Short Research Article

The risk model with a constant dividend barrier affected by a threshold value

Abstract

This paper considers a new risk model with a constant dividend barrier, which the claim amount affected by a threshold value. The hypothesis of the model is presented and the integro-differential equation for the Gerber-Shiu penalty function is given. Then the linear solution of the Gerber-Shiu discounted penalty function have been figured out. The paper also derives the integro-differential equation and the linear solution of the expected discounted dividend payments.

Keywords: Gerber-Shiu penalty function; Integro-differential equation; Linear solution; Expected discounted dividend payments

1 Introduction

Ruin probability and related problems in the classical risk model have been studied extensively. But in the theory, the classical compound Possion risk model is independence between the claim amount and the interclaim time. It is not common in the real world for such an assumption. For example, in the natural catastrophic events, the total claim amount and the time elapsed since the previous catastrophes are dependent. See Boudreault(2003)[1] and Nikoloulopoulos and Karlis(2008)[2].

Since then, many authors fouced themselves on the dependent structure. Albrecher and Boxma (2004)[3]studied a dependency structure, in which the distribution of the time between two adjacent claims depends on the amount of the previous claim. M. Boudreault et al.(2006)[4] thought about a reverse dependence structure, that is, the time elapsed since the last claim determines the distribution of the next claim size. And Albrecher and Teugels (2006)[5] gave an arbitrary dependence structure expressed by a copula function. Cgadjiconstantinidis and Vrontos (2012)[6] fouced themselves on a renewal risk process,which is dependence under a Farlie-Gumbel-Morgenstern copula function and follows the Erlang(n) distribution. Guan and Hu(2021)[7] considered the risk model with INAR(1) (2021) processes.

Some authors also studied the model with a constant dividend barrier. De Finetti (1957)[8] proposed the dividend strategies for insurance risk models initially. After this, many good papers focus on finding the optiaml dividend strategy. Barrier strategies for the compounded Poisson risk have been considered by Dickson and Waters (2004)[9] and Lin et al.(2011)[10]. Li and Garrido(2004)[11] considered a renewal risk process in the presence of a constant dividend barrier in which the claim waiting times are generalized Erlang(n) distributed. Some other papers thought about the constant dividend too. See Liu and Dan(2014)[12] and Zhang Lianzeng and Liu He(2020)[13].

In this paper, the risk model in which the distribution of the claim size is controlled by a threshold value M is considered in the presence of a constant dividend barrier. If the claim arrive times T is smaller than M, then the following claim size X_i has density function $f_1(x)$, otherwise its density function is $f_2(x)$.

The paper is organsized as follows. The risk model with a threshold value in the presence of a constant dividend barrier is introduced in section 2. In section 3, we derive an integro-differential equation for the Gerber-Shiu penalty function and the linear solution to Gerber-Shiu penalty function. We analyze the expected discounted dividend payments in section 4.

2 The model

We introduce the model in this part. The new surplus process $\{U(t), t \geq 0\}$ defined as follows

$$U(t) = u + qt - \sum_{i=1}^{N(t)} X_i,$$

where $u = U(0) \geq 0$ is the initial surplus and $q(q > 0)$ is the premium rate. The claim number process $\{N(t), t \geq 0\}$ is a homogeneous Possion process. $\{W_i\}_{i=1}^{\infty}$ is a sequence of independent and identically distributed(i.i.d.) interclaim times and the claim arrival times is $T_j, j \in N^+$ which $T_j = W_1 + \ldots + W_j$, and the random variable (r.v.) W_i has an Erlang(2) distribution with expectation $1/k, k > 0$. The probability distribution function (p.d.f.) gives

$$f_W(t) = k^2 te^{-kt}.t \geq 0$$

The random variable(r.v.) X_i represents the size of the ith claim. We assume that $M_i, i = 1, 2, \ldots$ a sequence of i.i.d. non-negative random variables distributed as M with exponentially distribution with expectation $1/l, l > 0$ with p.d.f. given by

$$x(t) = le^{-lt}, t \geq 0.$$

Then the claim sizes are determined as follows: If T_i is smaller than M_i, then the following claim size X_i has density function $f_1(x)$, otherwise its density function is $f_2(x)$. Here $M_i, i = 1, 2, \ldots$ are independent of T_i and X_i. From above notations, we get that

$$P(M \leq T) = 1 - e^{-lt},$$
$$P(M > T) = e^{-lt}.$$

Assuming that the insurance company needs dividends, we consider the Barrier strategy in this paper. That is, if the surplus reaches b, all this part will be distributed to shareholders, if the surplus is less than b, no dividend will be distributed. Let D(t) denote the dividend from time 0 to time t, and $U_b(t)$ denotes the surplus process at time t under this barrier strategy,

$$U_b(t) = U(t) - D(t), \quad t \geq 0.$$

Let u is the initial capital, that is $u = U_b(0)$. Corrected surplus process satisfied

$$dU_b(t) = \begin{cases} qdt - \mathrm{d}S(t), & U_b(t) < b \\ -\mathrm{d}S(t), & U_b(t) = b \end{cases}$$

where $S(t) = \sum_{i=1}^{N(t)} X_i$.

We ask $\rho = inf_{t\geq 0}\{t, U_t < 0\}$ to be the ruin time which $\rho = \infty$ if $X_t \geq 0$. The deficit at ruin is denoted by $|U_\rho|$ and $U_{\rho-}$ is the surplus just prior to ruin. The Gerber-Shiu discounted penalty function $m_\theta(u)$ is defined as

$$m_\theta(u) = E[e^{-\theta\rho} w(U_{\rho-}, |U_\rho|) 1_{\rho<\infty} | U_0 = u],$$

where $\theta > 0, w : R^+ \times R^+ \to R^+$ is the penalty function. And the expected discounted dividend payments function is defined as

$$V_{b,\theta}(t) = E[D],$$

where

$$D = \int_0^T e^{\theta t} dD(t).$$

And also $m_\theta(u)$ a defective renewal equation in section 6. Especially, the infinite-time ruin probability is $\psi(u) = Pr(\rho < \infty)$.

3 The integro-differential equation of the $m_{b,\theta}(u)$

In this section, we want to derive the integro-differential equation of the Gerber-Shiu penalty function $m_{b,\theta}(u)$. In order to do so, we should obtain $m_{b,\theta}(u)$ at first. For $0 \leq u \leq b$, we have

$$\begin{aligned}
m_{b,\theta}(u) &= E[e^{-\theta\rho_b} w(U_b(\rho_b^-), |U_b(\rho_b)|) 1_{\rho_b<\infty} | U_0 = u] \\
&= \int_0^{\frac{b-u}{q}} e^{-\theta t} f_W(t) P(M > t) \int_0^{u+qt} m_{b,\theta}(u+qt-x) f_1(x) dx dt \\
&+ \int_0^{\frac{b-u}{q}} e^{-\theta t} f_W(t) P(M > t) \int_{u+ct}^{\infty} w(u+ct, x-(u+qt)) f_1(x) dx dt \\
&+ \int_0^{\frac{b-u}{q}} e^{-\theta t} f_W(t) P(M \leq t) \int_0^{u+qt} m_{b,\theta}(u+qt-x) f_2(x) dx dt \\
&+ \int_0^{\frac{b-u}{q}} e^{-\theta t} f_W(t) P(M \leq t) \int_{u+qt}^{\infty} w(u+ct, x-(u+qt)) f_2(x) dx dt \\
&+ \int_{\frac{b-u}{q}}^{\infty} e^{-\theta t} f_W(t) P(M > t) \int_0^{b} m_{b,\theta}(b-x) f_1(x) dx dt \\
&+ \int_{\frac{b-u}{q}}^{\infty} e^{-\theta t} f_W(t) P(M > t) \int_b^{\infty} w(b, x-b) f_1(x) dx dt \\
&+ \int_{\frac{b-u}{q}}^{\infty} e^{-\theta t} f_W(t) P(M \leq t) \int_0^{b} m_{b,\theta}(b-x) f_2(x) dx dt \\
&+ \int_{\frac{b-u}{q}}^{\infty} e^{-\theta t} f_W(t) P(M \leq t) \int_b^{\infty} w(b, x-b) f_2(x) dx dt \\
&= \int_0^{\frac{b-u}{q}} k^2 t e^{-(\theta+l+k)t} \gamma_{1,b,\theta}(u+qt) dt \\
&+ \int_0^{\frac{b-u}{q}} k^2 t e^{-(\theta+k)t} (1-e^{-lt}) \gamma_{2,b,\theta}(u+qt) dt \\
&+ \int_{\frac{b-u}{q}}^{\infty} k^2 t e^{-(\theta+l+k)t} \gamma_{1,b,\theta}(b) dt \\
&+ \int_{\frac{b-u}{q}}^{\infty} k^2 t e^{-(\theta+k)t} (1-e^{-lt}) \gamma_{2,b,\theta}(b) dt, \qquad (1)
\end{aligned}$$

where

$$\zeta_i(u) = \int_u^{\infty} w(u, x-u) f_i(x) dx,$$

and

$$\gamma_{i,b,\theta}(u) = \int_0^{u} m_{b,\theta}(u-x) f_i(x) dx + \zeta_i(u), \qquad (2)$$

for i=1,2. Simple modifications of (1) lead to

$$\begin{aligned} m_{b,\theta}(u) &= \frac{k^2}{q^2}\int_u^b (t-u)e^{-(\theta+l+k)\frac{t-u}{q}}\left(\gamma_{1,b,\theta}(t)-\gamma_{2,b,\theta}(t)\right)dt \\ &+ \frac{k^2}{q^2}\int_u^b (t-u)e^{-(\theta+k)\frac{t-u}{q}}\gamma_{2,b,\theta}(t)dt \\ &+ \frac{k^2}{q^2}\int_b^\infty (t-u)e^{-(\theta+l+k)\frac{t-u}{q}}\left(\gamma_{1,b,\theta}(b)-\gamma_{2,b,\theta}(b)\right)dt \\ &+ \frac{k^2}{q^2}\int_b^\infty (t-u)e^{-(\theta+k)\frac{t-u}{q}}\gamma_{2,b,\theta}(b)dt \\ &= \frac{k^2}{q^2}\int_u^\infty (t-u)e^{-(\theta+l+k)\frac{t-u}{q}}\left(\gamma_{1,b,\theta}(t\wedge b)-\gamma_{2,b,\theta}(t\wedge b)\right)dt \\ &+ \frac{k^2}{q^2}\int_u^\infty (t-u)e^{-(\theta+k)\frac{t-u}{q}}\gamma_{2,b,\theta}(t\wedge b)dt, \end{aligned} \tag{3}$$

for $0 \le u \le b$ where $t \wedge b = min(t,b)$.

In the following, for simiplicity we denote I and D to be the identity and the differential operators.

Theorem 1. Let $\gamma_{1,b,\theta}(u)$ be differentiable with respect to (w.r.t.) u. In the risk model with the claim amount affected by a threshold value and a constant dividend b, the Gerber-Shiu expected discount penalty function $m_{b,\theta}(u)$ satisfies the following integro-differential equation:

$$\begin{aligned} &\left(\frac{\theta+l+k}{q}I-D\right)\left(\frac{\theta+k}{q}I-D\right)\left(\frac{\theta+k}{q}I-D\right)\left(\frac{\theta+l+k}{q}I-D\right)m_{b,\theta}(u) \\ &= \frac{k^2}{q}\left[\left(\frac{\theta+k}{q}\right)^2 I - \frac{2(\theta+k)}{q}D + D^2\right]\gamma_1(t\wedge b) + \frac{k^2}{q}\frac{l}{q}\left[\frac{2(\theta+k)+l}{q}I - 2D\right]\gamma_2(t\wedge b), \end{aligned} \tag{4}$$

for $0 \le u \le b < \infty$ with boundary conditions:

$$\begin{aligned} m_{b,\theta}^{'}(b) &= m_{b,\theta}^{''}(b) = 0, \\ m_{b,\theta}^{(3)}(b) &= \frac{k^2}{q^2}\gamma_{1,b,\theta}^{'}(b), \\ m_{b,\theta}^{(4)}(b) &= -\frac{k^2}{q^2}\frac{2(k+\theta)}{q}\gamma_{1,b,\theta}^{'}(b) - \frac{k^2}{q^2}\frac{2l}{q}\gamma_{2,b,\theta}^{'}(b) + \frac{k^2}{q^2}\gamma_{1,b,\theta}^{''}(b). \end{aligned} \tag{5}$$

Proof. By looking at the equation above, we can differentiate Eq(3) and put (3) into the result, we have

$$\begin{aligned} \frac{dm_{b,\theta}(u)}{du} &= \frac{l+k+\theta}{q}m_{b,\theta}(u) - \frac{k^2}{q^2}\frac{l}{q}\int_u^\infty (t-u)e^{-\frac{k+\theta}{q}(t-u)}\gamma_{2,b,\theta}(t\wedge b)\,dt \\ &- \frac{k^2}{q^2}\int_u^\infty e^{-\frac{l+k+\theta}{q}(t-u)}\Big[\gamma_{1,b,\theta}(t\wedge b)-\gamma_{2,b,\theta}(t\wedge b)\Big]dt \\ &- \frac{k^2}{q^2}\int_u^\infty\int_u^\infty e^{-\frac{k+\theta}{q}(t-u)}\gamma_{2,b,\theta}(t\wedge b)\,dt. \end{aligned} \tag{6}$$

From Eq(6), let $f_{b,\theta}(u) = \frac{l+k+\theta}{q} m_\theta(u) - m_\theta^{'}(u)$ and differentiating $f_{b,\theta}(u)$ w.r.t. u, then we have

$$\begin{aligned} f_{b,\theta}^{'}(u) &= \frac{k+\theta}{q} f_{b,\theta}(u) - \frac{k^2}{q^2}\frac{l}{c}\int_u^\infty \int_u^\infty e^{-\frac{k+\theta}{q}(t-u)} \gamma_{2,b,\theta}(t \wedge b)\, dt \\ &+ \frac{k^2}{q^2}\frac{l}{q}\int_u^\infty e^{-\frac{l+k+\theta}{q}(t-u)} \Big[\gamma_{1,b,\theta}(t \wedge b) - \gamma_{2,b,\theta}(t \wedge b)\Big]\, dt - \frac{k^2}{q^2}\gamma_{1,b,\theta}(u). \end{aligned} \tag{7}$$

From Eq(7), let $g_{b,\theta}(u) = \frac{k+\theta}{q} f_{b,\theta}(u) - f_{b,\theta}^{'}(u)$, and differentiating $g_{b,\theta}(u)$ w.r.t. u, then we have

$$\begin{aligned} g_{b,\theta}^{'}(u) &= \frac{k+\theta}{q} g_{b,\theta}(u) - \frac{k^2}{q^2}\frac{l}{q}\frac{l}{q}\int_u^\infty e^{-\frac{l+k+\theta}{q}(t-u)} [\gamma_{1,b,\theta}(t \wedge b) - \gamma_{2,b,\theta}(t \wedge b)]\, dt \\ &+ \frac{k^2}{q^2}\frac{l-k-\theta}{q}\gamma_{1,b,\theta}(u) - 2\frac{k^2}{q^2}\frac{l}{q}\gamma_{2,b,\theta}(u) + \frac{k^2}{q^2}\gamma_{1,b,\theta}^{'}(u). \end{aligned} \tag{8}$$

From Eq(8), let $h_{b,\theta}(u) = \frac{k+\theta}{q} g_{b,\theta}(u) - g_{b,\theta}^{'}(u)$, and differentiating $h_{b,\theta}(u)$ w.r.t. u, then we have

$$\begin{aligned} h_{b,\theta}^{'}(u) &= \frac{l+k+\theta}{q} h_{b,\theta}(u) - \frac{k^2}{q^2}\frac{(k+\theta)^2}{q^2}\gamma_{1,b,\theta}(u) - \frac{k^2}{q^2}\frac{l(l+2k+2\theta)}{q^2}\gamma_{2,b,\theta}(u) \\ &+ \frac{k^2}{q^2}\frac{2(k+\theta)}{q}\gamma_{1,b,\theta}^{'}(u) + 2\frac{k^2}{q^2}\frac{l}{q}\gamma_{2,b,\theta}^{'}(u) - \frac{k^2}{q^2}\gamma_{1,b,\theta}^{''}(u). \end{aligned} \tag{9}$$

From Eq(9), let $p_{b,\theta}(u) = \frac{l+k+\theta}{q} h_{b,\theta}(u) - h_{b,\theta}^{'}(u)$, then we have

$$\begin{aligned} p_{b,\theta}(u) &= \frac{k^2}{q^2}\left[\frac{(k+\theta)^2}{q^2} I - \frac{2(k+\theta)}{q} D + D^2\right]\gamma_{1,b,\theta}(u) \\ &+ \frac{k^2}{q^2}\left[\frac{l(l+2k+2\theta)}{q^2} I - 2\frac{l}{q} D\right]\gamma_{2,b,\theta}(u). \end{aligned} \tag{10}$$

Above all, using the identity and differentiation operators, we can easily get the Eq (4) and the boundary condition when $u = b$.

3.1 Linear solution to $m_{b,\theta}(u)$

We can know that Eq (4) doesn't depend on the dividend boundary b, so we can obtain the Gerber-Shiu expected discount penalty function $m_{\infty,\theta}(u)$ with no dividend boundary satisfies the following inhomogeneous integro-differential equation:

$$\begin{aligned} &\Big(\frac{\theta+l+k}{q} I - D\Big)\Big(\frac{\theta+k}{q} I - D\Big)\Big(\frac{\theta+k}{q} I - D\Big)\Big(\frac{\theta+l+k}{q} I - D\Big) m_{\infty,\theta}(u) \\ &= \frac{k^2}{q}\left[\Big(\frac{\theta+k}{q}\Big)^2 I - \frac{2(\theta+k)}{q} D + D^2\right] (\int_0^u m_{\infty,\theta}(u-x) f_1(x) dx \\ &+ \frac{k^2}{q}\frac{l}{q}\left[\frac{2(\theta+k)+l}{q} I - 2D\right]\int_0^u m_{\infty,\theta}(u-x) f_2(x) dx. \end{aligned} \tag{11}$$

The defective equation can be obtained by Laplace transform of the $m_{\infty,\theta}(u)$ when $0 \le u < \infty$. From Theorem 1, we can see that the $m_{b,\theta}(u)$ can be expressed as a combination of a particular solution and four linearly independent solutions, where the four linearly independent solutions satisfy the following integro-differential equation:

$$\begin{aligned}
&\left(\frac{\theta+l+k}{q}I-D\right)\left(\frac{\theta+k}{q}I-D\right)\left(\frac{\theta+k}{q}I-D\right)\left(\frac{\theta+l+k}{q}I-D\right)y(u)\\
&=\frac{k^2}{q}\left[\left(\frac{\theta+k}{q}\right)^2 I-\frac{2(\theta+k)}{q}D+D^2\right]\int_0^u y(u-x)f_1(x)dx\\
&+\frac{k^2}{q}\frac{l}{q}\left[\frac{2(\theta+k)+l}{q}I-2D\right]\int_0^u y(u-x)f_2(x)dx. \qquad (12)
\end{aligned}$$

Let

$$\hat{f}_i(s)=\int_0^\infty e^{-sx}f_i(x)dx, \quad i=1,2.$$

and

$$\begin{aligned}
G&=\left(\frac{l+k+\theta}{q}-s\right)\left(\frac{k+\theta}{q}-s\right)\left(\frac{k+\theta}{q}-s\right)\left(\frac{l+k+\theta}{q}-s\right)\\
&-\frac{k^2}{q^2}\left[\frac{(k+\theta)^2}{q^2}-\frac{2(k+\theta)}{q}s+s^2\right]\hat{f}_1(s)-\frac{k^2}{q^2}\left[\frac{l(l+2k+2\theta)}{q^2}-\frac{2ls}{q}\right]\hat{f}_2(s).
\end{aligned}$$

In order to get the four solutions, we take the Laplace transform of the Eq(12):

$$\hat{y}(s)=\int_0^\infty e^{-su}y(u)du.$$

Let's say the four linearly independent solutions as $\{y_{1,\theta}(u)\},\{y_{2,\theta}(u)\},\{y_{3,\theta}(u)\},\{y_{4,\theta}(u)\}$, where

$$\begin{aligned}
G*\hat{y}_{1,\theta}(s)&=s^3-2\frac{l+2k+2\theta}{q}s^2+\left[\frac{(k+\theta)^2}{q^2}+4\frac{k+\theta}{q}\frac{l+k+\theta}{q}+\frac{(l+k+\theta)^2}{q^2}\right]s\\
&-2\frac{k+\theta}{q}\frac{l+k+\theta}{q}\frac{l+2k+2\theta}{q}
\end{aligned}$$

$$G*\hat{y}_{2,\theta}(s)=s^2-2\frac{l+2k+2\theta}{q}s+\left[\frac{(k+\theta)^2+(l+k+\theta)^2}{q^2}+4\frac{(k+\theta)(l+k+\theta)}{q^2}\right]$$

$$G*\hat{y}_{3,\theta}(s)=s-2\frac{l+2k+2\theta}{q}$$

$$G*\hat{y}_{4,\theta}(s)=1.$$

Theorem 2. One expression for the Gerber-Shiu expected penalty function $m_{b,\theta}(u)$ is:

$$m_{b,\theta}(u)=m_{\infty,\theta}(u)+S_1y_{1,\theta}(u)+S_2y_{2,\theta}(u)+S_3y_{3,\theta}(u)+S_4y_{4,\theta}(u), \quad 0\le u\le b, \qquad (13)$$

where S_1, S_2, S_3, S_4 are the solutions of the following linear equations:

$$S_1 y_{1,\theta}^{'}(b) + S_2 y_{2,\theta}^{'}(b) + S_3 y_{3,\theta}^{'}(b) + S_4 y_{4,\theta}^{'}(b) = -m_{\infty,\theta}^{'}(b) \tag{14}$$

$$S_1 y_{1,\theta}^{''}(b) + S_2 y_{2,\theta}^{''}(b) + S_3 y_{3,\theta}^{''}(b) + S_4 y_{4,\theta}^{''}(b) = -m_{\infty,\theta}^{''}(b) \tag{15}$$

$$\begin{aligned}
& S_1 \left[y_{1,\theta}^{(3)}(b) - \frac{k^2}{q^2} D \int_0^u y_{1,\theta}(b-x) f_1(x)\, dx \right] \\
& + S_2 \left[y_{2,\theta}^{(3)}(b) - \frac{k^2}{q^2} D \int_0^u y_{2,\theta}(b-x) f_1(x)\, dx \right] \\
& + S_3 \left[y_{3,\theta}^{(3)}(b) - \frac{k^2}{q^2} D \int_0^u y_{3,\theta}(b-x) f_1(x)\, dx \right] \\
& + S_4 \left[y_{4,\theta}^{(3)}(b) - \frac{k^2}{q^2} D \int_0^u y_{4,\theta}(b-x) f_1(x)\, dx \right] \\
& = \frac{k^2}{q^2} D \int_0^u m_{\infty,\theta}(b-x) f_1(x)\, dy + \frac{k^2}{q^2} \zeta_1^{'}(b) - m_{\infty,\theta}^{(3)}(b).
\end{aligned} \tag{16}$$

$$\begin{aligned}
& S_1 \Big[y_{1,\theta}^{(4)}(b) + \frac{k^2}{q^2} \frac{2(k+\theta)}{q} D \int_0^u y_{1,\theta}(b-x) f_1(x)\, dx + \frac{k^2}{q^2} \frac{2l}{q} D \int_0^u y_{1,\theta}(b-x) f_2(x)\, dx \\
& - \frac{k^2}{q^2} D^2 \int_0^u y_{1,\theta}(b-x) f_1(x)\, dx \\
& + S_2 \Big[y_{2,\theta}^{(4)}(b) + \frac{k^2}{q^2} \frac{2(k+\theta)}{q} D \int_0^u y_{2,\theta}(b-x) f_1(x)\, dx + \frac{k^2}{q^2} \frac{2l}{q} D \int_0^u y_{2,\theta}(b-x) f_2(x)\, dx \\
& - \frac{k^2}{q^2} D^2 \int_0^u y_{2,\theta}(b-x) f_1(x)\, dx \\
& + S_3 \Big[y_{3,\theta}^{(4)}(b) + \frac{k^2}{q^2} \frac{2(k+\theta)}{q} D \int_0^u y_{3,\theta}(b-x) f_1(x)\, dx + \frac{k^2}{q^2} \frac{2l}{q} D \int_0^u y_{3,\theta}(b-x) f_2(x)\, dx \\
& - \frac{k^2}{q^2} D^2 \int_0^u y_{3,\theta}(b-x) f_1(x)\, dx \\
& + S_4 \Big[y_{4,\theta}^{(4)}(b) + \frac{k^2}{q^2} \frac{2(k+\theta)}{q} D \int_0^u y_{4,\theta}(b-x) f_1(x)\, dx + \frac{k^2}{q^2} \frac{2l}{q} D \int_0^u y_{4,\theta}(b-x) f_2(x)\, dx \\
& - \frac{k^2}{q^2} D^2 \int_0^u y_{4,\theta}(b-x) f_1(x)\, dx \\
& = \frac{k^2}{q^2} \frac{2l}{q} \int_0^u m_{\infty,\theta}(b-u) f_2(x)\, dx + \frac{k^2}{q^2} D^2 \int_0^\infty m_{\infty,b}(u-x) f_1(x)\, dx - \frac{k^2}{q^2} \frac{2(k+\theta)}{q} \zeta_2^{'}(b) \\
& + \frac{k^2}{q^2} \zeta_1^{''}(b) - m_{\infty,\theta}^{(4)}(b).
\end{aligned} \tag{17}$$

Proof. Since the $m_{b,\theta}(u)$ satifies the the boundary condition (5), then we can get the Eq(14) and Eq(15). Differentiating the Eq(2), we have

$$
\begin{aligned}
D\gamma_{i,b,\theta}(u) &= D\Big(\int_0^u m_{b,\theta}(u-x)f_i(x)dx + \zeta_i(u)\Big) \\
&= S_1 D\int_0^u y_{1,\theta}(u-x)f_i(x)dx + S_2 D\int_0^u y_{2,\theta}(u-x)f_i(x)dx \\
&+ S_3 D\int_0^u y_{3,\theta}(u-x)f_i(x)dy + S_4 D\int_0^u y_{4,\theta}(u-x)f_i(x)dx \\
&+ D\int_0^u m_{\infty,\theta}(u-x)f_i(x)dy + D\zeta_i(u),
\end{aligned}
$$

and

$$
\begin{aligned}
D^2\gamma_{i,b,\theta}(u) &= D^2\Big(\int_0^u m_{b,\theta}(u-x)f_i(x)dx + \zeta_i(u)\Big) \\
&= S_1 D^2\int_0^u y_{1,\theta}(u-x)f_i(x)dx + S_2 D^2\int_0^u y_{2,\theta}(u-x)f_i(x)dx \\
&+ S_3 D^2\int_0^u y_{3,\theta}(u-x)f_i(x)dx + S_4 D^2\int_0^u y_{4,\theta}(u-x)f_i(x)dx \\
&+ D^2\int_0^u m_{\infty,\theta}(u-x)f_i(x)dy + D^2\zeta_i(u).
\end{aligned}
$$

Then using them, we can get Eq(16) and Eq(17) at u=b.

4 Analysis of the expected discounted dividend payments

In this section, we analyze the expect discounted dividends $V_{b,\theta}$ before ruin. In order to find condition $V_{b,\theta}$ on the first time T and the amount of the claim X, when $0 \le u \le b$, we get that

$$
\begin{aligned}
V_{b,\theta}(u) =& \int_0^{\frac{b-u}{q}} k^2 t e^{-(\theta+k)t} P(M>t) \int_0^{u+ct} V_{b,\theta}(u+qt-x) f_1(x) dx dt \\
&+ \int_0^{\frac{b-u}{q}} k^2 t e^{-(\theta+k)t} P(M\le t) \int_0^{u+ct} V_{b,\theta}(u+qt-x) f_2(x) dx dt \\
&+ \int_{\frac{b-u}{q}}^{\infty} k^2 t e^{-kt} P(M>t) \int_0^{b} \Big(q e^{-\theta\left(\frac{b-u}{q}\right)} \overline{a}_{\overline{t-\frac{b-u}{q}|}} + e^{-\theta t} V_{b,\theta}(b-x) \Big) f_1(x) dx dt \\
&+ \int_{\frac{b-u}{q}}^{\infty} k^2 t e^{-kt} P(M\le t) \int_0^{b} \Big(q e^{-\theta\left(\frac{b-u}{q}\right)} \overline{a}_{\overline{t-\frac{b-u}{q}|}} + e^{-\theta t} V_{b,\theta}(b-x) \Big) f_2(x) dx dt \\
&+ \int_{\frac{b-u}{q}}^{\infty} k^2 t e^{-kt} P(M>t) \int_b^{\infty} \Big(q e^{-\theta\left(\frac{b-u}{q}\right)} \overline{a}_{\overline{t-\frac{b-u}{q}|}} \Big) f_1(x) dx dt \\
&+ \int_{\frac{b-u}{q}}^{\infty} k^2 t e^{-kt} P(M\le t) \int_b^{\infty} \Big(q e^{-\theta\left(\frac{b-u}{q}\right)} \overline{a}_{\overline{t-\frac{b-u}{q}|}} \Big) f_2(x) dx dt. \\
&= \frac{k^2}{q^2} \int_u^{\infty} (t-u) e^{-\frac{l+k+\theta}{q}(t-u)} \Big[\alpha_{1,b,\theta}(t\wedge b) - \alpha_{2,b,\theta}(t\wedge b) \Big] \, dt \\
&+ \frac{k^2}{q^2} \int_u^{\infty} (t-u) e^{-\frac{k+\theta}{q}(t-u)} \alpha_{2,b,\theta}(t\wedge b) \, dt \\
&+ \Big[\Big(\frac{q}{k} - \frac{q}{k+\theta} \Big)(b-u) + \Big(\frac{q}{k}\Big)^2 - \Big(\frac{q}{k+\theta}\Big)^2 \Big] e^{-\frac{k+\theta}{q}(b-u)},
\end{aligned}
\tag{18}
$$

where

$$
\alpha_{i,b,\theta}(u) = \int_0^u V_{b,\theta}(u-y) f_i(y) dy, \quad i = 1, 2.
$$

Theorem 3. In this risk model with a constant dividend barrier b, the expect discounted dividends $V_{b,\theta}$ satifies the following integro-differential equation:

$$
\begin{aligned}
&\Big(\frac{\theta+l+k}{q} I - D\Big)\Big(\frac{\theta+k}{q} I - D\Big)\Big(\frac{\theta+k}{q} I - D\Big)\Big(\frac{\theta+l+k}{q} I - D\Big) V_{b,\theta}(u) \\
&= \frac{k^2}{q} \Big[\Big(\frac{\theta+k}{q}\Big)^2 I - \frac{2(\theta+k)}{q} D + D^2 \Big] \int_0^u V_{b,\theta}(u-x) f_1(x) dx \\
&+ \frac{k^2}{q} \frac{l}{q} \Big[\frac{2(\theta+k)+l}{q} I - 2D \Big] \int_0^u V_{b,\theta}(u-x) f_2(x) dx, \quad 0 \le u \le \infty,
\end{aligned}
\tag{19}
$$

where the boundary condition is:

$$
\begin{aligned}
V'_{b,\theta}(b) &= \frac{\theta q}{k^2}, \\
V''_{b,\theta}(b) &= \frac{\theta^2}{k^2}, \\
V^{(3)}_{b,\theta}(b) &= \frac{k^2}{q^2} \alpha'_{1,b,\theta}(b) + \frac{l\theta}{qk}, \\
V^{(4)}_{b,\theta}(b) &= -\frac{k^2}{q^2} \frac{2(k+\theta)}{q} \alpha'_{1,b,\theta}(b) - \frac{k^2}{q^2} \frac{2l}{q} \alpha'_{2,b,\theta}(b) + \frac{k^2}{q^2} \alpha''_{1,b,\theta}(b).
\end{aligned}
\tag{20}
$$

Proof. By looking at the equation above, we can differentiate Eq(19) and put Eq(19) into the result, we have

$$\begin{aligned}
&\frac{dV_{b,\theta}(u)}{du}\\
&= \frac{l+k+\theta}{q}V_{b,\theta}(u) - \frac{k^2}{q^2}\frac{l}{q}\int_u^\infty (t-u)e^{-\frac{k+\theta}{q}(t-u)}\alpha_{2,b,\theta}(t\wedge b)\,dt\\
&- \frac{k^2}{q^2}\int_u^\infty e^{-\frac{l+k+\theta}{q}(t-u)}\Big[\alpha_{1,b,\theta}(t\wedge b) - \alpha_{2,b,\theta}(t\wedge b)\Big]\,dt\\
&- \frac{k^2}{q^2}\int_u^\infty\int_u^\infty e^{-\frac{k+\theta}{q}(t-u)}\alpha_{2,b,\theta}(t\wedge b)\,dt\\
&- \left\{\frac{l}{q}\left(\frac{q}{k} - \frac{q}{k+\theta}\right)(b-u) + \frac{l}{q}\left[\left(\frac{q}{k}\right)^2 - \left(\frac{q}{k+\theta}\right)^2\right] + \left(\frac{q}{k} - \frac{q}{k+\theta}\right)\right\}e^{-\frac{k+\theta}{q}(b-u)}. \qquad (21)
\end{aligned}$$

From Eq(21), let $f_{b,\theta}(u) = \frac{l+k+\theta}{q}V_{b,\theta}(u) - V^{'}_{b,\theta}(u)$ and differentiating $f_{b,\theta}(u)$ w.r.t. u, then we have

$$\begin{aligned}
f^{'}_{b,\theta}(u) &= \frac{k^2}{q^2}\frac{l}{q}\frac{k+\theta}{q}\int_u^\infty (t-u)e^{-\frac{k+\theta}{q}(t-u)}\alpha_{2,b,\theta}(t\wedge b)\,dt\\
&- \frac{k^2}{q^2}\frac{l-k-\theta}{q}\int_u^\infty e^{-\frac{k+\theta}{q}(t-u)}\alpha_{2,b,\theta}(t\wedge b)\,dt\\
&+ \frac{k^2}{q^2}\frac{l+k+\theta}{q}\int_u^\infty e^{-\frac{l+k+\theta}{q}(t-u)}\Big[\alpha_{1,b,\theta}(t\wedge b) - \alpha_{2,b,\theta}(t\wedge b)\Big]\,dt\\
&- \frac{k^2}{q^2}\alpha_1(t\wedge b) + \left[-\frac{l}{q}\left(\frac{q}{k} - \frac{q}{k+\theta}\right)\right]e^{-\frac{k+\theta}{q}(b-u)}\\
&+ \frac{k+\theta}{q}\left\{\frac{l}{q}\left(\frac{q}{k} - \frac{q}{k+\theta}\right)(b-u) + \frac{l}{q}\left[\left(\frac{q}{k}\right)^2 - \left(\frac{q}{k+\theta}\right)^2\right] + \left(\frac{q}{k} - \frac{q}{k+\theta}\right)\right\}e^{-\frac{k+\theta}{q}(b-u)}\\
&= \frac{k+\theta}{q}f_{b,\theta}(u) - \frac{k^2}{q^2}\frac{l}{q}\int_u^\infty\int_u^\infty e^{-\frac{k+\theta}{q}(t-u)}\gamma_{2,b,\theta}(t\wedge b)\,dt\\
&+ \frac{k^2}{q^2}\frac{l}{q}\int_u^\infty e^{-\frac{l+k+\theta}{q}(t-u)}\Big[\alpha_{1,b,\theta}(t\wedge b) - \alpha_{2,b,\theta}(t\wedge b)\Big]\,dt - \frac{k^2}{q^2}\alpha_{1,b,\theta}(u).\\
&- \frac{l}{q}\left(\frac{q}{k} - \frac{q}{k+\theta}\right)e^{-\frac{k+\theta}{q}(b-u)}. \qquad (22)
\end{aligned}$$

From Eq(22), let $g_{b,\theta}(u) = \frac{k+\theta}{q}f_{b,\theta}(u) - f^{'}_{b,\theta}(u)$, and differentiating $g_{b,\theta}(u)$ w.r.t. u, then we have

$$\begin{aligned}
g^{'}_{b,\theta}(u) &= \frac{k+\theta}{q}g_{b,\theta}(u)\\
&- \frac{k^2}{q^2}\frac{l}{q}\frac{l}{q}\int_u^\infty e^{-\frac{l+k+\theta}{q}(t-u)}[\alpha_{1,b,\theta}(t\wedge b) - \alpha_{2,b,\theta}(t\wedge b)]\,dt\\
&+ \frac{k^2}{q^2}\frac{l-k-\theta}{q}\alpha_{1,b,\theta}(u) - 2\frac{k^2}{q^2}\frac{l}{q}\alpha_{2,b,\theta}(u) + \frac{k^2}{q^2}\alpha^{'}_{1,b,\theta}(u).
\end{aligned} \qquad (23)$$

From Eq(23), let $h_{b,\theta}(u) = \frac{k+\theta}{q} g_{b,\theta}(u) - g^{'}_{b,\theta}(u)$, and differentiating $h_{b,\theta}(u)$ w.r.t. u, then we have

$$
\begin{aligned}
h^{'}_{b,\theta}(u) &= \frac{k^2}{q^2}\frac{l}{q}\frac{l}{q}\frac{l+k+\theta}{q}\int_u^\infty e^{-\frac{l+k+\theta}{q}(t-u)}[\alpha_{1,b,\theta}(t\wedge b) - \alpha_{2,b,\theta}(t\wedge b]\,dt \\
&- \frac{k^2}{q^2}\frac{l}{q}\frac{l}{q}[\alpha_{1,b,\theta}(u) - \alpha_{2,b,\theta}(u)] - \frac{k^2}{q^2}\frac{l-k-\theta}{q}\alpha^{'}_{1,b,\theta}(u) \\
&+ 2\frac{k^2}{q^2}\frac{l}{q}\alpha^{'}_{2,b,\theta}(u) - \frac{k^2}{q^2}\alpha^{"}_{1,b,\theta}(u). \\
&= \frac{l+k+\theta}{q}h_{b,\theta}(u) - \frac{k^2}{q^2}\frac{(k+\theta)^2}{q^2}\alpha_{1,b,\theta}(u) - \frac{k^2}{q^2}\frac{l(l+2k+2\theta)}{q^2}\gamma_{2,b,\theta}(u) \\
&+ \frac{k^2}{q^2}\frac{2(k+\theta)}{q}\alpha^{'}_{1,b,\theta}(u) + 2\frac{k^2}{q^2}\frac{l}{q}\alpha^{'}_{2,b,\theta}(u) - \frac{k^2}{q^2}\alpha^{"}_{1,b,\theta}(u).
\end{aligned}
\tag{24}
$$

From Eq(24), let $p_{b,\theta}(u) = \frac{l+k+\theta}{q} h_{b,\theta}(u) - h^{'}_{b,\theta}(u)$, then we have

$$
\begin{aligned}
p_{b,\theta}(u) &= \frac{k^2}{q^2}\frac{(k+\theta)^2}{q^2}\alpha_{1,b,\theta}(u) - \frac{k^2}{q^2}\frac{l(l+2k+2\theta)}{q^2}\alpha_{2,b,\theta}(u) + \frac{k^2}{q^2}\frac{2(k+\theta)}{q}\alpha^{'}_{1}(u) \\
&- 2\frac{k^2}{q^2}\frac{l}{q}\alpha^{'}_{2,b,\theta}(u) - \frac{k^2}{q^2}\alpha^{"}_{1,b,\theta}(u) \\
&= \frac{k^2}{q^2}\left[\frac{(k+\theta)^2}{q^2}I - \frac{2(k+\theta)}{q}D + D^2\right]\alpha_{1,b,\theta}(u) \\
&+ \frac{k^2}{q^2}\left[\frac{l(l+2k+2\theta)}{q^2}I - 2\frac{l}{q}D\right]\alpha_{2,b,\theta}(u).
\end{aligned}
\tag{25}
$$

So, we can get that the Eq(19), and the boundary condition (20) when $u = b$.

4.1 Linear solution to $V_{b,\theta}(u)$

Theorem 4. In this risk model which satisfying the preceding conditions, there is a fixed dividend boundary b,

$$V_{b,\theta} = \eta_1 y_{1,\theta}(u) + \eta_2 y_{2,\theta}(u) + \eta_3 y_{3,\theta}(u) + \eta_4 y_{4,\theta}(u), \quad 0 \le u \le b \tag{26}$$

where constants $\eta_1, \eta_2, \eta_3, \eta_4$ are solutions to the following system of linear equations:

$$\eta_1 y^{'}_{1,\theta}(b) + \eta_2 y^{'}_{2,\theta}(b) + \eta_3 y^{'}_{3,\theta}(b) + \eta_4 y^{'}_{4,\theta}(b) = \frac{\theta q}{k^2} \tag{27}$$

$$\eta_1 y^{"}_{1,\theta}(b) + \eta_2 y^{"}_{2,\theta}(b) + \eta_3 y^{"}_{3,\theta}(b) + \eta_4 y^{"}_{4,\theta}(b) = \frac{\theta^2}{k^2} \tag{28}$$

$$
\begin{aligned}
&\eta_1\left[y_{1,\theta}^{(3)}(b)-\frac{k^2}{q^2}D\int_0^u y_{1,\theta}(b-x)f_1(x)\,dx\right]\\
&+\eta_2\left[y_{2,\theta}^{(3)}(b)-\frac{k^2}{q^2}D\int_0^u y_{2,\theta}(b-x)f_1(x)\,dx\right]\\
&+\eta_3\left[y_{3,\theta}^{(3)}(b)-\frac{k^2}{q^2}D\int_0^u y_{3,\theta}(b-x)f_1(x)\,dx\right]\\
&+\eta_4\left[y_{4,\theta}^{(3)}(b)-\frac{k^2}{q^2}D\int_0^u y_{4,\theta}(b-x)f_1(x)\,dx\right]\\
&=\frac{l\theta}{k^2}.
\end{aligned}
\tag{29}
$$

$$
\begin{aligned}
&\eta_1\Big[y_{1,\theta}^{(4)}(b)+\frac{k^2}{q^2}\frac{2(k+\theta)}{q}D\int_0^u y_{1,\theta}(b-x)f_1(x)\,dx+\frac{k^2}{q^2}\frac{2l}{q}D\int_0^u y_{1,\theta}(b-y)f_2(x)\,dx\\
&-\frac{k^2}{q^2}D^2\int_0^u y_{1,\theta}(b-x)f_1(x)\,dx\\
&+\eta_2\Big[y_{2,\theta}^{(4)}(b)+\frac{k^2}{q^2}\frac{2(k+\theta)}{q}D\int_0^u y_{2,\theta}(b-x)f_1(x)\,dx+\frac{k^2}{q^2}\frac{2l}{q}D\int_0^u y_{2,\theta}(b-x)f_2(x)\,dx\\
&-\frac{k^2}{q^2}D^2\int_0^u y_{2,\theta}(b-x)f_1(x)\,dx\\
&+\eta_3\Big[y_{3,\theta}^{(4)}(b)+\frac{k^2}{q^2}\frac{2(k+\theta)}{q}D\int_0^u y_{3,\theta}(b-x)f_1(x)\,dx+\frac{k^2}{q^2}\frac{2l}{q}D\int_0^u y_{3,\theta}(b-x)f_2(x)\,dx\\
&-\frac{k^2}{q^2}D^2\int_0^u y_{3,\theta}(b-x)f_1(x)\,dx\\
&+\eta_4\Big[y_{4,\theta}^{(4)}(b)+\frac{k^2}{q^2}\frac{2(k+\theta)}{q}D\int_0^u y_{4,\theta}(b-x)f_1(x)\,dx+\frac{k^2}{q^2}\frac{2l}{q}D\int_0^u y_{4,\theta}(b-x)f_2(x)\,dx\\
&-\frac{k^2}{q^2}D^2\int_0^u y_{4,\theta}(b-x)f_1(x)\,dx\\
&=0
\end{aligned}
\tag{30}
$$

Proof. It's kind of a theorem 2.

5 Conclusion

In this paper, we have considered a new risk model with a constant dividend barrier, which the claim amount affected by a threshold value. We derived the Gerber-Shiu penalty function at first, and then the integro-differential equation has given. Then the linear solution of the Gerber-Shiu discounted penalty function have been figured out. The paper also derived the integro-differential equation and the linear solution of the expected discounted dividend payments function.

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