 \leq Re(s) < 1$; $0 < \alpha \leq 2, \alpha \neq 1, b > 0$ *are MGFs of GαL laws.*

Proof. Let the *r.v.* X be α-stable with MGF $\exp\{-b(1-\alpha)s^\alpha\}$. Then for $c > 0$, the MGF of $c^{1/\alpha}X$ is $\exp\{-c\, b(1-\alpha)s^\alpha\}$. Let c be a *r.v.* having the unit exponential law. Then the MGF of $c^{1/\alpha}X$ is $E_c\left[e^{-c\ b(1-\alpha)s^\alpha}\right] = \frac{1}{1+b(1-\alpha)s^\alpha}$. □

We are finding the MGF of the scale mixture of α-stable laws where the mixing distribution is unit exponential. If the MGF of the *r.v.* Y is $M(s)$ and $E \sim Exp(1)$, then $Y = E^{1/\alpha}X$ is the stochastic representation of Y.

Proposition 3.3. *GαL laws are geometric(p)-sum of its own type for every* $p \in (0, 1)$. *Hence they are geometrically infinitely divisible, infinitely divisible and also self-decomposable.*

Proof. The probability generating function (PGF) of a geometric(p) law on $\{1, 2, 3, ...\}$ is $P(s) = \frac{ps}{1-(1-p)s}$. Hence the MGF of the geometric(p)-sum is; $P(M(s)) = \frac{pM(s)}{1-(1-p)M(s)}$. Taking $M(s)$ as the MGF of GαL we have,

$$\begin{aligned} P(M(p^{1/\alpha}s)) &= \frac{p/[1 + b(1-\alpha)(p^{1/\alpha}s)^\alpha]}{1 - (1-p)/[1 + b(1-\alpha)(p^{1/\alpha}s)^\alpha]} \\ &= \frac{p}{p + b(1-\alpha)ps^\alpha} \\ &= \frac{1}{1 + b(1-\alpha)s^\alpha}. \end{aligned}$$

Since $0 < p^{1/\alpha} < 1$, and this is true for any $p \in (0, 1)$, GαL laws are geometric(p)-sum of its own type for every $p \in (0, 1)$. Hence they are geometrically infinitely divisible and infinitely divisible, Sandhya and Pillai (1999).

Now, rewriting the third and first lines we have,

$$\frac{1}{1+b(1-\alpha)s^{\alpha}} = \frac{1}{[1+b(1-\alpha)(p^{1/\alpha}s)^{\alpha}]} \times \frac{p}{1-(1-p)/[1+b(1-\alpha)(p^{1/\alpha}s)^{\alpha}]}$$
$$That\ is,\ M(s) = M(p^{1/\alpha}s) \times P_1(M(p^{1/\alpha}s)),$$

where P_1 is the PGF of the geometric law on $\{0,1,2,...\}$. Since $P_1(M(p^{1/\alpha}s))$ is also a CF, $0 < p^{1/\alpha} < 1$ and the above equation is true for any $p \in (0,1)$, GαL laws are self-decomposable. □

Definition 3.4. *Lévy processes $\{X(t); t \geq 0\}$ are generalized αLLP (GαLLP), if the distribution of $X(1)$ has MGF $M(s) = \frac{1}{1+b(1-\alpha)s^{\alpha}}; 0 \leq Re(s) < 1;\ 0 < \alpha \leq 2, \alpha \neq 1, b > 0$.*

Since the location parameter is zero for the generalized α-Laplace laws considered here, the GαLLP has zero drift. We now derive the FPTD of GαLLP using standard arguments based on optional sampling theorem applied to the following martingale of $\{X(t)\}$.

Proposition 3.5. *For the GαLLP $\{X(v), v \geq 0\}$, $W(v) = \exp\{sX(v) - \theta v\}$, $s > 0$ a constant, is a martingale, where $\theta = -\ln[1+b(1-\alpha)s^{\alpha}]$.*

Proof. Since, $E\left(e^{sX(v)}\right) = e^{\theta v}$, $E(|W(v)|) = E(W(v)) = e^{-\theta v}E\left(e^{sX(v)}\right) = 1 < \infty$. Since Lévy processes have stationary and independent increments, for $u \leq v$, $X(v) - X(u)$ is independent of $\mathcal{F}_u$, the filtration up to time u. Now,

$$\begin{aligned} E\left(W(v)/\mathcal{F}_u\right) &= E\left(\exp\{sX(v) - \theta v/\mathcal{F}_u\}\right) \\ &= e^{-\theta v}E\left(e^{s[X(v)-X(u)]}/\mathcal{F}_u\right)E\left(e^{sX(u)}/\mathcal{F}_u\right) \\ &= e^{-\theta v}E\left(e^{sX(v-u)}\right)e^{sX(u)} \\ &= e^{-\theta v}e^{\theta(v-u)}e^{sX(u)} \\ &= e^{sX(u)-\theta u} = W(u), \end{aligned}$$

that completes the proof. □

Theorem 3.6. *The FPTD of GαLLP for $1 < \alpha \leq 2$, is discrete $\frac{1}{\alpha}$-stable.*

Proof. Let the *r.v.* $T(\lambda) = T$ denote the FPT for the GαLLP $\{X(t), t \geq 0\}$ to reach or cross $\lambda > 0$. We saw that for $\{X(t)\}$, $W(t) = \exp\{sX(t) - \theta t\}$ is a martingale, where $\theta = -\ln[1 + b(1-\alpha)s^{\alpha}]$. For a martingale $\{W(t)\}$ and for the FPT T (which is a stopping time), $E\{W(0)\} = E\{W(T \wedge t)\}$. As $X(0) = 0$, $W(0) = 1$ and hence $E\{W(T \wedge t)\} = 1$. That is,

$$E\left[\exp\{sX(T \wedge t) - \theta(T \wedge t)\}\right] = 1, \tag{1}$$

Note that for $\alpha > 1$; $\theta = -\ln[1 + b(1-\alpha)s^{\alpha}] > 0$, and so $0 \le W(T \wedge t) \le e^{s\lambda}$.

Now assuming $P\{T < \infty\} = 1$ (we will justify this at the end of the proof) we may pass to the limit as $t \to \infty$ under the expectation in (1) by the optional sampling theorem, yielding;

$$1 = \lim_{t \to \infty} E\left[\exp\{sX(T \wedge t) - \theta(T \wedge t)\}\right] = e^{s\lambda} E\left[e^{-\theta T}\right] \implies E\left[e^{-\theta T}\right] = e^{-s\lambda}.$$

Now $\theta = -\ln[1 + b(1-\alpha)s^{\alpha}] \implies s = \left\{\frac{e^{-\theta}-1}{b(1-\alpha)}\right\}^{1/\alpha} = \left\{\frac{1-e^{-\theta}}{b(\alpha-1)}\right\}^{1/\alpha}$, and we get the LT of the FPT as,

$$E\left[e^{-\theta T}\right] = \exp\left[\frac{-\lambda(1-e^{-\theta})^{1/\alpha}}{[b(\alpha-1)]^{1/\alpha}}\right] = \exp\left[-\beta(1-e^{-\theta})^{1/\alpha}\right],$$

which is that of discrete $\frac{1}{\alpha}$-stable law, see Steutel and van Harn (1979).

Finally, since $P\{T < \infty\} = \lim_{\theta \downarrow 0} E\left[e^{-\theta T}\right] = 1$, T has a proper distribution, justifying our assumption $P\{T < \infty\} = 1$. □

Remark 3.1. $E\left[e^{-\theta T}\right] = e^{-\beta(1-e^{-\theta})^{1/\alpha}}$ *is the LT of a probability distribution only when* $0 < 1/\alpha < 1 \implies \alpha > 1$ *(Satheesh and Nair, 2002, Feller, 1971, p.448) and by the one-to-one correspondence* $P(e^{-\theta}) = L(\theta)$; $\theta \ge 0$, *between the probability generating function* P *and the LT* L *of a discrete distribution. Also, in the proof here we need* $\theta > 0 \implies \alpha > 1$. *These are the reasons for restricting the range of* α *to* $1 < \alpha \le 2$ *in the above theorem.*

Corollary 3.7. *When* $\alpha = 2$, *we have the Laplace process and the LT of* T *is* $E\left[e^{-\theta T}\right] = e^{\left[\frac{-\lambda}{\sqrt{b}}\right](1-e^{\theta})^{1/2}}$ *which is that of discrete* $\frac{1}{2}$*-stable. Recall that the FPTD of BM is* $\frac{1}{2}$*-stable.*

Remark 3.2. *That the FPTD of Laplace process is* ***discrete*** $\frac{1}{2}$*-stable has intrigued the author for long, because it is the distribution of time, that is continuous for the process. If one defines an exponential Lévy process on the* *same lines and find its FPTD as in theorem 3.6, it is Poisson. This is not entirely surprising, knowing the close relation between exponential and Poisson laws in the context of renewal processes. But here, we need an interpretation for this conclusion. Note that the increase in an exponential Lévy process is in jumps and hence* T *represents the number of jumps, which is discrete, to reach or cross the barrier* λ*. Thus one possible reason is that the change (increase/ decrease as Laplace law is difference of identical exponential laws) in Laplce process is in jumps. This intuition is substantiated by the result that every Lévy process* $X(t)$ *can be decomposed as* $X(t) = \mu t + \sigma B(t) + S(t)$*, where* $\mu \in \mathbb{R}$ *is the drift,* $B(t)$ *is the standard BM and* $S(t)$ *is a pure jump process, (see theorem 2.262 in Capasso and Bakstein, 2012, p.160). Thus among Lévy processes only BM has almost sure continuity of paths. Thus the changes in the* α*LLP are also in jumps and so what* T*, the FPT, represents here is the number of jumps required to reach or cross the barrier. One may also note that the structure of the martingale* $W(t)$ *here is comparable with that of the corresponding Wald's martingale, see Karlin and Taylor (1975, p.243).*

Remark 3.3. *We saw that the FPTD of exponential Lévy process is Poisson. Along with the discussion in Karlin and Taylor (1975, p.321), this is a martingale proof of the inter-arrival time characterization of Poisson process.*

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